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# THERMOELASTIC ANALYSIS OF FUNCTIONALLY GRADED CYLINDRICAL SHELLS

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We perform the analytic investigation of the stress-strain state of a functionally graded cylindrical shell of finite length heated by a two-dimensional temperature field. The properties of the shell material are regarded as analytic functions of the thickness coordinate. In our investigations, we use the equations of the refined theory of shells that takes into account the deformation of transverse shear and the transverse normal deformation. The heat-conduction equation is deduced under the assumption of linear temperature distribution over the thickness of the shell. For the boundary conditions of simply supported shell, the quasistatic uncoupled problem of thermoelasticity is solved by the methods of Fourier and Laplace transforms. Numerical examples are presented and discussed to show that it is important to take into account the influence of inhomogeneity of the properties of materials of the metal–ceramics composites.

**Keywords:** thermoelasticity, cylindrical shell, functionally graded material, thermal loads.

## Introduction

Thin-walled structural elements in the form of plates and shells made of inhomogeneous composite materials are often used in the contemporary engineering [1–3, 15]. Much attention is given to the investigation of functionally graded (FG) composites with continuous inhomogeneity. These materials are heat- and fracture-resistant, capable to operation under the action of elevated thermal stresses, and not susceptible to corrosion and erosion. Therefore, they are suitable for application in advanced technological processes. Functionally graded materials are, as a rule, alloys of ceramics and metal or combinations of different materials. Their physicomechanical characteristics undergo smooth and continuous variations from plane to plane. To apply nontraditional materials for the production of thin-walled structural elements in the form of plates and shells, it is necessary to develop new models and methods for their numerical analyses. Hence, the investigations carried out in this field are important and actual.

In recent years, the FG plates and shells were studied fairly extensively. Thus, in particular, the influence of the inhomogeneity of material on the limit equilibrium of a shell weakened by a surface crack was investigated in [3]. The nonstationary thermoelastic response of a cylindrical panel to the action of inhomogeneous thermal loads was theoretically investigated in [11]. An exact solution of the equations of thermoelasticity of the shear theory of FG cylindrical shells with finite length subjected to the action of thermal loading, internal pressure, and axial forces was found in [7]. The classical theory, as well as various refined theories, was used for this purpose in [6, 8, 12, 14]. The thermomechanical behavior of FG cylindrical shells was analyzed in [5, 16]

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by using the equations of coupled thermoelasticity and the finite-element method. For FG shells and plates, analytic solutions were constructed on the basis of 3D equations of thermoelasticity in [4, 9, 13, 19]. In [17], the loss of stability of FG cylindrical shells under thermal and force loads was analyzed by using the methodology based on the Hamiltonian principle. The optimal compositions of FG materials, which enable one to lower the level of thermal stresses and improve their heat resistance were analyzed in [10]. A more detailed survey of various theories aimed at modeling and investigation of FG shells and plates can be found in [18]. At the same time, the stress-strain state of inhomogeneous shells formed under the action of temperature fields determined from the heat conduction equation with regard for the heat transfer is studied quite poorly.

The aim of the present work is to determine the thermoelastic state of an isotropic FG circular cylindrical shell heated by a two-dimensional temperature field given at the initial time by using the equations of thermoelasticity of the refined theory of shells and heat conduction equations with regard for the presence of convective heat exchange with ambient medium.

## 1. Formulation of the Problem and Basic Equations

Consider an inhomogeneous isotropic circular cylindrical shell of constant thickness $2h$, length $l$, and radius of the middle surface $R$. We refer the points of the shell volume to a cylindrical coordinate system $x$, $\theta$, $z$ whose coordinates correspond to the axial, circumferential, and radial directions, respectively. In what follows, we denote these coordinates by subscripts 1, 2, and 3, respectively.

We assume that the shell is made of a metal–ceramics composite. Then the effective properties of the composite material $P_{ef}$ can be expressed via the characteristics of the ceramics $P_c$ and metal $P_m$ as follows:

$$P_{ef}(z) = P_c f_c + P_m f_m ,$$

where $f_c$ and $f_m$ are the relative fractions of the ceramics and the metal in the composite, respectively, whose distributions over the thickness should be specified and, in addition, $f_c + f_m = 1$. We specify the following power law of distribution [5, 8]:

$$f_c = f_c^- + (f_c^+ - f_c^-)\left(\frac{z}{2h} + \frac{1}{2}\right)^k ,$$

where $f_c^+$ and $f_c^-$ are, respectively, the fractions of ceramics $f_c$ on the upper $z = h$ and lower $z = -h$ surfaces and $k$ is the parameter of inhomogeneity that describes the variations of the fraction of material across the thickness and takes the values $k \geq 0$. Varying the values of this parameter, we can get the optimal properties of the composite. In a special case where $f_c^- = 0$ and $f_c^+ = 1$, we find

$$f_c = \left(\frac{z}{2h} + \frac{1}{2}\right)^k ,$$

and the formula for the effective characteristics of the material takes the form

$$P_{ef}(z) = P_m + (P_c - P_m)\left(\frac{z}{2h} + \frac{1}{2}\right)^k . \quad (1)$$

It is clear that, as the parameter $k$ decreases down to zero, the material of the shell acquires the properties of pure ceramics. At the same time, as the parameter infinitely increases, the characteristics of the material approach the characteristics of the pure metal. We assume that Young's modulus $E(z)$, the coefficient of linear thermal expansion $\alpha^t(z)$, and heat-conduction coefficient $\lambda(z)$ are described by Eq. (1) and that Poisson's ratio $\nu = \text{const}$.

Suppose that the shell is heated either by thermal sources or by a temperature field given at the initial time and that convective heat exchange takes place between the surfaces $z = \pm h$ and the ambient medium. As a result, temperature strains and stresses are formed in the shell. For the investigation of the stress-strain state of the shell, we apply the mathematical model proposed in [1, 2] and based on the assumptions that the dependences of the components of the vector of displacements $U_j$ and temperature $t$ on the radial coordinate $z$ are linear, i.e.,

$$U_j(x,\theta,z,\tau) = u_j(x,\theta,\tau) + z\gamma_j(x,\theta,\tau) \qquad (j = 1,2,3), \tag{2}$$

$$t(x,\theta,z,\tau) = T_1(x,\theta,\tau) + \frac{z}{h}T_2(x,\theta,\tau), \tag{3}$$

where $u_j$ are the components of displacements of points of the middle surface; $\gamma_1$ and $\gamma_2$ are the angles of rotation of the normal; $\gamma_3$ is the radial normal strain, and

$$T_i = \frac{2i-1}{2h^i}\int_{-h}^{h} t z^{i-1} dz \quad (i = 1,2)$$

are the integral characteristics of temperature.

The proposed model is formed by the system of heat-conduction and thermoelasticity equations. In the general case, these equations are coupled. If we neglect the influence of deformation on the variations of temperature field, then the corresponding systems become independent and the heat conduction equations for the integral characteristics of temperature $T_1$ and $T_2$ take the form

$$\begin{gathered}\Delta_{(1)}T_1 - \varepsilon_1^t T_1 + \Delta_{(2)}T_2 + \left(\frac{\lambda^{(1)}}{hR} - \varepsilon_2^t\right)T_2 - C^{(1)}\dot{T}_1 - C^{(2)}\dot{T}_2 = -f_1 - W_1^t, \\ \Delta_{(2)}T_1 - \varepsilon_2^t T_1 + \Delta_{(3)}T_2 + \left(\frac{\lambda^{(2)}}{hR} - \frac{\lambda^{(1)}}{h^2} - \varepsilon_1^t\right)T_2 - C^{(2)}\dot{T}_1 - C^{(3)}\dot{T}_2 = -f_2 - W_2^t,\end{gathered} \tag{4}$$

where

$$\Delta_{(j)} = \lambda^{(j)}\left(\partial_{11}^2 + \frac{1}{R^2}\partial_{22}^2\right), \quad \left\{\lambda^{(j)}, C^{(j)}\right\} = \int_{-h}^{h}\{\lambda(z), c_\varepsilon(z)\}\left(\frac{z}{h}\right)^{j-1} dz, \quad (j = 1,2,3);$$

$$W_i^t = \int_{-h}^{h} w_t\left(\frac{z}{h}\right)^{i-1} dz, \quad f_i = t_1^z\varepsilon_i^t + t_2^z\varepsilon_{3-i}^t \quad (i = 1,2),$$

$$\varepsilon_i^t = \left(\alpha^+ - (-1)^i \alpha^-\right), \qquad t_i^z = \frac{1}{2}\left(t_z^+ - (-1)^i t_z^-\right);$$

$\lambda(z)$ is the heat-conduction coefficient; $t_z^+$ and $t_z^-$ are the temperatures of the media surrounding the surfaces $z = h$ and $z = -h$, respectively; $\alpha^+$ and $\alpha^-$ are the heat-transfer coefficients for these surfaces; $c_\varepsilon(z)$ is the specific heat; $w_t$ is the intensity of heat sources, and the overdot in the notation $\dot{T}_i$ denotes the partial derivative of the function with respect to time $\tau$.

We now represent the system of thermoelasticity equations in generalized displacements in the operator form as follows:

$$\sum_{s=1}^{6} L_{rs} y_s = b_r \qquad (r, s = 1, 2, \ldots, 6), \tag{5}$$

where

$$y_j = u_j, \qquad y_{3+j} = \gamma_j \qquad (j = 1, 2, 3).$$

The differential operators $L_{rs}$ $\left(L_{rs} = L_{sr}\right)$ and the free terms $b_r$ are given by the formulas

$$L_{11} = A_{11}\partial_{11}^2 + \frac{A_{66}}{R^2}\partial_{22}^2, \qquad L_{12} = \frac{A_{12} + A_{66}}{R}\partial_{12}^2,$$

$$L_{13} = \frac{A_{12}}{R}\partial_1, \qquad L_{14} = B_{11}\partial_{11}^2 + \frac{B_{66}}{R^2}\partial_{22}^2,$$

$$L_{15} = \frac{B_{12} + B_{66}}{R}\partial_{12}^2, \qquad L_{16} = \left(A_{13} + \frac{B_{12}}{R}\right)\partial_1,$$

$$L_{22} = A_{66}\partial_{11}^2 + \frac{A_{22}}{R^2}\partial_{22}^2 - k'\frac{A_{55}}{R^2}, \qquad L_{23} = \frac{A_{22} + k'A_{55}}{R^2}\partial_2,$$

$$L_{24} = \frac{B_{12} + B_{66}}{R}\partial_{12}^2, \qquad L_{25} = B_{66}\partial_{11}^2 + \frac{B_{22}}{R^2}\partial_{22}^2 + \frac{k'A_{55}}{R},$$

$$L_{26} = \left(\frac{A_{23}}{R} + \frac{B_{22} + k'B_{55}}{R^2}\right)\partial_2,$$

$$L_{33} = -k'A_{44}\partial_{11}^2 - \frac{k'A_{55}}{R^2}\partial_{22}^2 + \frac{A_{22}}{R^2}, \qquad L_{34} = \left(\frac{B_{12}}{R} - k'A_{44}\right)\partial_1,$$

$$L_{35} = \frac{B_{22}/R - k'A_{55}}{R}\partial_2, \qquad L_{36} = -k'B_{44}\partial_{11}^2 + \frac{B_{22} - k'B_{55}\partial_{22}^2}{R^2} + \frac{A_{23}}{R},$$

$$L_{44} = D_{11}\partial_{11}^2 + \frac{D_{66}}{R^2}\partial_{22}^2 - k'A_{44}, \quad L_{45} = \frac{D_{12}+D_{66}}{R}\partial_{12}^2,$$

$$L_{46} = \left(\frac{D_{12}}{R} + B_{13} - k'B_{44}\right)\partial_1,$$

$$L_{55} = D_{66}\partial_{11}^2 + \frac{D_{22}}{R^2}\partial_{22}^2 - k'A_{55}, \quad L_{56} = \left(\frac{B_{23}-k'B_{55}}{R} + \frac{D_{22}}{R^2}\right)\partial_2,$$

$$L_{66} = A_{33} + \frac{2B_{23}}{R} + \frac{D_{22}}{R^2} - k'D_{44}\partial_{11}^2 - \frac{k'D_{55}}{R^2}\partial_{22}^2,$$

$$b_1 = A_{11}^t\partial_1 T_1 + \frac{B_{11}^t}{h}\partial_1 T_2, \quad b_2 = \frac{A_{22}^t}{R}\partial_2 T_1 + \frac{B_{22}^t}{Rh}\partial_2 T_2,$$

$$b_3 = \frac{A_{22}^t}{R}T_1 + \frac{B_{22}^t}{Rh}T_2, \quad b_4 = B_{11}^t\partial_1 T_1 + \frac{D_{11}^t}{h}\partial_1 T_2,$$

$$b_5 = \frac{B_{22}}{R}\partial_2 T_1 + \frac{D_{22}^t}{Rh}\partial_2 T_2, \quad b_6 = \left(A_{33}^t + \frac{B_{22}^t}{R}\right)T_1 + \frac{D_{22}^t/R + B_{33}^t}{h}T_2,$$

where

$$\left\{A_{ii}, B_{ii}, D_{ii}\right\} = \frac{1-\nu}{(1+\nu)(1-2\nu)}\int_{-h}^{h} E(z)\left\{1, z, z^2\right\}dz \quad (i = 1,2,3),$$

$$\left\{A_{ij}, B_{ij}, D_{ij}\right\} = \frac{\nu}{(1+\nu)(1-2\nu)}\int_{-h}^{h} E(z)\left\{1, z, z^2\right\}dz \quad (i, j = 1,2,3), \quad (i \neq j),$$

$$\left\{A_{ii}^t, B_{ii}^t, D_{ii}^t\right\} = \frac{1}{1-2\nu}\int_{-h}^{h} E(z)\alpha^t(z)\left\{1, z, z^2\right\}dz \quad (i = 1,2,3),$$

$$\left\{A_{ii}, B_{ii}, D_{ii}\right\} = \frac{1}{2(1+\nu)}\int_{-h}^{h} E(z)\left\{1, z, z^2\right\}dz \quad (i = 4,5,6),$$

and

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial \theta},$$

$k'$ is the shear coefficient [2].

According to the known displacements and temperature fields, the forces and moments in the shell can be found as follows:

$$\begin{pmatrix} N_{11} \\ N_{22} \\ N_{33} \\ M_{11} \\ M_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} \\ A_{12} & A_{22} & A_{23} & B_{12} & B_{22} \\ A_{13} & A_{23} & A_{33} & B_{13} & B_{23} \\ B_{11} & B_{12} & B_{13} & D_{11} & D_{12} \\ B_{12} & B_{22} & B_{23} & D_{12} & D_{22} \end{pmatrix} \begin{pmatrix} \partial_1 u_1 \\ \dfrac{\partial_2 u_2 + u_3}{R} \\ \gamma_3 \\ \partial_1 \gamma_1 \\ \dfrac{\partial_2 \gamma_2 + \gamma_3}{R} \end{pmatrix} - \begin{pmatrix} A_{11}^t \\ A_{22}^t \\ A_{33}^t \\ B_{11}^t \\ B_{22}^t \end{pmatrix} T_1 - \begin{pmatrix} B_{11}^t \\ B_{22}^t \\ B_{33}^t \\ D_{11}^t \\ D_{22}^t \end{pmatrix} \frac{T_2}{h},$$

$$\begin{pmatrix} N_{12} \\ M_{12} \end{pmatrix} = \begin{pmatrix} A_{66} & B_{66} \\ B_{66} & D_{66} \end{pmatrix} \begin{pmatrix} \partial_1 u_2 + \dfrac{\partial_2 u_1}{R} \\ \partial_1 \gamma_2 + \dfrac{\partial_2 \gamma_1}{R} \end{pmatrix},$$

$$\begin{pmatrix} N_{13} \\ M_{13} \end{pmatrix} = k' \begin{pmatrix} A_{44} & B_{44} \\ B_{44} & D_{44} \end{pmatrix} \begin{pmatrix} \gamma_1 + \partial_1 u_3 \\ \partial_1 \gamma_3 \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} N_{23} \\ M_{23} \end{pmatrix} = k' \begin{pmatrix} A_{55} & B_{55} \\ B_{55} & D_{55} \end{pmatrix} \begin{pmatrix} \gamma_2 + \dfrac{\partial_2 u_3 - u_2}{R} \\ \dfrac{\partial_2 \gamma_3}{R} \end{pmatrix}.$$

## 2. Procedure of Solution

For the uniqueness of the solution of systems (4) and (5), it is necessary to impose the corresponding boundary conditions for the mechanical and temperature functions and also the initial conditions for temperature. Assume that the ends $x = 0$ and $x = l$ of the shell are simply supported and kept at temperature equal to zero. Then we have the following boundary conditions:

$$u_3 = u_2 = \gamma_3 = \gamma_2 = 0, \quad N_{11} = M_{11} = 0, \tag{7}$$

$$T_1 = T_2 = 0. \tag{8}$$

At the initial time, we specify the temperature characteristics as the following functions of coordinates:

$$T_1(x,\theta,0) = T_1^0(x,\theta), \quad T_2(x,\theta,0) = T_2^0(x,\theta). \tag{9}$$

It is assumed that the conditions of heat exchange on the surfaces $z = \pm h$ are identical: $\alpha^+ = \alpha^- = \alpha_z$, $t_z^+ = t_z^- = 0$, the heat sources are absent, and $c_\varepsilon = \text{const}$. Hence, by using the Laplace integral transformation and double finite Fourier transformation, in view of conditions (8) and (9), we find the solution of system (4) in the form

$$T_1 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i\neq j=1}^{2} \frac{(p_i - g_4)T_{1nm}^0 + g_2 T_{2nm}^0}{p_i - p_j} e^{-p_i\tau} \sin\frac{\pi n x}{l}\cos m\theta,$$

$$T_2 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i\neq j=1}^{2} \frac{(p_i - g_1)T_{2nm}^0 + g_3 T_{1nm}^0}{p_i - p_j} e^{-p_i\tau} \sin\frac{\pi n x}{l}\cos m\theta, \tag{10}$$

where

$$p_i = \frac{g_1 + g_4}{2} + (-1)^i \sqrt{\frac{(g_1 - g_4)^2}{4} + g_2 g_3}\,,$$

$$g_1 = \beta_1\xi + \frac{\mathrm{Bi}}{\delta^2}, \qquad g_2 = \beta_2\xi - \beta_1/\delta, \qquad g_3 = 3\beta_2\xi, \qquad g_4 = 3\left(\beta_3\xi + \frac{\beta_1}{\delta^2} - \frac{\beta_2}{\delta} + \frac{\mathrm{Bi}}{\delta^2}\right),$$

$$\xi = \mu_n^2 + m^2, \qquad \mu_n = \frac{\pi n R}{l}, \qquad \mathrm{Bi} = \frac{\alpha_z h}{\lambda_m}, \qquad \delta = \frac{h}{R},$$

$$\beta_1 = \frac{\lambda_c/\lambda_m + k}{k+1}, \qquad \beta_2 = \frac{(\lambda_c/\lambda_m - 1)k}{(k+1)(k+2)}, \qquad \beta_3 = \frac{3\left(k^2+k+2\right)\lambda_c/\lambda_m + k\left(k^2+3k+8\right)}{3(k+1)(k+2)(k+3)}.$$

Here,

$$T_{inm}^0 = \frac{k_0}{\pi l}\int_0^l\int_{-\pi}^{\pi} T_i^0(x,\theta)\sin\frac{\pi n}{l}x\cos m\theta\, dx\, d\theta, \qquad k_0 = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 \end{cases} \qquad (i = 1,2). \tag{11}$$

As an example, at the initial time, we specify a plane temperature field described by the following piecewise continuous function:

$$T_1^0(x,\theta) = T^\bullet\left(1 - \frac{(x - x_0)^2}{d^2}\right)\left(1 - \frac{\theta^2}{\eta^2}\right)$$

$$\times\left[S_-(x - x_0 + d) - S_+(x - x_0 - d)\right]\left[S_-(\theta + \eta) - S_+(\theta - \eta)\right],$$

$$T_2^0(x,\theta) = 0, \tag{12}$$

where $T^\bullet = \mathrm{const}$, $2d$ and $2\eta$ are, respectively, the width and angle of the domain of heating, $(x_0, 0)$ are the coordinates of the center of this domain, and

$$S_+(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0, \end{cases} \qquad S_-(x) = \begin{cases} 1, & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad \text{are unit functions.}$$

Relations (11) and (12) now imply the following expressions for the Fourier coefficients:

$$T_{1n0}^{0} = \frac{16}{3}\frac{\eta T^{*}}{\pi^{3}n^{2}(d/l)^{2}}\left(\frac{1}{\pi n}\sin\frac{\pi nd}{l} - \frac{d}{l}\cos\frac{\pi nd}{l}\right)\sin\frac{\pi n x_0}{l}, \qquad T_{2n0}^{0} = 0,$$

$$T_{1nm}^{0} = \frac{32T^{*}}{\pi^{3}n^{2}m^{2}\eta^{2}(d/l)^{2}}\left(\frac{1}{\pi n}\sin\frac{\pi nd}{l} - \frac{d}{l}\cos\frac{\pi nd}{l}\right)\left(\frac{1}{m}\sin m\eta - \eta\cos m\eta\right)\sin\frac{\pi n x_0}{l},$$

$$T_{2nm}^{0} = 0 \qquad (m \neq 0).$$

We find the components of generalized displacements caused by the temperature field (10) with regard for the boundary conditions (7) from the system of differential equations (5) also by the method of double finite Fourier transforms. The forces and moments are determined from Eqs. (6).

## 3. Analysis of Numerical Results

The numerical computations were carried out for a shell made of a metal–ceramics composite with the following physicomechanical characteristics [5]:

– metal: $\nu = 0.3$; $E_m = 66.2\,\text{GPa}$; $\alpha_m^t = 10.3\cdot 10^{-6}\,1/\text{K}$; $\lambda_m = 18.1\,\text{W/mK}$;

– ceramics: $\nu = 0.3$; $E_c = 117\,\text{GPa}$; $\alpha_c^t = 7.11\cdot 10^{-6}\,1/\text{K}$; $\lambda_c = 2.036\,\text{W/mK}$.

The values of the other parameters are as follows: $h/R = 0.05$, $l/R = 3$, $\eta = \pi/6$, $d/l = (R/l)\sin\eta$, $x_0 = l/2$, $k' = 5/6$, and $\text{Bi} = 0.2$.

For the given parameters, we computed the values of dimensionless deflections

$$w' = \frac{u_3}{R\alpha_m T^{*}},$$

normal forces

$$N_i' = \frac{N_{ii}}{E_m h \alpha_m T^{*}},$$

and bending moments

$$M_i' = \frac{M_{ii}}{E_m h^2 \alpha_m T^{*}}$$

for the dimensionless times $\tau' = \dfrac{\lambda_m \tau}{c_\varepsilon h^2}$ equal to 0.05 and 5 and also for parameter of inhomogeneity $k = 1, 5,$ and 20.

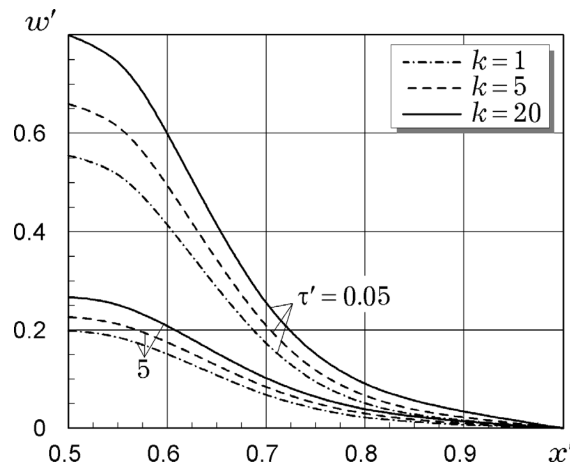


**Fig. 1.** Dependences of the deflection $w'$ on the axial coordinate $x'$.

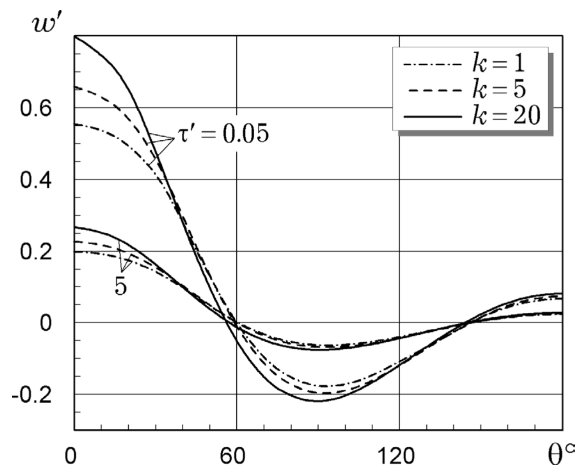


**Fig. 2.** Dependences of the deflection $w'$ on the circumferential coordinate $\theta$.

In Figs. 1 and 2, we show the variations of radial deflections $w'$ along the generatrix $\theta = 0$ and along the directrix $x' = 0.5$, respectively. We recorded the maximal deflections at the center of the domain of heating. Along the generatrix, these deflections monotonically decrease to zero on approaching the ends of the shell. At the same time, along the directrix, they oscillate between positive and negative values. Moreover, the deflections become larger as the parameter of inhomogeneity $k$ (the fraction of metal in the composite) increases.

The dependences of normal forces $N_1'$ and $N_2'$ on the axial $x'$ and circumferential $\theta$ coordinates are presented in Figs. 3–6. The normal forces acting at the center of the domain of heating are always compressive. Their variations along the directrix are oscillating, and the maximal positive values are attained on the boundary of the heated and unheated domains. The maximal values of the force $N_2'$ are higher than $N_1'$.

As shown in Figs. 7–10, the characters of changes in the bending moments $M_1'$ and $M_2'$ both along the generatrix and along the directrix are oscillating starting from the center of the heated domain. At the center of this domain, they take positive values. For the moment $M_1'$, these values are maximal. The maximal values of the moment $M_2'$ are first attained at a point (0.5; 30°) and then shift (with time) toward the center of the heated domain. In general, the maximal values of $M_1'$ are higher than $M_2'$. As the fraction of ceramics

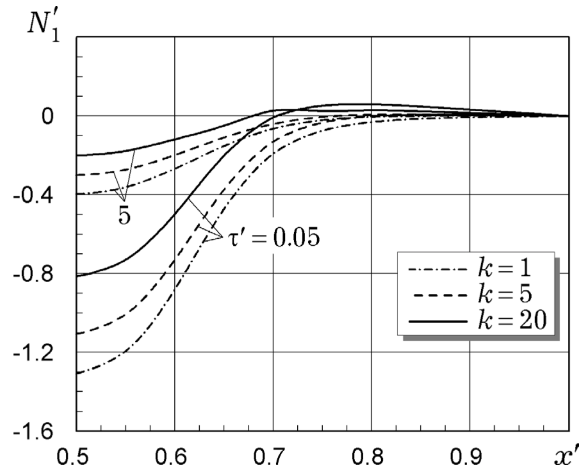


**Fig. 3.** Dependences of the normal force $N_1'$ on the axial coordinate $x'$.

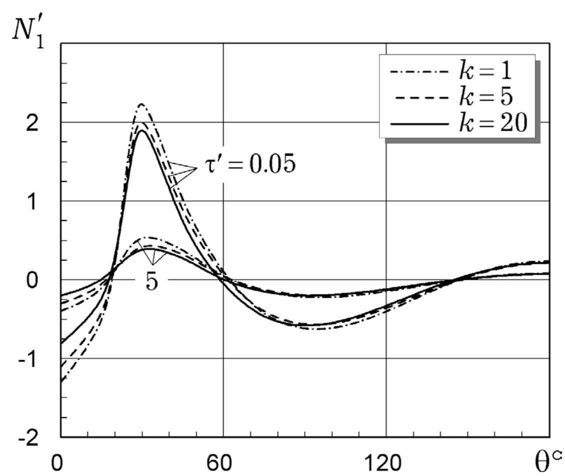


**Fig. 4.** Dependences of the normal force $N_1'$ on the circumferential coordinate $\theta$.

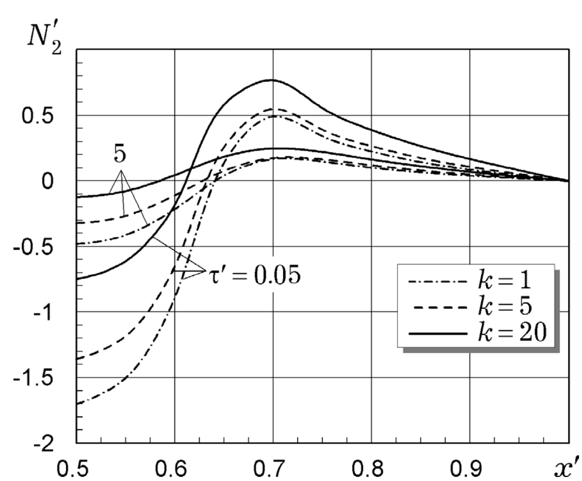


**Fig. 5.** Dependences of the normal force $N_2'$ on the axial coordinate $x'$.

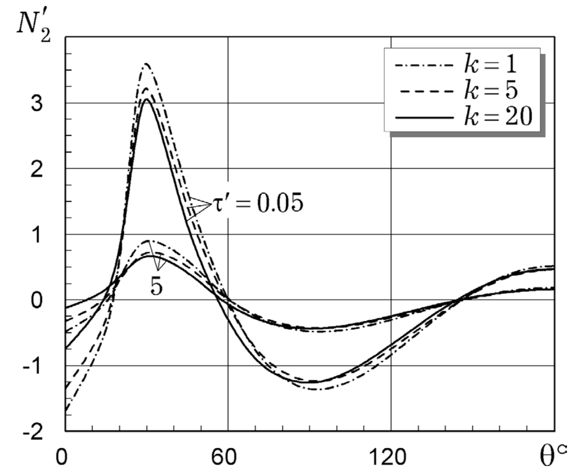


**Fig. 6.** Dependences of the normal force $N_2'$ on the circumferential coordinate $\theta$.

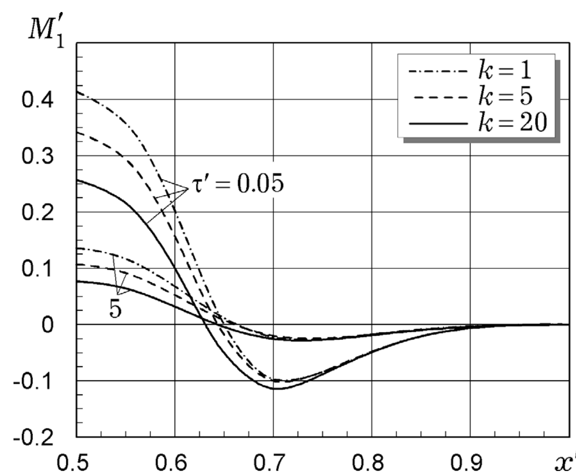


**Fig. 7.** Dependences of bending moment $M_1'$ on the axial coordinate $x'$.

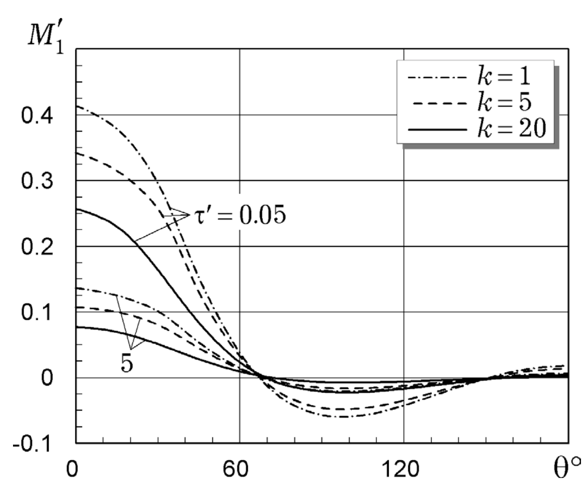


**Fig. 8.** Dependences of the bending moment $M_1'$ on the circumferential coordinate $\theta$.

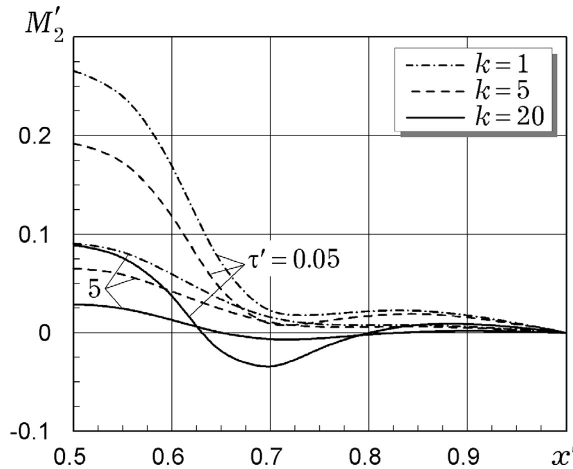


**Fig. 9.** Dependences of the bending moment $M_2'$ on the axial coordinate $x'$.

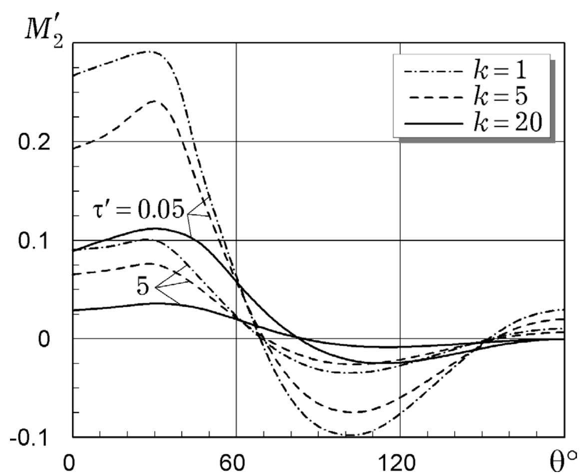


**Fig. 10.** Dependences of the bending moment $M_2'$ on the circumferential coordinate $\theta$.

in the composite increases (the parameter $k$ decreases), the forces and moments become higher because the Young modulus of the ceramics is higher than for the metal.

## CONCLUSIONS

By using the equations of the refined theory of uncoupled thermoelasticity, we analyze the stress-strain state of inhomogeneous (across the thickness) isotropic circular cylindrical shells subjected to the action of nonstationary local heating. By the methods of Fourier and Laplace transforms, we constructed, in the closed form, a solution of the quasistatic thermal-stress problem for a finite shell simply supported at its ends and subjected to heating by temperature fields specified at the initial time. We perform the numerical analysis for a metal–ceramics composite whose properties vary in the radial direction according to the power law from the ceramics on the outer surface of the shell to the metal on its inner surface. We also study the dependences of radial displacements, normal forces, and bending moments on the radial and circumferential coordinates for different times and different values of the parameter of inhomogeneity. The accumulated numerical results are presented in the graphical form.

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