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METHOD OF DIRECT CUTTING-OUT IN THE PROBLEMS OF ELASTIC EQUILIBRIUM OF ANISOTROPIC BODIES WITH CRACKS UNDER LONGITUDINAL SHEAR

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In the earlier developed method of direct cutting-out, we take into account the anisotropy of material. This method is based on the procedure of modeling of finite or bounded bodies with thin structural defects of any type and boundary conditions on its contour by an infinite space with the same inhomogeneities as in the original problem and additional thin inhomogeneities (cracks or absolutely rigid inclusions), which form the boundary of the investigated body. Thus, loaded cracks are used to model boundary conditions of the first kind, whereas absolutely rigid inclusions embedded in the matrix with certain tension model boundary conditions of the second kind. The developed approach is verified for several problems of longitudinal shear of an anisotropic half space, a layer, and a wedge in the presence of an internal crack under given boundary conditions of the first kind.

Keywords: anisotropy, orthotropic material, crack, half space, layer, wedge, stress intensity factor, method of direct cutting-out, longitudinal shear.

The observed rapid development of contemporary technologies and industry would be impossible without taking into account broad spectra of properties of both traditional and new materials. In the mathematical modeling of materials, it becomes customary not to use the hypothesis that the material is homogeneous and isotropic. Indeed, there exists an urgent need to take into account the variations of the elastic properties of materials in different directions, i.e., the property of anisotropy. This is important not only for the analysis of natural anisotropy (observed in wood, crystals, and rocks) but also in the case of structural or artificial anisotropy in inhomogeneous materials and products; in particular, in plates and shells reinforced by thin strip-like inhomogeneities, fibers, and fillers, in smart materials, etc.

The processes of production and operation of structural elements or other material objects are inevitably accompanied by the formation of various inhomogeneities, both predicted in the design of structures and unpredicted (i.e., cracks and elastic or thin rigid inclusions of different shapes). Therefore, it is necessary to know the distribution of stresses in bodies containing inhomogeneities of this kind and determine the applicability of these structures for subsequent operation.

At present, there are numerous well-developed and powerful methods and approaches aimed at the numerical-analytic or pure numerical analyses of the elastic equilibrium of bounded bodies with thin strip-like inhomogeneities. Note that the methods of integral transformations, methods of the theory of functions of complex variable, direct numerical methods, and their combinations prove to be among the most widespread approaches of this kind.

The antiplane problem of the theory of elasticity was solved by the methods of Fourier integral transforms and jump functions in [10] for a collection of anisotropic bands containing strip-like inhomogeneities with

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arbitrary elastic properties. In the limit cases, these inhomogeneities may correspond either to a crack or to an absolutely rigid thin interlayer. In the cited work, the numerical results were presented for the case of uniform loading of a piecewise-homogeneous space in the presence of an elastic band in one of the half spaces. The case of longitudinal shear of an anisotropic half space containing a thin anisotropic elastic inclusion and subjected to the action of uniform loading was investigated in [11]. The influence of screw dislocations and concentrated forces on layered structures was studied in [23].

An orthotropic elastic layer containing a crack perpendicular to the boundary was investigated in [18] with the help of Fourier series and integral transforms. The case of crack parallel to the boundary of the layer was studied in [14] by the methods of the theory of functions of complex variable. The solutions of these problems were obtained in closed form. In [13], the attention of the researchers was focused on the interaction of internal and edge cracks and cavities in the orthotropic layer in the presence of dislocations.

In [8], the relationship between the invariant *J*-integral and the stress function was established by analyzing the longitudinal shear of elastic anisotropic bodies with thin-walled inclusions. It was proposed to use this relationship in estimating the limit state and generalized stress intensity factors (SIF) for the corresponding problems. The relationship between the generalized SIF and the *J*-integral was found by the methods of the theory of functions of complex variable.

The antiplane deformation of anisotropic bodies with thin-walled structures was studied in [9, 12] with the help of a direct numerical approach, namely, by using the boundary-element method. In particular, an approach to the regularization of singular and quasisingular integrals was proposed in [9].

The problem of longitudinal shear of a finite one- or two-component anisotropic wedge containing an interface crack was investigated with the help of an analog of the finite Mellin transform in [15, 22]. The problem of elastic equilibrium of an orthotropic rectangular bimaterial with interface crack was solved in [17]. With the help of expansions of the solution in the Fourier series, this problem was reduced to a singular integral equation (SIE), which was solved numerically.

The analytic expressions for the SIF in a series of problems of longitudinal shear of bodies with edge cracks were obtained by the method of conformal mappings in [16, 21, 25]. In particular, the authors of the cited works studied the cases of a round shaft, a wedge, a piecewise homogeneous half space, and a two-layer structure containing edge cracks and subjected to the action of concentrated factors.

Unlike direct numerical methods, for the investigation of the stressed states of bodies of various geometries it is customary to use absolutely different numerical-analytic methods and approaches. In the general form, for the first time, it was proposed [4] to model the solutions of the problems for deformable bodies of complex geometric shapes with thin inhomogeneities by the method of direct cutting-out. This method is based on the formation of the investigated body with thin inhomogeneities with the help of an infinite homogeneous or piecewise-homogeneous space containing a system of thin defects some of which form the boundary of the body and the conditions imposed on this boundary. The boundary of the body with applied forces is formed by loaded cracks. At the same time, the displacements on the corresponding boundary are modeled by absolutely rigid inclusions inserted into the matrix with certain tension. In this method, the number of SIE somewhat increases but the procedure of their construction and solution is significantly simplified and unified. The indicated approach made it possible to successfully solve a series of problems of elastic equilibrium of various isotropic plane-layered structures (half spaces, spheres, layers, and two-layer structures) [3, 4], wedge structures [2, 3], and beams [1, 6] with cracks and inclusions under the action of uniform loads and concentrated factors.

In the present paper, we continue the investigations carried out in [2, 4] and take into account the anisotropy of the material. We analyze the stability of computational schemes applied for the solution of the systems of SIE. In view of the restricted volume of the present paper, we restrict ourselves to the investigation of the problems of longitudinal shear for the cases of an anisotropic half space, a layer, or a wedge containing internal cracks with boundary conditions of the first type (forces) imposed on the boundaries of the body.

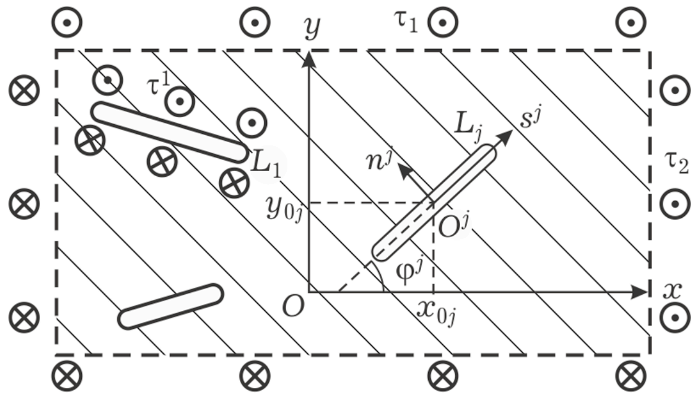

Fig. 1

The method of direct cutting-out makes it possible to reduce these problems to the analysis of the interaction of different systems of cracks in the anisotropic space.

As an alternative to the method of cutting-out, it is possible to consider the boundary-element method of jump functions, which is more complicated in realization but, at the same time, more universal. For the first time, it was successfully applied to the solution of antiplane and plane problems of anisotropic thermoelectromagnetoelasticity for bodies with thin strip-like inclusions [24]. Later, this method was used for thin shell-like inclusions in similar 3D-structures [19, 20].

1. Formulation of the Problem

Consider the problem of antiplane deformation of an anisotropic space with moduli of elasticity a_{km} $(k, m = 4, 5)$ containing N arbitrarily oriented plane tunnel cracks (mathematical cuts) L_j $(j = 1, \ldots, N)$ with local coordinate systems $s^j O^j n^j$ at their centers. The coordinates of the crack centers $z_{0j} = x_{0j} + i y_{0j}$, the angles of their rotation about the abscissa axis φ_j, and the crack lengths $2a_j$ $(j = 1, \ldots, N)$ are regarded as known. It is assumed that the cracks do not intersect and do not touch. The space can be loaded at infinity by uniform stresses

$$\sigma_{yz}^0 = \tau_1, \qquad \sigma_{xz}^0 = \tau_2.$$

Moreover, symmetric stresses τ^j may act on the top and bottom faces of the jth crack (Fig. 1). The stress-strain states in the planes perpendicular to the direction of shear are identical, and, therefore, we restrict ourselves to the analysis of fields in the plane xOy. The Oz-axis is directed in the direction of shear.

2. Construction of the Solution

According to the method of jump functions [7], we exclude thin inhomogeneities from consideration and replace their action upon the matrix by unknown functions of the jumps of stresses f_5^j and the derivatives of the jumps of displacements f_6^j in the median line $L_j' \equiv \left[-a^j, a^j\right]$ $\left(n^j = 0;\ -a^j \le s^j \le a^j\right)$ of inhomogeneity in its local coordinate system $s^j O^j n^j$:

$$\sigma_{nz}^{j-} - \sigma_{nz}^{j+} = f_5^j(s^j), \qquad \frac{\partial}{\partial s}\left(w^{j-} - w^{j+}\right) = f_6^j(s^j), \qquad s^j \in L_j' \quad (j = 1, \ldots, N). \tag{1}$$

Here, $f_r^j = 0$ $(r = 5, 6)$ for $s^j \notin L_j'$. The superscripts (+) and (–) correspond to the top and bottom faces of the inhomogeneity. We represent the stressed state of the anisotropic space with thin inhomogeneity L_j in its local coordinate system via the unknown jump functions in the following form [7, 10, 23]:

$$\sigma_{nz}^{j,\text{in}} + i\sigma_{sz}^{j,\text{in}} = \frac{1}{4}\left(g_p^j t_5(z^j) - g_m^j t_5(\bar{z}^j)\right) + \frac{i}{4a_{55}^j \alpha^j}\left(g_p^j t_6(z^j) + g_m^j t_6(\bar{z}^j)\right). \tag{2}$$

Here, the superscript "in" stands for inhomogeneity and

$$t_r(z^j) = \frac{1}{\pi}\int_{L_j'} \frac{f_r^j(t)dt}{t - z^j} \quad (r = 5, 6),$$

$$g_p^j = \beta^j + i(1 + \alpha^j), \qquad g_m^j = \beta^j + i(1 - \alpha^j),$$

$$\alpha^j = \frac{\sqrt{a_{44}^j a_{55}^j - (a_{45}^j)^2}}{a_{55}^j}, \qquad \beta^j = \frac{a_{45}^j}{a_{55}^j}, \qquad z^j = s^j + \left(\beta^j + i\alpha^j\right) n^j.$$

The elastic constants a_{km}^j in the coordinate systems $s^j O^j n^j$ are given by the formulas [5, 7]:

$$a_{44}^j = a_{44}\cos^2\varphi_j - 2a_{45}\sin\varphi_j\cos\varphi_j + a_{55}\sin^2\varphi_j,$$

$$a_{45}^j = (a_{44} - a_{55})\sin\varphi_j\cos\varphi_j + a_{45}\left(\cos^2\varphi_j - \sin^2\varphi_j\right),$$

$$a_{55}^j = a_{44}\sin^2\varphi_j + 2a_{45}\sin\varphi_j\cos\varphi_j + a_{55}\cos^2\varphi_j.$$

By using the principle of superposition of solutions, expression (2), and the formulas of transformation for the stress tensor under changes of the coordinate system, we represent the stressed state of the anisotropic space containing a system of N arbitrarily oriented inhomogeneities in the form

$$\sigma_{yz} + i\sigma_{xz} = \sigma_{yz}^0 + i\sigma_{xz}^0 + \sum_{j=1}^{N}\left(\sigma_{nz}^{j,\text{in}} + i\sigma_{sz}^{j,\text{in}}\right)e^{-i\varphi_j},$$

$$s^j + in^j = \left(x + iy - z_{0j}\right)e^{-i\varphi_j}. \tag{3}$$

Here,

$$\sigma_{yz}^0 + i\sigma_{xz}^0 = \tau_1 + i\tau_2$$

are the homogeneous stresses for the space without defects but with a given external load.

We now pass to the coordinate system of the lth inhomogeneity

$$\sigma_{nz}^l + i\sigma_{sz}^l = \left(\sigma_{yz} + i\sigma_{xz}\right) e^{i\varphi_l},$$

$$x + iy = \left(s^l + in^l\right) e^{i\varphi_l} + z_{0l}$$

and determine the limit stresses $\sigma_{nz}^{l\pm}$ on its faces by the Sochocki–Plemelj formula [7]:

$$\sigma_{nz}^{l\pm} = \mp\frac{1}{2} f_5^l(s^l) - \frac{1}{2\pi\alpha^l a_{55}^l} \int_{L_j'} \frac{f_6^l(t)dt}{t - s^l}$$

$$+ \operatorname{Re}\left[\left(\sigma_{yz}^0 + i\sigma_{xz}^0 + \sum_{j=1, j\neq l}^{N} \left(\sigma_{nz}^{j,\mathrm{in}} + i\sigma_{sz}^{j,\mathrm{in}}\right) e^{-i\varphi_j}\right) e^{i\varphi_l}\right], \quad (4)$$

$$s^j + in^j = \left(s^l e^{i\varphi_l} + z_{0l} - z_{0j}\right) e^{-i\varphi_j}.$$

Here, the upper and lower signs correspond to the top and bottom faces of the inhomogeneity, respectively.

We now consider the stresses imposed on the crack faces. In view of the opposite directions of normal to the matrix and crack surfaces, we obtain

$$\sigma_{nz}^{l+} + \sigma_{nz}^{l-} = -2\tau^l, \qquad \sigma_{nz}^{l-} - \sigma_{nz}^{l+} = 0 \quad (l = 1, \ldots, N). \quad (5)$$

Substituting (4) in (5), we arrive at a system of $2N$ SIE for the unknown jump functions. The additional conditions of balance and single-valuedness of displacements in traversing each defect, namely,

$$\int_{-a_l}^{a_l} f_5^l(t)\, dt = 0, \qquad \int_{-a_l}^{a_l} f_6^l(t)\, dt = 0 \quad (l = 1, \ldots, N)$$

enable us to solve the resulting system of SIE, in particular, by the method of collocations with the use of expansion of the jump functions into finite sums of the series with isolated root singularity [7]:

$$f_r^l(t) = \sum_{m=0}^{M_l} \frac{A_m^{rl} T_m(t/a_l)}{\sqrt{1 - (t/a_l)^2}}.$$

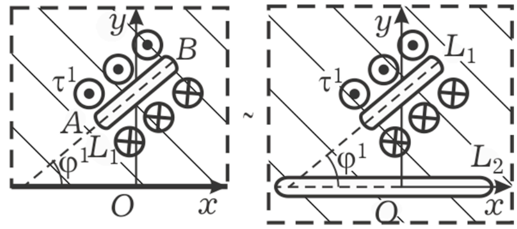

Fig. 2

Here, A_m^{rl} are the required coefficients, T_m are the Chebyshev polynomials of the first kind, and M_l is the number of terms of the expansion in series that are taken into account. Note that, it follows from expression (1) and the second relation in (5) that the jumps functions of stresses f_5^j $(j = 1, \ldots, N)$ are equal to zero. By using these functions, we can determine the stressed state of the anisotropic space with cracks according to relations (3) and (2).

The SIF are important parameters of the stressed state, which can be used to estimate the crack-growth resistance [7, 23]. For cracks in the anisotropic material, we compute the SIF at the tips A and B by the formulas [7, 23]

$$K_3^{lA,B} = \pm \frac{1}{a_{55}^l \alpha^l} p_6^{l\pm} \sqrt{\pi/2},$$

$$p_6^{l\pm} = \lim_{t \to \pm a_l} \left[\sqrt{|a_l \mp t|}\, f_6^l(\pm a_l) \right],$$

where the upper and lower signs correspond to the tips B and A, respectively.

Further, we apply the method of direct cutting-out to the solution of four characteristic problems of longitudinal shear of an anisotropic half space, of a layer, or of a wedge containing an internal crack. In this case, we restrict ourselves to the evaluation of the SIF

$$K_3^{0A,B} = K_3^{lA,B} / \tau^l \sqrt{\pi a_l}$$

normalized by the load and the half length of the crack. To obtain the SIF with an error not higher than 1%, it is sufficient to take into account $M = 80$ first terms of the expansion of jump functions in series in the Chebyshev polynomials for $a_l < 16$ and $M = 320$ for $a_l \geq 16$ $(l = 1, \ldots, N)$. We consider four types of materials in the basic coordinate system: isotropic (***I***) with $a_{44}/a_{55} = 1$ and $a_{45}/a_{55} = 0$; orthotropic (***II***) with $a_{44}/a_{55} = 2/3$ and $a_{45}/a_{55} = 0$; (***III***) orthotropic with $a_{44}/a_{55} = 3/2$ and $a_{45}/a_{55} = 0$, and anisotropic (orthotropic in the rotated coordinate system) (***IV***) with $a_{44}/a_{55} = 2/3$ and $a_{45}/a_{55} = 1/3$.

Example 1. Consider an anisotropic half space with load-free boundary containing a crack L_1 symmetrically loaded by shear tractions τ^1. Here, $(0, y_{01})$ are the coordinates of the crack center, $2a_1$ is its length, and φ^1 is the angle of orientation of the crack relative to the Ox-axis (left diagram in Fig. 2).

Table 1

a	0.5	1	2	4	8	16
(***I***)	1.0126	1.0440	1.0829	1.0906	**1.0912**	**1.0912**
(***II***)	1.0178	1.0652	1.1158	1.1227	**1.1230**	**1.1230**
(***III***)	1.0083	1.0284	1.0571	1.0654	**1.0662**	**1.0662**
(***IV***)	1.0292	1.0867	1.1336	1.1395	**1.1398**	**1.1398**

Applying the method of direct cutting-out, we reduce the original problem to the investigation of an anisotropic space with two cracks L_1 and L_2, where the load-free crack L_2 $(x_{02} = 0,\ y_{02} = 0,\ \varphi^2 = 0)$ simulates the load-free boundary of the body (right diagram in Fig. 2).

In Table 1, we present the results of evaluation of the normalized SIF for the crack parallel to the boundary of the modeled half space $(\varphi^1 = \varphi^2 = 0)$ for different relative lengths $a = a_2/a_1$ and the indicated four types of materials. The relative distance from crack center to the boundary of the domain is equal to one $(d = y_{01}/a_1 = 1)$. Since the problem is symmetric, the SIF for the left and right crack tips are identical $(K_3^{0A} = K_3^{0B})$.

As the relative length a increases, the SIF gradually approaches final values. Moreover, in the isotropic case, the result coincides with the known value from [4]. An error lower than 0.1% was obtained even for $a = 8$ in all analyzed cases [as the parameter a increases further, the first five significant digits of the solution (printed in bold face) do not change]. Note that, for the other limit case of the crack perpendicular to the boundary of the domain $(d = 2,\ \varphi^1 = \pi/2)$, the convergence rate of the method of direct cutting-out is lower. Indeed, in order to get an error lower than 1%, the relative crack length L_2 must be not smaller than $a = 16$ and, for an error of about 0.1%, the crack length should be increased to $a = 64$.

In Figs. 3 and 4, we show the dependences of the normalized SIF on the orientation angle φ^1 for the left (A) and right (B) crack tips, respectively, and different types of materials; here, $d = 2$ and $a = 64$.

For the orthotropic (***II***), (***III***) and isotropic (***I***) materials, the plots of the normalized SIF are symmetric about the value $\varphi^1 = \pi/2$. In view of the symmetry of our problem, this observation additionally confirms the reliability of the results. Note that, for the angle $\varphi^1 = \pi/2$, the values of SIF for different measures of orthotropy relative to the surface of the half space (***II***) and (***III***) and for the isotropic material (***I***) coincide to within the error of calculations.

In addition, the different measures of orthotropy of the material (***II***), (***III***) affect not only the quantitative but also the qualitative behavior of the plots of SIF. In particular, for the orthotropy of type (***III***), the SIF for the crack tip B increases with the angle φ^1 from zero to $\pi/2$. At the same time, for the orthotropy of type (***II***), it decreases. For the crack tip A, as the angle φ^1 increases from zero to $\pi/2$, the values of SIF become higher for all types of orthotropy. The maximal values of SIF were obtained for the anisotropic material

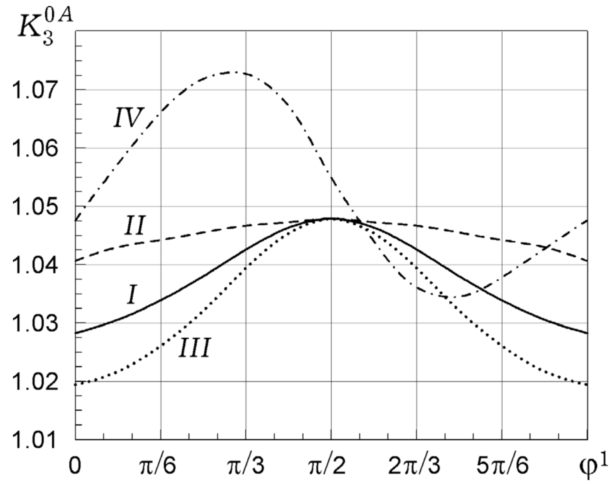

Fig. 3

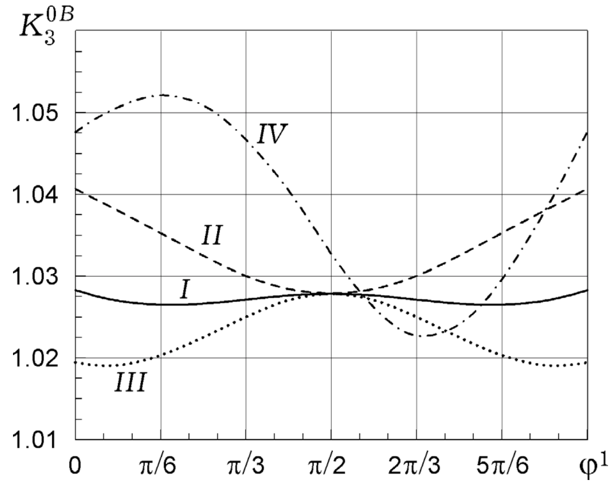

Fig. 4

of type (***IV***). For the isotropic material (***I***), the corresponding values coincide with the values obtained in [4, 7] as a result of the direct solution of the problem for a half space, which also confirms their reliability.

Example 2. Consider the case of longitudinal shear of a layer with load-free faces in the presence of an internal central crack L_1 symmetrically loaded by tractions τ^1 (left diagram in Fig. 5). The width of the layer is $2H$. The center of the crack L_1 is located at the point $(0, H)$; the crack length is equal to $2a_1$, and φ^1 is the angle of orientation relative to the Ox-axis.

Applying the method of direct cutting-out, we reduce the original problem to the investigation of the anisotropic space with three cracks L_1, L_2, and L_3, where the cracks L_2 and L_3 $\left(x_{02} = x_{03} = 0;\ y_{02} = 0;\ y_{03} = 2H;\ \varphi^2 = \varphi^3 = 0\right)$ simulate the load-free boundaries of the layer (right diagram in Fig. 5).

In Table 2, we present the results of evaluation of the normalized SIF

$$K_3^{0A,B} = K_3^{1A,B} \Big/ \tau^1 \sqrt{\pi a_1}$$

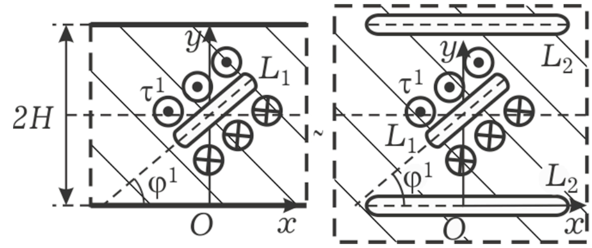

Fig. 5.

Table 2

a	0.5	1	2	4	8	16
(***I***)	1.0251	1.0847	1.1446	**1.1493**	**1.1493**	**1.1493**
(***II***)	1.0353	1.1246	1.1982	**1.2009**	**1.2009**	**1.2009**
(***III***)	1.0165	1.0548	1.1016	**1.1085**	**1.1085**	**1.1085**
(***IV***)	1.0378	1.1261	1.2238	**1.2280**	**1.2281**	**1.2277**

for a crack parallel to the boundary of the modeled layer $\left(\varphi^1 = 0\right)$ for different relative lengths $a = a_2/a_1 = a_3/a_1$ and four types of materials. The relative distance from crack center to the boundary of the domain is equal to one $\left(d = H/a_1 = 1\right)$.

By analogy with ***Example 1***, as the relative length a increases, the normalized SIF gradually approach certain specific values. Note that, in this case, an error lower than 0.1% is obtained even for $a = 4$ for all investigated types of materials.

If the crack is perpendicular to the boundary of the domain $(d = 2,\ \varphi^1 = \pi/2)$, then, in order to get an error lower than 1%, it is necessary to take the relative length L_2 not smaller than $a = 64$.

In Fig. 6, for $d = 2$ and $a = 64$, we present the dependences of the normalized SIF on the orientation angle φ^1 for different types of materials. Since the problem is symmetric, the SIF obtained for the left and right crack tips are identical:

$$K_3^{0A} = K_3^{0B}.$$

As in the problem for the half space, the plots of the normalized SIF are symmetric about the value $\varphi^1 = \pi/2$ for the orthotropic (in the basic coordinate system) (***II***) and (***III***) and isotropic (***I***) materials. As the angle φ^1 increases from zero to $\pi/2$, the values of SIF for the materials (***I***), (***II***), and (***III***) become higher. The inclination of the principal axes of orthotropy in the material (***IV***) violates this symmetry. For the angle $\varphi^1 = \pi/2$, as in ***Example 1***, the SIF for the materials (***I***), (***II***), and (***III***) coincide to within the error of calculations. A conclusion that the SIF is independent of the elastic constants of the material orthotropic with respect

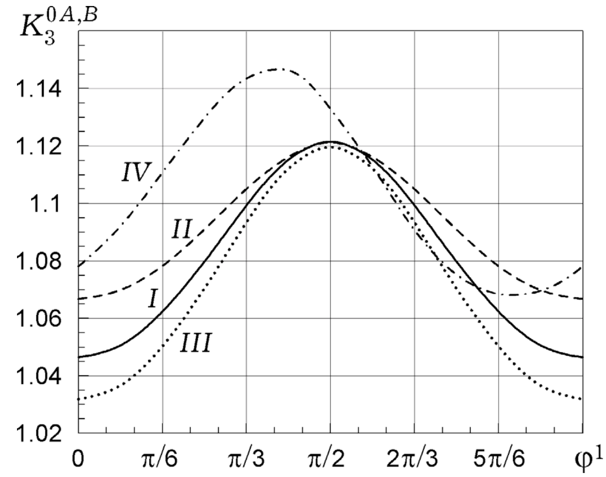

Fig. 6

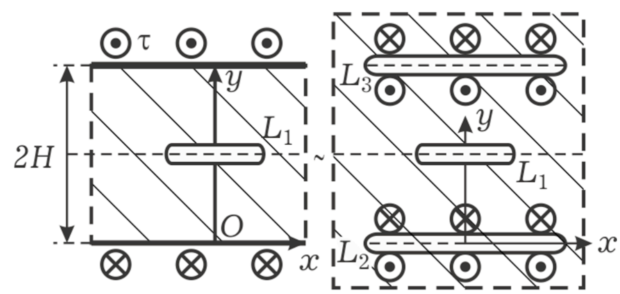

Fig. 7

to the crack axis for a symmetrically loaded central crack perpendicular to the boundaries of the layer was also made in [18].

Example 3. Consider ***Example 2*** in the case of loading of the surfaces of the layer by tractions τ when the internal central crack L_1 is free of loads and the external forces are applied to the outer surface (left diagram in Fig. 7). By the method of direct cutting-out, we reduce the original problem to the investigation of elastic balance of an anisotropic space with three cracks L_1, L_2, and L_3, where the cracks L_2 and L_3 symmetrically loaded by tractions $\tau^2 = \tau^3 = -\tau$ model the loaded boundaries of the layer (right diagram in Fig. 7).

In Table 3, we present the results of evaluation of the normalized SIF $K_3^{0A,B} = K_3^{1A,B} / \tau\sqrt{\pi a_1}$ for a crack parallel to the boundary of the modeled layer $(\varphi^1 = 0)$ for different relative lengths $a = a_2/a_1 = a_3/a_1$ and four types of materials. The relative distance from crack center to the boundary of the domain is equal to one $(d = H/a_1 = 1)$.

As the relative length a increases, the SIF gradually approaches its final values. Note that, even for $a = 8$, its values coincide with the values obtained for $a = 16$ with an error lower than 0.1%. Furthermore, the values of SIF for a symmetrically loaded crack in a layer with load-free faces (***Example 2***) also coincide (to within the error of calculations) with the results obtained in the same example for all analyzed materials.

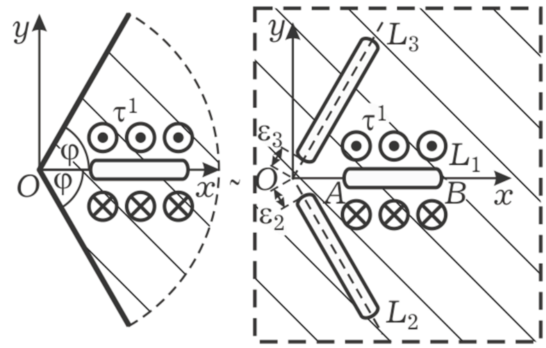

Fig. 8

Table 3

a	0.5	1	2	4	8	16
(***I***)	0.0910	0.3635	0.9146	1.1364	**1.1492**	**1.1492**
(***II***)	0.1022	0.4375	1.0407	1.1966	**1.2010**	**1.2010**
(***III***)	0.0775	0.2963	0.7870	1.0772	**1.1083**	**1.1085**
(***IV***)	0.1133	0.4446	1.0711	1.2251	**1.2281**	**1.2287**

Example 4. Consider the case of longitudinal shear of an anisotropic wedge containing a crack L_1 located on its bisectrix and symmetrically loaded by tractions τ^1. The boundaries of the wedge are free of loads. The opening angle of the wedge is 2φ. The center of the crack L_1 is located at a point $(x_{01}, 0)$ and its length is equal to $2a_1$ (left diagram in Fig. 8). By the method of direct cutting-out, we reduce the original problem to the investigation of an anisotropic space with three cracks L_1, L_2, and L_3, where L_2 and L_3 model the load-free boundary of the body (right diagram in Fig. 8).

By ε_2 and ε_3 we denote the distances from neighboring tips of the cracks L_2 and L_3 to the point of intersection of their axial lines. Suppose that $\varepsilon = \varepsilon_2/a_1 = \varepsilon_3/a_1$ is the relative distance to the virtual edge of the wedge and $a = a_2/a_1 = a_3/a_1$ is the relative length of simulating cracks.

We now study the influence of a on the rate of convergence of the method of direct cutting-out for an angle $\varphi = \pi/6$. Suppose that $\varepsilon = 0.001$ and the relative distance from the center of the crack L_1 to the edge of the wedge is $d = x_{01}/a_1 = 2$.

In Table 4, we present the results of evaluation of the normalized SIF for the left (A) and right (B) crack tips $\left(K_3^{0A,B} = K_3^{1A,B}/\tau^1\sqrt{\pi a_1}\right)$ for different relative lengths a and four types of materials.

It is easy to see that, as the relative length a increases, the normalized SIF approach their final values. Even for $a = 8$, they coincide with the results of evaluation of the normalized SIF for $a = 16$ with an error smaller than 0.1%, which enables us to treat this result as the actual value of the normalized SIF for a loaded crack in the wedge.

Table 4

K_3^{0A}						
a	0.5	1	2	4	8	16
(*I*)	1.0055	1.1042	1.2148	**1.2215**	**1.2216**	**1.2216**
(*II*)	1.0097	1.1193	1.2708	**1.2775**	**1.2776**	**1.2776**
(*III*)	1.0044	1.0922	1.1712	**1.1775**	**1.1777**	**1.1777**
(*IV*)	1.0119	1.1475	1.3015	**1.3101**	**1.3101**	**1.3101**
K_3^{0B}						
(*I*)	1.0036	1.0195	1.1238	**1.1374**	**1.1376**	**1.1376**
(*II*)	1.0055	1.0179	1.1595	**1.1753**	**1.1754**	**1.1754**
(*III*)	1.0020	1.0217	1.0972	**1.1083**	**1.1087**	**1.1087**
(*IV*)	1.0075	1.0259	1.1800	**1.1983**	**1.1985**	**1.1985**

Table 5

	K_3^{0A}				K_3^{0B}			
ε	0.1	0.01	0.001	0.0001	0.1	0.01	0.001	0.0001
(*I*)	1.9658	**2.2013**	**2.2015**	**2.2015**	1.3966	**1.4037**	**1.4037**	**1.4037**
(*II*)	2.1544	**2.3619**	**2.3617**	**2.3627**	1.4741	**1.4795**	**1.4793**	**1.4798**
(*III*)	1.8020	**2.0652**	**2.0657**	**2.0653**	1.3323	**1.3421**	**1.3426**	**1.3425**
(*IV*)	2.1731	**2.4425**	**2.4474**	**2.4469**	1.5141	**1.5208**	**1.5211**	**1.5209**

We now analyze the influence of the relative distance to the virtual edge of the wedge ε on the rate of convergence of method of direct cutting-out. Suppose that $a = 16$ and relative distance from the center of the crack L_1 to edge of the wedge is $d = 1.1$ because it is clear that the stress concentration becomes weaker as the distance between the tip of the crack L_1 and the edge of the wedge O increases.

In Table 5, we illustrate the results of evaluation of the normalized SIF K_3^{0A} and K_3^{0B} for different values of the parameter ε and four types of materials. High accuracy of the results is attained even for $\varepsilon = 0.01$. The results obtained for the isotropic material and presented in Tables 4 and 5 coincide (to within the error of calculations) with the data obtained earlier [2], which additionally confirms the reliability of the performed calculations.

Conclusions

The method of direct cutting-out developed in our previous works is generalized by taking into account the anisotropy of materials in the problems of longitudinal shear of homogeneous bodies weakened by cracks. By analyzing several illustrative examples of investigation of the elastic balance of an anisotropic half space, a layer, or a wedge containing cracks, we confirm the efficiency and reliability of this method. In the problems of antiplane deformation of a half space (or a layer) in the presence of defects, for certain geometric parameters and types of loading of the crack, it was discovered that the SIF do not depend on the elastic constants of the material orthotropic with respect to the crack axis. The obtained results of evaluation of the normalized SIF coincide with the data known from the literature with an error that does not exceed 1%. It was established that the measure of anisotropy and orientation of the principal axes of anisotropy in numerous cases strongly affect the values of the stress intensity factors.

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