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INITIAL-VALUE PROBLEM FOR A HIGHER-ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATION

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Abstract. We examine an initial-value problem for a certain higher-order quasilinear partial differential equation. Expressing the partial differential operator as the superposition of first-order operators, we apply methods of solution of first-order equations. We prove the unique solvability of the initial-value problem considered.

***Keywords and phrases*:** initial-value problem, characteristics, directional derivative, superposition of differential operators, unique solvability.

***AMS Subject Classification*:** 35A30, 35C15, 35G55, 35L30

1. Statement of the problem. Higher-order partial differential equations are of great interest in numerous physical applications (see [1, 4–8, 11–13]. First-order partial differential equations can be locally solved by the methods of the theory of ordinary differential equations by the reduction to characteristic systems. Application of the method of characteristics allows one to reduce the study of the evolution of waves to the study of particle propagation (see [2]). In [3], a technique for integrating first-order nonlinear partial differential equations was developed. In [9, 10], inverse problems for first-order quasilinear partial differential equations were considered.

In the domain $\Omega \equiv \Omega_T \times \mathbb{R}$, we consider the quasilinear equation

$$\left(\frac{\partial^2}{\partial t^2} - \alpha^2 \frac{\partial^2}{\partial x^2}\right)^n \left(\frac{\partial}{\partial t} + \int\limits_0^1 u(t,\sigma)\,d\sigma \frac{\partial}{\partial x}\right)^m u(t,x) = f\big(t,x,u(t,x)\big) \tag{1}$$

with the initial conditions

$$u(t,x)\big|_{t=0} = \varphi_1(x), \quad \frac{\partial^i}{\partial t^i} u(t,x)\big|_{t=0} = \varphi_{i+1}(x), \quad x \in \mathbb{R}, \quad i = \overline{1, 2n+m-1}, \tag{2}$$

where $u(t,x)$ is the unknown function, $f(t,x,u) \in C(\Omega_T \times \mathbb{R}^2)$, $\varphi_i(x) \in C(\mathbb{R})$, $i = \overline{1, 2n+m}$, $\Omega_T \equiv [0;T]$, $0 < T < \infty$, $\mathbb{R} \equiv (-\infty;\infty)$, $0 < \alpha = \text{const}$; n and m are arbitrary natural numbers.

2. Reduction of the initial-value problem to an integral equation.

Lemma 1. *The initial-value problem* (1), (2) *is equivalent to the following integral equation*:

$$u(t,x) \equiv \Theta(t,x;u,r) = \sum_{i=1}^m \varphi_i\big(r(t,0,x)\big)\frac{t^{m-i}}{(m-i)!}$$
$$+ \frac{1}{2}\sum_{j=1}^n \int\limits_0^t \frac{(t-s)^{m-1}}{(m-1)!}\Big[\varphi_{m+2j-1}\big(r(t,s,x)-\alpha s\big) + \varphi_{m+2j-1}\big(r(t,s,x)+\alpha s\big)\Big]\frac{s^{n-j}}{(n-j)!}\,ds$$

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 156, Mathematical Analysis, 2018.

$$+\frac{1}{2}\sum_{k=1}^{n}\int_0^t \frac{(t-s)^{n+m-1}}{(n+m-1)!}\Big[\varphi_{m+2k}\big(r(t,s,x)-\alpha(t-2s)\big)+\varphi_{m+2k}\big(r(t,s,x)+\alpha(t-2s)\big)\Big]\frac{s^{n-k}}{(n-k)!}\,ds$$

$$+\int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} f\Big(s, r(t,s,x), u(s, r(t,s,x))\Big)\,ds, \quad (3)$$

where

$$r(t,s,x) = x - \int_s^t \int_0^1 u(\theta,\sigma)\,d\sigma\,d\theta$$

and x plays the role of a parameter.

Proof. We rewrite the left-hand side of Eq. (1) in the form

$$\left(\frac{\partial^2}{\partial t^2} - \alpha^2\frac{\partial^2}{\partial x^2}\right)^n \left(\frac{\partial}{\partial t} + \int_0^1 u(t,\sigma)d\sigma\frac{\partial}{\partial x}\right)^m u$$

$$= \left(\frac{\partial}{\partial t} - \alpha\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial t} + \int_0^1 u(t,\sigma)\,d\sigma\frac{\partial}{\partial x}\right)^m u = D_2^n D_1^n D_0^m[u],$$

where

$$D_2[u] \equiv u_t - \alpha u_x, \quad D_1[u] \equiv u_t + \alpha u_x, \quad D_0[u] \equiv u_t + \int_0^1 u(t,\sigma)\,d\sigma u_x.$$

Then Eq. (1) takes the form

$$D_2^n D_1^n D_0^m[u] = f\big(t, x, u(t,x)\big). \quad (4)$$

From (4) we see that Eq. (1) has the characteristics

$$x + \alpha t = C_1, \quad x - \alpha t = C_2, \quad x - \int_0^t \int_0^1 u(s,\sigma)\,d\sigma\,ds = C_3,$$

where C_i are arbitrary constants, $i = \overline{1,3}$. The characteristics possess the following property: the differential expressions $D_1[u]$, $D_2[u]$, and $D_3[u]$ are (up to a multiplicative constant) the derivatives du/dl_1, du/dl_2, and du/dl_3 of the function u in the directions l_1, l_2, and l_3 along the characteristics. This allows one to represent Eq. (1) as an ordinary differential equation, which describes the variation of u along the characteristics.

Consider the expression $D_2[u] \equiv u_t - \alpha u_x$. Introduce the notation $p(t,s,x) = x + \alpha(t-s)$ and perform the change of variables $u(t,x) = \vartheta(t,z)$, $z = p(t,0,x)$. After differentiation we obtain

$$u_t(t,x) = \vartheta_t(t,z) + \vartheta_z(t,z)z_t.$$

Since $\vartheta_z(t,z) = u_x(t,x)$, $z_t = \alpha$, we obtain

$$\vartheta_t(t,z) = u_t(t,x) - \alpha u_x(t,x).$$

Taking into account the last relation and the formula $x = z - \alpha t$, we rewrite Eq. (4) in the form

$$\frac{\partial^n}{\partial t^n} D_1^n D_0^m[\vartheta(t,z)] = f\Big(t, z-\alpha t, \vartheta(t, z-\alpha t)\Big). \quad (5)$$

Integrating Eq. (5) n times, we have

$$\frac{\partial^{n-1}}{\partial t^{n-1}} D_1^n D_0^m[\vartheta(t,z)] = \Phi_1(z) + \int\limits_0^t f\Big(s, z-\alpha s, \vartheta(s, z-\alpha s)\Big)\, ds, \tag{6}$$

$$\frac{\partial^{n-2}}{\partial t^{n-2}} D_1^n D_0^m[\vartheta(t,z)] = \Phi_2(z) + \Phi_1(z)t + \int\limits_0^t (t-s) f\Big(s, z-\alpha s, \vartheta(s, z-\alpha s)\Big)\, ds, \tag{7}$$

$$\dots\dots\dots\dots\dots\dots$$

$$D_1^n D_0^m[\vartheta(t,z)] = \sum_{i=1}^n \Phi_i(z)\frac{t^{n-i}}{(n-i)!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!} f\Big(s, z-\alpha s, \vartheta(s, z-\alpha s)\Big)\, ds, \tag{8}$$

where $\Phi_i(z)$, $i = 1, 2, \dots, n$, are arbitrary constants along the first characteristic $x + \alpha t = C_1$ to be determined, $C_1 = \text{const}$. The initial conditions (2) for (6)–(8) have the form

$$\frac{\partial^{n-1}}{\partial t^{n-1}} D_1^n D_0^m[\vartheta(0,z)] = \varphi_{2n+m}(z), \qquad \frac{\partial^{n-2}}{\partial t^{n-2}} D_1^n D_0^m[\vartheta(0,z)] = \varphi_{2n+m-2}(z), \quad \dots,$$
$$\frac{\partial}{\partial t} D_1^n D_0^m[\vartheta(0,z)] = \varphi_{m+4}(z), \qquad D_1^n D_0^m[\vartheta(0,z)] = \varphi_{m+2}(z).$$

Due to these conditions, we conclude from (6)–(8) that

$$D_1^n D_0^m[\vartheta(t,z)] = \sum_{i=1}^n \varphi_{2i+m}(z)\frac{t^{n-i}}{(n-i)!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!} f\Big(s, z-\alpha s, \vartheta(s, z-\alpha s)\Big)\, ds. \tag{9}$$

Taking into account the relations $\vartheta(t,z) = u(t,x)$, $z = x + \alpha t$, and $z - \alpha s = p(t,s,x)$, we rewrite the integro-differential equation (9) in the form

$$D_1^n D_0^m[u(t,x)] = \sum_{i=1}^n \varphi_{2i+m}(x+\alpha t)\frac{t^{n-i}}{(n-i)!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!} f\Big(s, p(t,s,x), u\big(s, p(t,s,x)\big)\Big)\, ds, \tag{10}$$

where $p(t,s,x) = x + \alpha(t-s)$.

Now we consider the differential expression $D_1[u] \equiv u_t + \alpha u_x$. Introduce the notation $q(t,s,x) = x - \alpha(t-s)$ and perform the change of variables $u(t,x) = w(t,\eta)$, $\eta = q(t,0,x)$. After differentiation we have

$$u_t(t,x) = w_t(t,\eta) - \alpha w_\eta(t,\eta).$$

Since $w_\eta(t,\eta) = u_x(t,x)$, we obtain

$$w_t(t,\eta) = u_t(t,x) + \alpha u_x(t,x). \tag{11}$$

Taking into account (11) and the relation $x + \alpha t = \eta + 2\alpha t$, we rewrite Eq. (10) in the form

$$\frac{\partial^n}{\partial t^n} D_0^m[w(t,\eta)]$$
$$= \sum_{i=1}^n \varphi_{2i+m}(\eta+2\alpha t)\frac{t^{n-i}}{(n-i)!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!} f\Big(s, p(t,s,\eta+\alpha t), w\big(s, p(t,s,\eta+\alpha t)\big)\Big)\, ds. \tag{12}$$

Integrating Eq. (12), we obtain

$$\frac{\partial^{n-1}}{\partial t^{n-1}} D_0^m[w(t,\eta)] = \Phi_{n+1}(\eta) + \sum_{j=1}^{n} \int_0^t \varphi_{2j+m}(\eta + 2\alpha s) \frac{s^{n-j}}{(n-j)!}\, ds + \int_0^t \frac{(t-s)^n}{n!} f\Big(s, p(t,s,q), w\big(s, p(t,s,q)\big)\Big)\, ds, \quad (13)$$

$$\frac{\partial^{n-2}}{\partial t^{n-2}} D_0^m[w(t,\eta)] = \Phi_{n+2}(\eta) + \Phi_{n+1}(\eta)t + \sum_{j=1}^{n} \int_0^t (t-s)\varphi_{2j+m}(\eta + 2\alpha s) \frac{s^{n-j}}{(n-j)!}\, ds + \int_0^t \frac{(t-s)^{n+1}}{(n+1)!} f\Big(s, p(t,s,q), w\big(s, p(t,s,q)\big)\Big)\, ds, \quad (14)$$

. . . .

$$D_0^m w(t,\eta) = \sum_{i=n+1}^{2n} \Phi_i(\eta) \frac{t^{2n-i}}{(2n-i)!} + \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \varphi_{2j+m}(\eta + 2\alpha s) \frac{s^{n-j}}{(n-j)!}\, ds + \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} f\Big(s, p(t,s,q), w\big(s, p(t,s,q)\big)\Big)\, ds, \quad (15)$$

where $\Phi_i(\eta)$, $i = n+1, n+2, \ldots, 2n$, are arbitrary constants align the characteristic $x - \alpha t = C_2$ to be determined, $C_2 = \text{const}$. The initial conditions (2) for (13)–(15) have the form

$$\frac{\partial^{n-1}}{\partial t^{n-1}} D_0^m w(0,\eta) = \varphi_{2n+m-1}(\eta), \quad \frac{\partial^{n-2}}{\partial t^{n-2}} D_0^m w(0,\eta) = \varphi_{2n+m-3}(\eta), \quad \ldots, \quad D_0^m w(0,\eta) = \varphi_{m+1}(\eta).$$

Due to these conditions, from (13)–(15) we obtain

$$D_0^m w(t,\eta) = \sum_{i=1}^{n} \varphi_{2i+m-1}(\eta) \frac{t^{n-i}}{(n-i)!} + \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \varphi_{2j+m}(\eta + 2\alpha s) \frac{s^{n-j}}{(n-j)!}\, ds + \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} f\Big(s, p(t,s,q), w\big(s, p(t,s,q)\big)\Big)\, ds. \quad (16)$$

Taking into account the relations $w(t,\eta) = u(t,x)$, $\eta = q(t,0,x) = x - \alpha t$, $\eta + 2\alpha s = x - \alpha(t-2s)$, and $p(t,s,q) = x$, we rewrite Eqs. (18) in the form

$$D_0^m[u(t,x)] = \sum_{i=1}^{n} \varphi_{2i+m-1}(x - \alpha t) \frac{t^{n-i}}{(n-i)!} + \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \varphi_{2j+m}\big(x - \alpha(t-2s)\big) \frac{s^{n-j}}{(n-j)!}\, ds + \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} f\big(s, x, u(s,x)\big)\, ds, \quad (17)$$

where x plays the role of a parameter. Now Eq. (1), in contrast to Eq. (4), takes the form

$$D_1^n D_2^n D_0^m[u] = f\big(t, x, u(t,x)\big).$$

Repeating all procedures (5)–(16), similarly to (17), we obtain

$$D_0^m u(t,x) = \sum_{i=1}^{n} \varphi_{2i+m-1}(x+\alpha t)\frac{t^{n-i}}{(n-i)!} + \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\varphi_{2j+m}(x+\alpha(t-2s))\frac{s^{n-j}}{(n-j)!}\,ds + \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} f\big(s,x,u(s,x)\big)\,ds. \quad (18)$$

From (17) and (18) we conclude that

$$D_0^m u(t,x) = \frac{1}{2}\sum_{i=1}^{n} \Big[\varphi_{2i+m-1}(x-\alpha t) + \varphi_{2i+m-1}(x+\alpha t)\Big]\frac{t^{n-i}}{(n-i)!} + \frac{1}{2}\sum_{j=1}^{n}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\Big[\varphi_{2j+m}(x-\alpha(t-2s)) + \varphi_{2j+m}(x+\alpha(t-2s))\Big]\frac{s^{n-j}}{(n-j)!}\,ds + \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} f\big(s,x,u(s,x)\big)\,ds, \quad (19)$$

where x plays the role of a parameter.

Consider the differential expression

$$D_0[u] \equiv u_t + \int_0^1 u(t,\sigma)d\sigma u_x.$$

Introduce the notation

$$r(t,s,x) = x - \int_s^t \int_0^1 u(\theta,\sigma)\,d\sigma\,d\theta$$

and perform the change of variables $u(t,x) = h(t,\xi)$, $\xi = r(t,0,x)$. After differentiation we have

$$u_t(t,x) = h_t(t,\xi) - \int_0^1 u(t,\sigma)\,d\sigma\,h_\xi(t,\xi).$$

Since $h_\xi(t,\xi) = u_x(t,x)$, we obtain

$$h_t(t,\xi) = u_t(t,x) + \int_0^1 u(t,\sigma)\,d\sigma\,u_x(t,x).$$

Taking into account this relation and the formula

$$x = \xi + \int_0^t \int_0^1 u(\theta,\sigma)\,d\sigma\,d\theta = \xi + r(t,0,0),$$

we rewrite Eq. (19) in the form

$$
\begin{aligned}
\frac{\partial^m}{\partial t^m} h(t,\xi) = \frac{1}{2}\sum_{i=1}^{n} \Big[\varphi_{2i+m-1}\big(\xi + r(t,0,0) - \alpha t\big) + \varphi_{2i+m-1}\big(\xi + r(t,0,0) + \alpha t\big)\Big] \frac{t^{n-i}}{(n-i)!} \\
+ \frac{1}{2}\sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \Big[\varphi_{2j+m}\big(\xi + r(s,0,0) - \alpha(t-2s)\big) \\
+ \varphi_{2j+m}\big(\xi + r(s,0,0) + \alpha(t-2s)\big)\Big] \frac{s^{n-j}}{(n-j)!}\, ds \\
+ \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} f\Big(s, \xi + r(s,0,0), h\big(s, \xi + r(s,0,0)\big)\Big)\, ds. \quad (20)
\end{aligned}
$$

Integrating Eq. (20) along the third characteristic m times, we obtain

$$
\begin{aligned}
\frac{\partial^{m-1}}{\partial t^{m-1}} h(t,\xi) = \Phi_{2n+1}(\xi) \\
+ \frac{1}{2}\sum_{i=1}^{n} \int_0^t \Big[\varphi_{2i+m-1}\big(\xi + r(s,0,0) - \alpha s\big) + \varphi_{2i+m-1}\big(\xi + r(s,0,0) + \alpha s\big)\Big] \frac{s^{n-i}}{(n-i)!}\, ds \\
+ \frac{1}{2}\sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n}}{n!} \Big[\varphi_{2j+m}\big(\xi + r(s,0,0) - \alpha(t-2s)\big) \\
+ \varphi_{2j+m}\big(\xi + r(s,0,0) + \alpha(t-2s)\big)\Big] \frac{s^{n-j}}{(n-j)!}\, ds \\
+ \int_0^t \frac{(t-s)^{2n}}{(2n)!} f\Big(s, \xi + r(s,0,0), h\big(s, \xi + r(s,0,0)\big)\Big)\, ds, \quad (21)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^{m-2}}{\partial t^{m-2}} h(t,\xi) = \Phi_{2n+2}(\xi) + \Phi_{2n+1}(\xi) t \\
+ \frac{1}{2}\sum_{i=1}^{n} \int_0^t (t-s)\Big[\varphi_{2i+m-1}\big(\xi + r(s,0,0) - \alpha s\big) + \varphi_{2i+m-1}\big(\xi + r(s,0,0) + \alpha s\big)\Big] \frac{s^{n-i}}{(n-i)!}\, ds \\
+ \frac{1}{2}\sum_{j=1}^{n} \int_0^t \frac{(t-s)^{n+1}}{(n+1)!} \Big[\varphi_{2j+m}\big(\xi + r(s,0,0) - \alpha(t-2s)\big) \\
+ \varphi_{2j+m}\big(\xi + r(s,0,0) + \alpha(t-2s)\big)\Big] \frac{s^{n-j}}{(n-j)!}\, ds \\
+ \int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} f\Big(s, \xi + r(t,0,0), h\big(s, \xi + r(t,0,0)\big)\Big)\, ds, \quad (22)
\end{aligned}
$$

....

$$h(t,\xi) = \sum_{i=1}^{m} \Phi_{2n+i}(\xi) \frac{t^{m-i}}{(m-i)!}$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \Big[\varphi_{2j+m-1}\big(\xi + r(s,0,0) - \alpha s\big) + \varphi_{2j+m-1}\big(\xi + r(s,0,0) + \alpha s\big)\Big] \frac{s^{n-j}}{(n-j)!}\, ds$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \int_0^t \frac{(t-s)^{n+m-1}}{(n+m-1)!} \Big[\varphi_{2k+m}\big(\xi + r(s,0,0) - \alpha(t-2s)\big)$$

$$+ \varphi_{2k+m}\big(\xi + r(s,0,0) + \alpha(t-2s)\big)\Big] \frac{s^{n-k}}{(n-k)!}\, ds$$

$$+ \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} f\Big(s, \xi + r(s,0,0), h\big(s, \xi + r(s,0,0)\big)\Big)\, ds, \quad (23)$$

where Φ_i, $i = \overline{2n+1, 2n+m}$, are arbitrary constants along the third characteristic to be determined.

The initial conditions (2) for (21)–(23) have the form

$$\frac{\partial^{m-1}}{\partial t^{m-1}} h(0,\xi) = \varphi_m(\xi), \quad \frac{\partial^{m-2}}{\partial t^{m-2}} h(0,\xi) = \varphi_{m-1}(\xi), \quad \dots, \quad h(0,\xi) = \varphi_1(\xi).$$

Due to these conditions, from (21)–(23) we obtain

$$h(t,\xi) = \sum_{i=1}^{m} \varphi_i(\xi) \frac{t^{m-i}}{(m-i)!}$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \Big[\varphi_{2j+m-1}\big(\xi + r(s,0,0) - \alpha s\big) + \varphi_{2j+m-1}\big(\xi + r(s,0,0) + \alpha s\big)\Big] \frac{s^{n-j}}{(n-j)!}\, ds$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \int_0^t \frac{(t-s)^{n+m-1}}{(n+m-1)!} \Big[\varphi_{2k+m}\big(\xi + r(s,0,0) - \alpha(t-2s)\big)$$

$$+ \varphi_{2k+m}\big(\xi + r(s,0,0) + \alpha(t-2s)\big)\Big] \frac{s^{n-k}}{(n-k)!}\, ds$$

$$+ \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} f\Big(s, \xi + r(s,0,0), h\big(s, \xi + r(s,0,0)\big)\Big)\, ds. \quad (24)$$

Taking into account the relations

$$h(t,\xi) = u(t,x), \quad \xi = r(t,0,x) = x - \int_0^t \int_0^1 u(\theta,\sigma)\, d\sigma\, d\theta,$$

$$\xi + r(s,0,0) = \xi + \int_0^s \int_0^1 u(\theta,\sigma)\, d\sigma\, d\theta = x - \int_s^t \int_0^1 u(\theta,\sigma)\, d\sigma\, d\theta = r(t,s,x),$$

from (24) we obtain the integral equation (3). Differentiating $(2n+m)$ times along the corresponding characteristics, we obtain from (3) the following differential equation:

$$\frac{d^{2n+m} u(t,x)}{dt^{2n+m}} = f\big(t, x, u(t,x)\big), \quad (25)$$

where x plays the role of a parameter.

For the left-hand side of (25), the following relation is valid along the characteristic:

$$\frac{d^{2n+m}u(t,x)}{dt^{2n+m}} = \left(\frac{\partial}{\partial t} - \alpha\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial t} + \int_0^1 u(\theta,\sigma)\, d\sigma\, d\theta \frac{\partial}{\partial x}\right)^m u(t,x)$$

$$= \left(\frac{\partial^2}{\partial t^2} - \alpha^2\frac{\partial^2}{\partial x^2}\right)^n \left(\frac{\partial}{\partial t} + \int_0^1 u(\theta,\sigma)\, d\sigma\, d\theta \frac{\partial}{\partial x}\right)^m u(t,x).$$

Therefore, the nonlinear integral equation (3) is equivalent to the initial-value problem (1), (2) along the characteristics. The lemma is proved. □

3. Analysis of the integral equation (3).

Lemma 2. *Assume that the following conditions are fulfilled*:

1. $$0 < \max_{x\in\mathbb{R}} \sum_{i=1}^{m} \left|\varphi_i(x)\right| \frac{T^{m-i}}{(m-i)!} + \max_{(t,x)\in\Omega} \sum_{j=1}^{n} \left|\varphi_{m+2j-1}(x)\right| \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}\frac{s^{n-j}}{(n-j)!}\, ds +$$
$$+ \max_{(t,x)\in\Omega} \sum_{k=1}^{n} \left|\varphi_{m+2k}(x)\right| \int_0^t \frac{(t-s)^{n+m-1}}{(n+m-1)!}\frac{s^{n-k}}{(n-k)!} ds \le \Delta_0 < \infty;$$
2. $\left|\varphi_i(x_1) - \varphi_i(x_2)\right| \le \chi_i \left|x_1 - x_2\right|, \quad 0 < \chi_i = \text{const} < \infty, \quad i = \overline{1, 2n};$
3. $\max_{x\in\mathbb{R}} \left|f(t,x,u)\right| \le M(t), \quad 0 < M(t) \in C(\Omega_T);$
4. $\left|f(t,x_1,u_1) - f(t,x_2,u_2)\right| \le Q(t)\left|x_1 - x_2\right| + N(t)\left|u_1 - u_2\right|,$
$0 < Q(t) \in C(\Omega_T), \quad 0 < N(t) \in C(\Omega_T);$
5. $$0 < \max_{t\in\Omega_T} \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} M(s)\, ds \le \Delta_1.$$

Then the integral equation (3) *has a unique solution in the domain* Ω, *which can be found by the method of successive approximations*:

$$u_0(t,x) = 0, \quad u_{\tau+1}(t,x) \equiv \Theta(t,x;u_\tau,p_\tau), \quad \tau = 0,1,\dots, \tag{26}$$

where

$$p_0(s,t,x) = x, \quad p_\tau(s,t,x) = x - \int_s^t \int_0^1 u_\tau(\theta,\sigma)\, d\sigma\, d\theta.$$

Proof. Due to the conditions of the lemma, we conclude that the first difference approximation (26) satisfies the following estimate:

$$\left|u_1(t,x) - u_0(t,x)\right| \le \sum_{i=1}^{m} \left|\varphi_i(x)\right| \frac{T^{m-i}}{(m-i)!} + \sum_{j=1}^{n} \left|\varphi_{m+2j-1}(x)\right| \max_{t\in\Omega_T} \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}\frac{s^{n-j}}{(n-j)!} ds$$

$$+ \sum_{k=1}^{n} \left|\varphi_{m+2k}(x)\right| \max_{t\in\Omega_T} \int_0^t \frac{(t-s)^{n+m-1}}{(n+m-1)!}\frac{s^{n-k}}{(n-k)!}\, ds + \max_{t\in\Omega_T} \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} M(s)\, ds \le \Delta_0 + \Delta_1. \tag{27}$$

Taking into account (27) and the conditions of the lemma, from (26) we conclude that the second difference approximation (26) satisfies the following estimate:

$$\big|u_2(t,x)-u_1(t,x)\big| \le \sum_{i=1}^{m}\frac{\chi_i T^{m-i}}{(m-i)!}\int\limits_0^t \big|u_1(s,x)-u_0(s,x)\big|\,ds$$
$$+\sum_{j=1}^{n}\chi_{m+2j-1}\max_{t\in\Omega_T}\int\limits_0^t\frac{(t-s)^{m-1}}{(m-1)!}\frac{s^{n-j}}{(n-j)!}\int\limits_s^t\big|u_1(\theta,x)-u_0(\theta,x)\big|\,d\theta\,ds$$
$$+\sum_{k=1}^{n}\chi_{m+2k}\max_{t\in\Omega_T}\int\limits_0^t\frac{(t-s)^{n+m-1}}{(n+m-1)!}\frac{s^{n-k}}{(n-k)!}\int\limits_s^t\big|u_1(\theta,x)-u_0(\theta,x)\big|\,d\theta\,ds$$
$$+\int\limits_0^t\frac{(t-s)^{2n+m-1}}{(2n+m-1)!}\left[Q(s)\int\limits_s^t\big|u_1(\theta,x)-u_0(\theta,x)\big|\,d\theta+N(s)\big|u_1(s,x)-u_0(s,x)\big|\right]ds$$
$$\le\int\limits_0^t H(t,s)\big|u_1(s,x)-u_0(s,x)\big|\,ds\le\big(\Delta_0+\Delta_1\big)\int\limits_0^t H(t,s)ds, \quad (28)$$

where

$$H(t,s)=\sum_{i=1}^{m}\frac{\chi_i T^{m-i}}{(m-i)!}+\sum_{j=1}^{n}\chi_{m+2j-1}\frac{(t-s)^m}{(m-1)!}\frac{s^{n-j}}{(n-j)!}$$
$$+\sum_{k=1}^{n}\chi_{m+2k}\frac{(t-s)^{n+m}}{(n+m-1)!}\frac{s^{n-k}}{(n-k)!}+\frac{(t-s)^{2n+m-1}}{(2n+m-1)!}\Big[Q(s)(t-s)+N(s)\Big].$$

Taking into account (28), for the third difference approximation (26) we obtain the estimate

$$\big|u_3(t,x)-u_2(t,x)\big|\le\int\limits_0^t H(t,s)\big|u_2(s,x)-u_1(s,x)\big|\,ds$$
$$\le\big(\Delta_0+\Delta_1\big)\int\limits_0^t H(t,s)\int\limits_0^s H(s,\theta)\,d\theta\,ds=\frac{\Delta_0+\Delta_1}{2!}\left[\int\limits_0^t H(t,s)ds\right]^2.$$

Continuing this process, we inductively obtain

$$\big|u_{\tau+1}(t,x)-u_\tau(t,x)\big|\le\int\limits_0^t H(t,s)\big|u_\tau(s,x)-u_{\tau-1}(s,x)\big|\,ds\le\frac{\Delta_0+\Delta_1}{\tau!}\left[\int\limits_0^t H(t,s)\,ds\right]^\tau. \quad (29)$$

The estimate (29) implies that the sequence of functions $\big\{u_\tau(t,x)\big\}_{\tau=1}^{\infty}$ defined by (26) converges absolutely and uniformly in the domain Ω.

Now we assume that the integral equation (3) has two solutions $u(t,x)$ and $\vartheta(t,x)$ in the domain Ω. Then the absolute value of their difference satisfies the estimate

$$\big|u(t,x)-\vartheta(t,x)\big|\le\int\limits_0^t H(t,s)\big|u(s,x)-\vartheta(s,x)\big|\,ds.$$

Applying the Grönwall–Bellman inequality, we see that

$$\big|u(t,x) - \vartheta(t,x)\big| \equiv 0$$

in the domain Ω. The lemma is proved. $\square$

Lemma 3. *Assume that the conditions of Lemma 2 are fulfilled. Then for the iterative process* (26), *the following estimate of the convergence rate is valid*:

$$\big|u_\tau(t,x) - u(t,x)\big| \le \big(\Delta_0 + \Delta_1\big)\frac{\omega^\tau}{\tau!}\cdot \exp\omega, \tag{30}$$

where

$$\omega = \max_{t\in\Omega_T}\int_0^t H(t,s)\,ds < \infty.$$

Proof. Indeed, due to the conditions of the lemma, taking into account (29), we have the estimate

$$\big|u_\tau(t,x) - u(t,x)\big| \le \big|u_{\tau+1}(t,x) - u_\tau(t,x)\big| + \big|u_{\tau+1}(t,x) - u(t,x)\big|$$
$$\le \big(\Delta_0 + \Delta_1\big)\frac{\omega^\tau}{\tau!} + \int_0^t H(t,s)\big|u_\tau(s,x) - u(s,x)\big|\,ds.$$

Applying the Grönwall–Bellman inequality, we arrive at the estimate (30). $\square$

Lemma 4. *Assume that the conditions of Lemma 2 be fulfilled. Then for any $x_1, x_2 \in \mathbb{R}$, the following estimate is valid*:

$$\big|u(t,x_1) - u(t,x_2)\big| \le \Psi(t)\big|x_1 - x_2\big|, \tag{31}$$

where

$$\Psi(t) = \mu\exp\left\{\int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!}N(s)\,ds\right\} < \infty,$$

$$\mu = \max_{t\in\Omega_T}\int_0^t \Bigg[\sum_{i=1}^m \frac{\chi_i T^{m-i}}{(m-i)!} + \sum_{j=1}^n \chi_{m+2j-1}\frac{(t-s)^m}{(m-1)!}\frac{s^{n-j}}{(n-j)!}$$
$$+ \sum_{k=1}^n \chi_{m+2k}\frac{(t-s)^{n+m}}{(n+m-1)!}\frac{s^{n-k}}{(n-k)!} + \frac{(t-s)^{2n+m-1}}{(2n+m-1)!}Q(s)(t-s)\Bigg]\,ds.$$

Proof. Indeed, due to the conditions of the lemma, we have the estimate

$$\big|u(t,x_1) - u(t,x_2)\big| \le \sum_{i=1}^m \frac{\chi_i T^{m-i}}{(m-i)!}\int_0^t \big|x_1 - x_2\big|\,ds$$
$$+ \sum_{j=1}^n \chi_{m+2j-1}\max_{t\in\Omega_T}\int_0^t \frac{(t-s)^{m-1}}{(m-1)!}\frac{s^{n-j}}{(n-j)!}\int_s^t \big|x_1 - x_2\big|d\theta ds$$
$$+ \sum_{k=1}^n \chi_{m+2k}\max_{t\in\Omega_T}\int_0^t \frac{(t-s)^{n+m-1}}{(n+m-1)!}\frac{s^{n-k}}{(n-k)!}\int_s^t \big|x_1 - x_2\big|\,d\theta\,ds$$

$$+\int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} Q(s) \int_s^t |x_1 - x_2|\, d\theta\, ds + \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} N(s) \big| u_1(s,x) - u_0(s,x) \big|\, ds$$

$$\leq \mu \cdot |x_1 - x_2| + \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} N(s) \big| u(s,x_1) - u(s,x_2) \big|\, ds,$$

where

$$\mu = \max_{t\in\Omega_T} \int_0^t \Bigg[\sum_{i=1}^m \frac{\chi_i T^{m-i}}{(m-i)!} + \sum_{j=1}^n \chi_{m+2j-1} \frac{(t-s)^m}{(m-1)!} \frac{s^{n-j}}{(n-j)!}$$
$$+ \sum_{k=1}^n \chi_{m+2k} \frac{(t-s)^{n+m}}{(n+m-1)!} \frac{s^{n-k}}{(n-k)!} + \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} Q(s)(t-s) \Bigg] ds.$$

Applying the Grönwall–Bellman inequality to the last estimate, we obtain

$$\big| u(t,x_1) - u(t,x_2) \big| \leq \mu |x_1 - x_2| \cdot \exp\left\{ \int_0^t \frac{(t-s)^{2n+m-1}}{(2n+m-1)!} N(s)\, ds \right\}.$$

This implies the estimate (31). □

The lemmas proved above imply the following theorem.

Theorem 1. *Assume that the conditions of Lemma* 2 *are fulfilled. Then the initial-value problem* (1), (2) *has a unique solution in the domain* Ω. *This solution can be found by the Picard iterative process* (26). *The solution of the initial-value problem* (1), (2) *satisfies the estimates* (30) *and* (31).

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