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Distribution of random motion at renewal instants in three-dimensional space

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(Presented by O. Dovgoshey)

Abstract. In physics, chemistry, and mathematics, the process of Brownian motion is often identified with the Wiener process that has infinitesimal increments. Recently, many models of Brownian motion with finite velocity have been intensively studied. We consider one of such models, namely, a generalization of the Goldstein–Kac process to the three-dimensional case with the Erlang-2 and Maxwell–Boltzmann distributions of velocities alternations. Despite the importance of having a three-dimensional isotropic random model for the motion of Brownian particles, numerous research efforts did not lead to an expression for the probability of the distribution of the particle position, the motion of which is described by the three-dimensional telegraph process. The case where a particle carries out its movement along the directions determined by the vertices of a regular $n+1$-hedron in the n-dimensional space was studied in [13], and closed-form results for the distribution of the particle position were obtained. Here, we obtain expressions for the distribution function of the norm of the vector that defines particle's position at renewal instants in semi-Markov cases of the Erlang-2 and Maxwell–Boltzmann distributions and study its properties. By knowing this distribution, we can determine the distribution of particle positions, since the motion of a particle is isotropic, i.e., the direction of its movement is uniformly distributed on the unit sphere in $\mathbb{R}^3$. Our results may be useful in studying the properties of an ideal gas.

Keywords. Transport process, telegraph equation, Erlang distribution; Maxwell–Boltzmann distribution, 3-D random motion.

1. Introduction

Let us consider the renewal process $\nu(t) = \max\{m \ge 0 : \tau_m \le t\}$, $t \ge 0$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \ge 0$, $k = 1, 2, \dots$, are i.i.d. random variables with a distribution function $G(t)$ and a probability density function (pdf) $g(t) = \frac{d}{dt} G(t)$.

We assume that a particle starting from the coordinate origin $(0,0,0)$ of the space $\mathbb{R}^3$, at the time $t=0$, continues its motion with a velocity $v > 0$ along the direction of $\boldsymbol{\eta}_1$, where $\boldsymbol{\eta}_1 = (x_1, x_2, x_3)$ is a random 3-dimensional vector uniformly distributed on the unit sphere $\Omega_1^2 = \left\{(x_1,x_2,x_3) : x_1^2 + x_2^2 + x_3^2 = 1\right\}$.

At the instant τ_1, the particle changes its direction to $\boldsymbol{\eta}_2$, where $\boldsymbol{\eta}_2$ and $\boldsymbol{\eta}_1$ are independent and identically distributed on Ω_1^2, and continues its motion with a velocity v along the direction of $\boldsymbol{\eta}_2$. Then, at the instant τ_2, the particle changes its direction to $\boldsymbol{\eta}_3$, where $\boldsymbol{\eta}_3$ is also uniformly distributed on Ω_1^2 and independent of $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, and continues its motion with a velocity v along the direction of $\boldsymbol{\eta}_3$, and so on.

Denote, by $\mathbf{x}(t)$, $t \ge 0$, the particle position at the time t. We have that

$$\mathbf{x}(t) = v \sum_{j=1}^{\nu(t)} \boldsymbol{\eta}_j (\tau_j - \tau_{j-1}) + v\, \boldsymbol{\eta}_{\nu(t)+1} \left(t - \tau_{\nu(t)}\right). \tag{1}$$

Here, we assume that $\sum_{j=1}^{0} \boldsymbol{\eta}_{j-1} (\tau_j - \tau_{j-1}) = 0$.

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Basically, this equation determines the transport or storage process with semi-Markov switches $\nu(t)$. It is easily seen that $\nu(t)$ is the number of velocity alternations occurred in the interval $(0, t)$.

The standard generalization of the Goldstein–Kac telegraph process to a 3D-space is obtained, by assuming that θ_k is exponentially distributed. However, despite the efforts of many scholars in obtaining the closed-form results for the distribution of $\mathbf{x}(t)$, it is not represented in terms of elementary functions [2–6].

Let us denote $\mathbf{x}_n = v\sum_{j=1}^{n} \boldsymbol{\eta}_j (\tau_j - \tau_{j-1})$. It is easily seen that $\mathbf{x}_n$ is $\mathbf{x}(t)$ at the instant of n-th velocity alternation. Denote, by $x_n = v\sum_{j=1}^{n} \eta_{j-1}(\tau_j - \tau_{j-1})$, the projection on a line, where η_j is the projection of $\boldsymbol{\eta}_j$ on the line.

In this paper, we will find the distribution $F_{\|\mathbf{x}_n\|}(x)$ and the corresponding probability density function of $\|\mathbf{x}_n\| = v\left\|\sum_{j=1}^{n} \boldsymbol{\eta}_j (\tau_j - \tau_{j-1})\right\|$, where $\|\mathbf{x}_n\|$ is the norm of the vector $\mathbf{x}_n$, by using the distribution $F_{|x_n|}(x)$ of the length of x_n, i.e., $|x_n| = v\left|\sum_{j=1}^{n} \eta_j (\tau_j - \tau_{j-1})\right| = v\left|\sum_{j=1}^{n} \eta_j\theta_j\right|$.

Lemma 1. *The probabilistic properties of a random vector $\mathbf{x}_n$ are completely determined by those of its projection x_n on a line.*

Proof. Indeed, it is easily verified that $\mathbf{x}_n$ is isotropic, and the characteristic function φ_n of variable $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ depends only on $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, i.e.,

$$\varphi_n(\|\boldsymbol{\alpha}\|) = \mathbf{E}\left[\exp\left\{i(\boldsymbol{\alpha}, \mathbf{x}_n)\right\}\right].$$

Hence, we can show that

$$\varphi_n(\|\boldsymbol{\alpha}\|) = \mathbf{E}\left[\exp\left\{i(\boldsymbol{\alpha}, \mathbf{x}_n)\right\}\right] = \mathbf{E}\left[\exp\left\{i\|\boldsymbol{\alpha}\|(\mathbf{e}, \mathbf{x}_n)\right\}\right] = \mathbf{E}\left[\exp\left\{i\|\boldsymbol{\alpha}\| x_n\right\}\right],$$

and $\mathbf{E}\left[\exp\left\{i\|\boldsymbol{\alpha}\| x_n\right\}\right]$ is the characteristic function of x_n.

Thus, if we have the characteristic function of x_n, we have the characteristic function of $\mathbf{x}_n$.

It is well known that the projection η_1 on a line is uniformly distributed on $[-1, 1]$ [1].

Let us denote, by $G_\theta(x)$, the distribution function of θ_1 and, by $G_{\eta\theta}(x)$, the distribution function of $\eta_1\theta_1$.

Lemma 2. *The relationship between $G_{\eta\theta}$ and G_θ is as follows*:

$$G_{\eta\theta}(x) = \frac{1}{2} - \frac{1}{2}\int_{-1}^{0} G_\theta(x/y)\, dy\, \mathrm{I}_{\{x<0\}} + \frac{1}{2}\int_{0}^{1} G_\theta(x/y)\, dy\, \mathrm{I}_{\{x\geq 0\}}. \tag{2}$$

Proof. Taking into account that $\frac{d}{dx}P(\eta_1 \leq y) = \frac{1}{2}\mathrm{I}_{\{-1\leq y\leq 1\}}$, it is easily seen that

$$G_{\eta\theta}(x) = P(\eta_1\theta_1 \leq x) = \frac{1}{2}\int_{-1}^{0} P(\theta_1 \geq x/y)\, dy\, \mathrm{I}_{\{x<0\}} + \frac{1}{2}\int_{0}^{1} P(\theta_1 \leq x/y)\, dy\, \mathrm{I}_{\{x\geq 0\}}.$$

This concludes the proof.

By differentiating Eq. (2), we obtain the following equation for the pdf $g_{\eta\theta}$ of $\eta_1\theta_1$:

$$g_{\eta\theta}(x) = \frac{d}{dx}G_{\eta\theta}(x) = \frac{1}{2}\int_{|x|}^{\infty} g_\theta(y)\frac{dy}{y}, \tag{3}$$

where $g_\theta(x) = \frac{d}{dx}G_\theta(x)$ is the pdf of θ_1.

By differentiating Eq. (3), we obtain

$$g_\theta(x) = \frac{d}{dx} G_\theta(x) = -\frac{x}{2}\frac{d}{dx} g_{\eta\theta}(x), \ x \in \mathbb{R}. \tag{4}$$

2. Erlang-2 case

Let us consider the case where θ_1 has the Erlang-2 distribution, i.e., $g_\theta(y) = \lambda^2 y e^{-\lambda y}\,\mathrm{I}_{\{y\geq 0\}}$, with scale parameter λ.

Lemma 3. *The characteristic function $\varphi_{x_n}(z)$ of x_n is of the following form:*

$$\varphi_{x_n}(z) = \left(\frac{\lambda^2}{\lambda^2 + v^2 z^2}\right)^n.$$

Proof. It follows from Eq. (3) that

$$g_{\eta\theta}(x) = \frac{1}{2}\int_{|x|}^{\infty} \lambda^2 y e^{-\lambda y}\frac{dy}{y} = \frac{\lambda}{2} e^{-\lambda|x|}, \quad x \in \mathbb{R}.$$

Therefore, $\eta_1\theta_1$ has the Laplace or double exponential distribution. It is well known that

$$\eta_1\theta_1 = \xi_1 - \varsigma_1,$$

where ξ_1 and ς_1 are independent exponentially distributed random variables with parameter λ.

Thus, there exist a set ξ_j, ς_j, $j = 1, 2, \ldots, n$, of independent exponentially distributed random variables with scale parameter λ such that

$$x_n = v\sum_{j=1}^{n} \eta_j\theta_j = v\sum_{j=1}^{n} (\xi_j - \varsigma_j) = v\left(\sum_{j=1}^{n}\xi_j - \sum_{j=1}^{n}\varsigma_j\right).$$

It is also well known that $\sum_{j=1}^{n}\xi_j$ and $\sum_{j=1}^{n}\varsigma_j$ have the Erlang-n distribution with the pdf

$$f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}\,\mathrm{I}_{\{x\geq 0\}}.$$

The characteristic function of the Erlang-n pdf is

$$\varphi_n(z) = \left(\frac{\lambda}{\lambda - iz}\right)^n.$$

Therefore, the characteristic function of x_n is

$$\varphi_{x_n}(z) = \left(\frac{\lambda}{\lambda - ivz}\right)^n\left(\frac{\lambda}{\lambda + ivz}\right)^n = \left(\frac{\lambda^2}{\lambda^2 + v^2 z^2}\right)^n.$$

Hence, the lemma is proved.

The function $\varphi_{x_n}(z) = \left(\frac{\lambda^2}{\lambda^2+v^2z^2}\right)^n$ is the characteristic function of the bilateral double (BD) gamma distribution [7], whose pdf is sometimes denoted as $\mathrm{BD}\Gamma(n, \lambda/v; n, \lambda/v)(x)$.

It is easily seen that $\mathrm{BD}\Gamma(n, \lambda/v; n, \lambda/v)(x)$ is the convolution

$$\mathrm{BD}\Gamma(n, \lambda/v; n, \lambda/v)(x) = (\Gamma(n, \lambda/v) * \Gamma(n, -\lambda/v))(x),$$

where $\Gamma(n, \lambda/v)(x)$ is the pdf of a gamma distribution with parameters $n, \lambda/v$.

The explicit expression for $\mathrm{BD}\Gamma(n_1, \alpha_1; n_2, \alpha_2)(x)$ was obtained in [8]. In our case, the pdf $\mathrm{BD}\Gamma(n, \lambda/v; n, \lambda/v)(x)$ has the form

$$\mathrm{BD}\Gamma(n, \lambda/v; n, \lambda/v)(x) = \frac{|x|^{n-1} e^{-\frac{\lambda}{v}|x|}}{(v/\lambda)^n \Gamma(n)} \int_0^\infty e^{-t} t^{n-1} \left(1 + \frac{vt}{2\lambda|x|}\right)^{n-1} dt. \tag{5}$$

Formula (5) was obtained from formulas (5) and (41) in [8]. In formula (41), we took $\lambda_1 = \lambda_2 = v/\lambda$, $\alpha_1 = \alpha_2 = n$, since the authors of [8] used the parameter $1/\lambda$ instead of the common notation λ in the gamma distribution.

The pdf $f_{x_n}(x)$ of x_n is the inverse Fourier transform of $\varphi_{x_n}(z)$. Hence,

$$f_{x_n}(x) = \mathcal{F}^{-1}(\varphi_{x_n}(z)) = \mathrm{BD}\Gamma(n, \lambda/v; n, \lambda/v)(x).$$

Theorem 1. *The pdf* $f_{\|\mathbf{x}_n\|}$ *of* $\|\mathbf{x}_n\|$ *in the case where* θ_1 *has the Erlang-2 distribution of the form*

$$f_{\|\mathbf{x}_n\|}(x) = (4n-2)\frac{\lambda^{n-1}}{v^{n-1}}\frac{\partial^2}{\partial x^2}\left(\frac{x^n e^{-\frac{\lambda}{v}x}}{n!}\int_0^\infty e^{-t}t^n\left(1+\frac{vt}{2\lambda x}\right)^n dt\right)$$

$$+2\frac{\lambda^{n+1}}{v^{n+1}}\frac{x^n e^{-\frac{\lambda}{v}x}}{n!}\int_0^\infty e^{-t}t^n\left(1+\frac{vt}{2\lambda x}\right)^n dt, \quad x \geq 0. \tag{6}$$

Proof. Considering the symmetry of x_n with respect to 0, it is easily verified that

$$f_{|x_n|}(x) = 2 f_{x_n}(x), \quad x \geq 0. \tag{7}$$

We note that $|\eta_1|$ is uniformly distributed on $[0, 1]$. Similarly to Eq. (4), we can show that the pdf $f_{\|\mathbf{x}_n\|}(x)$ can be expressed in terms of the pdf $f_{|x_n|}(x)$ as follows:

$$f_{\|\mathbf{x}_n\|}(x) = -x\frac{d}{dx}f_{|x_n|}(x), \quad x \geq 0.$$

Let us assume $d(x) = -x\frac{d}{dx}f_{x_n}(x), \quad x \in \mathbb{R}$. It is easily seen that $d(x)$ is the probability distribution, and $f_{\|\mathbf{x}_n\|}(x) = 2d(x), \quad x \geq 0$.

The characteristic function $\varphi_d(z)$ of $d(x)$ can be written as

$$\varphi_d(z) = z\frac{\partial}{\partial z}\varphi_{x_n}(z) + \varphi_{x_n}(z) = \left(\frac{\lambda^2}{\lambda^2 + v^2 z^2}\right)^n \left(1 - \frac{2nv^2z^2}{\lambda^2 + v^2z^2}\right). \tag{8}$$

Therefore, making the inverse Fourier transformation of Eq. (8), we obtain

$$f_{\|\mathbf{x}_n\|}(x) = 2\mathcal{F}^{-1}(\varphi_d(z)) = (4n-2)\frac{v^2}{\lambda^2}\frac{\partial^2}{\partial x^2}\mathcal{F}^{-1}\left(\varphi_{x_{n+1}}(z)\right)(x) + 2\mathcal{F}^{-1}\left(\varphi_{x_{n+1}}(z)\right)(x) \tag{9}$$

$$= (4n-2)\frac{v^2}{\lambda^2}\frac{\partial^2}{\partial x^2}\mathrm{BD}\Gamma(n+1, \lambda/v; n+1, \lambda/v)(x) + 2\,\mathrm{BD}\Gamma(n+1, \lambda/v; n+1, \lambda/v)(x), \quad x \geq 0.$$

Taking Eq. (5) into account, we complete the proof of the theorem.

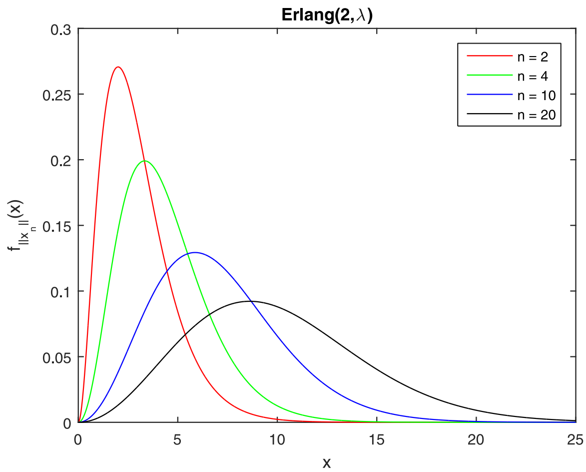

Figure 1: Probability density function $f_{||x_n||}(x)$ for $\lambda = 1$ and $v = 1$.

In Fig. 1, several probability density functions are shown for the values of $n = 2, 4, 10$, and 20. These pdfs were calculated, by using the MatLab package and Eqs. (8) and (9).

Remark. Applying the binomial theorem, we obtain

$$\int_0^\infty e^{-t}t^n\left(1+\frac{vt}{2\lambda x}\right)^n dt = \int_0^\infty e^{-t}t^n\sum_{k=0}^{n}\left(\frac{vt}{2\lambda x}\right)^k dt$$

$$= \sum_{k=0}^{n}\left(\frac{v}{2\lambda x}\right)^k\int_0^\infty e^{-t}t^{n+k}\, dt = \sum_{k=0}^{n}\left(\frac{v}{2\lambda x}\right)^k\Gamma(n+k+1).$$

Substituting this expression in Eq. (6), we have the following formula for the pdf of $\|\mathbf{x}_n\|$:

$$f_{\|\mathbf{x}_n\|}(x) = e^{-\frac{\lambda}{v}x}\left(\frac{\lambda}{v}\right)^{n-1}\sum_{k=0}^{n}\left(\frac{v}{2\lambda}\right)^k\frac{(n+k)!}{n!}x^{n-k-2}$$

$$\times\left[(4n-2)(n-k)\left(n-k-1-\frac{2\lambda}{v}x\right)+4n\left(\frac{\lambda}{v}\right)^2x^2\right].$$

3. Maxwell–Boltzmann distribution

Let θ_1 have the Maxwell–Boltzmann probability density

$$g_\theta(x) = \sqrt{\frac{2}{\pi}}\frac{x^2e^{-\frac{x^2}{2a^2}}}{a^3}, \quad x \geq 0.$$

This density is used for describing the particle speeds in ideal gases, where the particles move freely. Thus, it is a reasonable model for the distribution of the free path of a particle.

Theorem 2. *The pdf $f_{\|\mathbf{x}_n\|}$ of $\|\mathbf{x}_n\|$ in the case where θ_1 has the Maxwell–Boltzmann probability density is of the following form*:

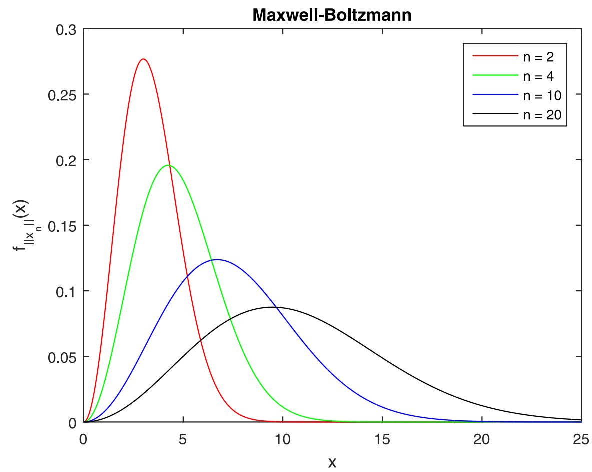

Figure 2: Probability density function $f_{||x_n||}(x)$ for $a = 1.5$ and $v = 1$.

$$f_{\|\mathbf{x}_n\|}(x) = -x\frac{d}{dx}f_{|x_n|}(x) = \frac{2x^2}{(av)^3\sqrt{2\pi n^3}}\, e^{-\frac{x^2}{2n(av)^2}}, \quad x \geq 0. \tag{10}$$

Proof. By using Eq. (3), we have

$$g_{\eta\theta}(x) = \frac{1}{2}\int_{|x|}^{\infty} g_\theta(y)\,\frac{dy}{y} = \frac{1}{\sqrt{2\pi}a}e^{-\frac{x^2}{2a^2}}, \quad x \in \mathbb{R}.$$

In this case, the pdf $f_{x_n}(x)$ of $x_n = v\sum_{j=1}^{n}\eta_j\theta_j$ is as follows:

$$f_{x_n}(x) = \frac{1}{\sqrt{2\pi(va)^2 n}}e^{-\frac{x^2}{2n(av)^2}}.$$

Hence,

$$f_{|x_n|}(x) = \frac{2}{\sqrt{2\pi(va)^2 n}}e^{-\frac{x^2}{2n(av)^2}}, \quad x \geq 0.$$

Therefore,

$$f_{\|\mathbf{x}_n\|}(x) = -x\frac{d}{dx}f_{|x_n|}(x) = \frac{2x^2}{(av)^3\sqrt{2\pi n^3}}\, e^{-\frac{x^2}{2n(av)^2}}, \quad x \geq 0.$$

In Fig. 2, several probability density functions are shown for the values of $n = 2, 4, 10$, and 20. These pdfs were calculated with the use of the MatLab package and Eq. (10).

Hence, the mean distance $\|x_n\|$ from the starting point to the position of the particle at the n-th change of a movement direction (i.e., the mean of $\|x_n\|$) is

$$\mathbf{E}\left[\|x_n\|\right] = \frac{2}{(av)^3\sqrt{2\pi n^3}}\int_0^{\infty} x^3 e^{-\frac{x^2}{2n(av)^2}}\,dx = \frac{4av\sqrt{n}}{\sqrt{2\pi}}.$$

It is not difficult to prove that the following integral has the closed-form solution:

$$\mathbf{E}\left[\|x_n\|^{m-2}\right] = \frac{2}{(av)^3\sqrt{2\pi n^3}}\int_0^\infty x^m e^{-\frac{x^2}{2n(av)^2}}\,dx$$

$$= \frac{(av)^{m-2}(2n)^{\frac{m+1}{2}}}{\sqrt{2\pi n^3}}\,\Gamma\left(\frac{m+1}{2}\right).$$

For $m = 3$, it reduces to the mean value $\mathbf{E}[\|x_n\|] = \frac{4av\sqrt{n}}{\sqrt{2\pi}}$. The second moment is given, when $m = 4$, and can be expressed as $\mathbf{E}\left[\|x_n\|^2\right] = 3(av)^2 n$. As a consequence, the variance is given by

$$\mathrm{Var}\left(\|x_n\|\right) = (av)^2 n\left(3 - \frac{8}{\pi}\right) \approx 0.453521\ (av)^2 n.$$

It is easily seen that $f_{\|x_n\|}$ is a unimodal pdf [9], and its mode m_f can be obtained from $\frac{d}{dx}f_{\|x_n\|}(x) = 0$, which gives

$$m_f = \sqrt{2}av\sqrt{n}.$$

With regard for the Gauss inequality for unimodal distributions [10], we have

$$P\left(|\|x_n\| - m_f| > 2\sqrt{(\mathbf{E}[\|x_n\|]-m_f)^2 + \mathrm{Var}\left(\|x_n\|\right)}\right) \le \frac{1}{9},$$

if $2\sqrt{(\mathbf{E}[\|x_n\|]-m_f)^2 + \mathrm{Var}\left(\|x_n\|\right)} \ge \frac{2}{\sqrt{3}}\sqrt{\mathrm{Var}\left(\|x_n\|\right)}$.

The last inequality can be easily verified:

$$\left(\sqrt{2}av\sqrt{n} - \frac{4av\sqrt{n}}{\sqrt{2\pi}}\right)^2 + (av)^2 n\left(3 - \frac{8}{\pi}\right) - \frac{1}{3}(av)^2 n\left(3 - \frac{8}{\pi}\right)$$

$$= (av)^2 n^2\left(4 + \frac{8}{3\pi} - \frac{8}{\sqrt{\pi}}\right) > 0.$$

Therefore, with a probability of $\frac{8}{9}$, the distance from the starting point to a particle position at the instant of the n-th change of a movement direction satisfies the inequality

$$\|x_n\| \le m_f + 2\sqrt{(\mathbf{E}\left[\|x_n\|\right] - m_f)^2 + \mathrm{Var}\left(\|x_n\|\right)}$$

$$= \sqrt{2}av\sqrt{n} + av\sqrt{n}\sqrt{\left(\sqrt{2} - \frac{4}{\sqrt{2\pi}}\right)^2 + \left(3 - \frac{8}{\pi}\right)}$$

$$= \left(\sqrt{2} + \sqrt{5 - \frac{8}{\sqrt{\pi}}}\right)av\sqrt{n} = 2.111697\,av\sqrt{n}.$$

We can find a double-sided tighter bound, by using the well-known Vysochanskij–Petunin inequality for the unimodal pdf of a random variable X [11, 12]

$$P\left(|X - \mathbf{E}[X]| > \lambda\sqrt{\mathrm{Var}\left(\|X\|\right)}\right) \le \frac{4}{9\lambda^2}, \qquad \lambda > \sqrt{8/3}.$$

By using the Vysochanskij–Petunin inequality for $\lambda = 2$, we obtain with a probability of greater than $\frac{8}{9}$ that the distance from the starting point to a particle position at the instant of the n-th change of a movement direction satisfies the inequalities

$$\mathbf{E}\left[\|x_n\|\right] - 2\sqrt{\mathrm{Var}\left(\|X\|\right)} < \|x_n\| < \mathbf{E}\left[\|x_n\|\right] + 2\sqrt{\mathrm{Var}\left(\|X\|\right)}.$$

or, equivalently,

$$P\left(\left(\frac{4}{\sqrt{2\pi}} - 2\sqrt{3 - \frac{8}{\pi}}\right) av\sqrt{n} \le \|x_n\| \right.$$
$$\left. \le \left(\frac{4}{\sqrt{2\pi}} + 2\sqrt{3 - \frac{8}{\pi}}\right) av\sqrt{n}\right) \ge \frac{8}{9}.$$

Thus, almost all particles of such an ideal gas after n collisions will be located in the ring between two circles with the radii

$$r_1 = \left(\frac{4}{\sqrt{2\pi}} - 2\sqrt{3 - \frac{8}{\pi}}\right) av\sqrt{n} = 0.248889\, av\sqrt{n}$$

and

$$r_2 = \left(\frac{4}{\sqrt{2\pi}} + 2\sqrt{3 - \frac{8}{\pi}}\right) av\sqrt{n} = 2.942648\, av\sqrt{n}$$

centered at 0.

As a final remark, we mention that these models can approximate the random motion of some birds, animals in the ocean, flying objects (airplanes), and mobile users in a shopping mall.

REFERENCES

1. W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York, 1971, Vol. 2.
2. E. V. Tolubinskii, *The Theory of Transfer Processes* [in Russian], Naukova Dumka, Kiev, 1969.
3. W. Stadje, “Exact probability distributions for non-correlated random walk models,” *J. Stat. Phys.*, **56**, 415–435 (1989).
4. E. Orsingher and A. De Gregorio, “Random flights in higher spaces,” *J. Theor. Prob.*, **20**, 769–806 (2007).
5. R. Garra and E. Orsingher, “Random flights govern by Klein–Gordon-type partial differential equations,” *Stoch. Process. Their Appl.*, **122**, 676–713 (2014).
6. A. Pogorui and R. M. Rodríguez-Dagnino, “Goldstein–Kac telegraph equations and random flights in higher dimensions,” *Appl. Math. Comput.*, **361**, 617–629 (2019).
7. U. Kuchler and S. Tappe, “Bilateral gamma distributions and processes in financial mathematics,” *Stoch. Process. Their Appl.*, **118**(2), 261–283 (2008).
8. A. Tashkandy Yusra, A. Omair Maha, and A. Alzaid Abdulhamid, “Bivariate and bilateral distribution,” *Int. J. Statist. Probab.*, **7**(2), 66–79 (2018).
9. T. M. Sellke and S. H. Sellke, “Chebyshev inequalities for unimodal distributions,” *Amer. Statistician*, **51**(1), 34–40 (1997).
10. G. Upton and I. Cook, *Gauss Inequality. A Dictionary of Statistics.* Oxford Univ. Press, Oxford, 2008.
11. F. Pukelsheim, “The three sigma rule,” *Amer. Statistician*, **48**(2), 88–91 (1994).

12. D. F. Vysochanskij and Y. I. Petunin, “Justification of the 3σ rule for unimodal distributions,” *Theor. Probab. Math. Statist.*, **21**, 25–36 (1980).

13. I. V. Samoilenko, “Distribution function of Markovian random evolution in R^n,” (2009), https://arxiv.org/pdf/0911.0165.

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