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ON ENTROPY SOLUTIONS OF ANISOTROPIC ELLIPTIC EQUATIONS WITH VARIABLE NONLINEARITY INDICES IN UNBOUNDED DOMAINS

L. M. Kozhevnikova UDC 517.956.25

Abstract. For a class of second-order anisotropic elliptic equations with variable nonlinearity indices and summable right-hand sides, we consider the Dirichlet problem in arbitrary unbounded domains. We prove the existence and uniqueness of entropy solutions in anisotropic Sobolev spaces with variable exponents.

CONTENTS

Introduction

Let Ω be an arbitrary domain of the space $\mathbb{R}^n = \{\mathrm{x} = (x_1, x_2, \dots, x_n)\}$, $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$. We consider the Dirichlet problem for equations of the kind

$$\sum_{i=1}^{n} (a_i(\mathrm{x}, \nabla u))_{x_i} = |u|^{p_0(x)-2}u + a(\mathrm{x}, u), \quad \mathrm{x} \in \Omega, \tag{1}$$

with the homogeneous boundary condition

$$u\Big|_{\partial\Omega} = 0. \tag{2}$$

Nonlinear second-order elliptic equations

$$\sum_{i=1}^{n} (a_i(\mathrm{x}, u, \nabla u))_{x_i} - a_0(\mathrm{x}, u, \nabla u) = f, \tag{3}$$

where f belongs to L_1 or is a measure, have been actively investigated since the end of the previous century. For equations of the kind (3) with power nonlinearities and $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, weak solutions in the whole space $\mathbb{R}^n$ were investigated, e.g., in [7, 14, 16]. In [12, 13], the existence of weak solutions of the Dirichlet problem in a bounded domain Ω was proved for elliptic equations under the assumption that the right-hand side f belongs to $L_1(\Omega)$ or is a bounded Radon measure.

In [9], for elliptic equations with power nonlinearities and right-hand sides from L_1, the entropy solution of the Dirichlet problem was introduced and its existence and uniqueness were proved. Instead of the entropy solution introduced in [27] for first-order equations, one can consider renormalized

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solutions as well. Such solutions are elements of the same class of functions as entropy solutions; however, unlike the latter ones, they satisfy another family of integral relations. There are cases where the notion of entropy solutions is equivalent to the notion of renormalized ones.

Summability properties and estimates of entropy solutions of the Dirichlet problem for Eq. (3) with the degenerate coercivity condition in bounded domains were established by Kovalevskiy (see [22]). For second-order nonlinear elliptic equations with degenerate coercivities and right-hand sides from L_1, the existence of entropy solutions of the Dirichlet problem was proved in [2]; for more general cases, it was proved by Kovalevskiy in [23].

In [11], local entropy solutions were introduced for the following equation with the p-Laplacian, absorbtion, and Radon measure f:

$$\Delta_p u - |u|^{p_0-2}u = f, \quad p \in (1, n), \quad p < p_0. \tag{4}$$

In particular, for the case where $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, the existence of a local entropy solution of Eq. (4) in the space $\mathbb{R}^n$ is proved by Bidaut-Veron.

In [1, 10, 20], the existence and uniqueness of renormalized and entropy solutions of the Dirichlet problem were investigated in Orlich spaces for second-order elliptic equations with nonlinearities different from power functions (nonpower nonlinearities) and right-hand sides from $L_1(\Omega)$, where Ω is a bounded domain. In [24, 25], for a class of anisotropic elliptic equations with nonpower nonlinearities, theorems on existence and uniqueness of entropy solutions were proved for the Dirichlet problem in arbitrary domains.

Nowadays, differential equations and variational problems related to the $p(\mathrm{x})$-growth conditions are broadly studied. This interest is motivated by the fact that such equations can be used to simulate phenomena arising in mathematical physics. For example, they are required for the investigation of such physical fields as electro-rheological and thermo-rheological liquids (see [21]). Other important applications are related to the image processing and elasticity.

In [3, 5, 6, 8, 15, 28, 29], for equations with variable powers of the nonlinearities, existence and uniqueness theorems were proved for renormalized and entropy solutions of the Dirichlet problem in bounded domains. Papers [15, 28] are the closest to the results presented here. Namely, in [15], in a bounded subset $\Omega \subset \mathbb{R}^n$, $n \geq 3$, Bonzi and Ouaro considered the Dirichlet problem with condition (2) for the isotropic equation

$$\sum_{i=1}^{n} (a_i(\mathrm{x}, \nabla u))_{x_i} = f + a(u),$$

where $\sum\limits_{i=1}^{n} (a_i(\mathrm{x}, \nabla u))_{x_i}$ is an analog of the $p(\mathrm{x})$-Laplacian, $p : \Omega \to (1, \infty)$ is a measurable function, and $a : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function. The existence and uniqueness of an entropy solution were proved under the assumption that $f \in L_1(\Omega)$.

In [28], for the anisotropic equation

$$\sum_{i=1}^{n} (a_i(\mathrm{x}, u_{x_i}))_{x_i} = f$$

with $f \in L_1(\Omega)$, Ouaro proved the existence and uniqueness of an entropy solution of the Dirichlet problem with condition (2) for bounded subsets $\Omega \in \mathbb{R}^n$, $n \geq 3$. Rather restrictive assumptions were imposed on the Caratheodory functions $a_i(\mathrm{x}, s) : \Omega \times \mathbb{R} \to \mathbb{R}$. For example, one can take the functions $a_i(\mathrm{x}, s) = |s|^{p_i(\mathrm{x})-2}s$, $i = 1, \ldots, n$, where $p_i : \overline{\Omega} \to [2, n)$ are continuous functions such that

$$\frac{\overline{p}^-(n-1)}{n(\overline{p}^- - 1)} < p_i^- < \frac{\overline{p}^-(n-1)}{n - \overline{p}^-}, \quad \sum_{i=1}^{n} \frac{1}{p_i^-} > 1, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p}^- - n}{\overline{p}^-(n-1)},$$

$\overline{p}^- = n\left(\sum\limits_{i=1}^{n} 1/p_i^-\right)^{-1}$, $p_i^- = \inf\limits_{\mathrm{x}\in\Omega} p_i(\mathrm{x})$, and $p_i^+ = \sup\limits_{\mathrm{x}\in\Omega} p_i(\mathrm{x})$, $i = 1, \ldots, n$.

Note that a lot of papers from this research direction were devoted to the Neumann boundary condition, including nonlinear Neumann conditions (see, e.g., [18]). We do not discuss these results here.

Thus, in publications known by the author, the results are obtained for entropy and renormalized solutions of elliptic problems in bounded domains (except for [9, 11]). In the present paper, the existence and uniqueness of entropy solutions of problem (1)-(2) are proved in anisotropic Sobolev spaces with variable exponents such that no boundedness of the domain Ω is assumed and the admissible class of equations is substantially broader than the one from [15, 28] (see conditions (2.1)–(2.7) below).

1. Anisotropic Sobolev Spaces with Variable Exponents

Let $Q \subsetneq \mathbb{R}^n$ be an arbitrary domain. Introduce the notation

$$C^+(\overline{Q}) = \{p \in C(\overline{Q}) : \ 1 < p^- \leq p^+ < +\infty\},$$

where $p^- = \inf\limits_{\mathrm{x}\in Q} p(\mathrm{x})$ and $p^+ = \sup\limits_{\mathrm{x}\in Q} p(\mathrm{x})$.

If $p \in C^+(\overline{Q})$, then the following Young inequality holds:

$$|zy| \leq |y|^{p(\mathrm{x})} + |z|^{p'(\mathrm{x})}, \quad z, y \in \mathbb{R}, \quad \mathrm{x} \in Q, \quad p'(\mathrm{x}) = p(\mathrm{x})/(p(\mathrm{x}) - 1). \tag{1.5}$$

By virtue of the convexity, the following inequality holds:

$$|y + z|^{p(\mathrm{x})} \leq 2^{p^+-1}(|y|^{p(\mathrm{x})} + |z|^{p(\mathrm{x})}), \quad z, y \in \mathbb{R}, \quad \mathrm{x} \in Q. \tag{1.6}$$

Define the Lebesgue space $L_{p(\cdot)}(Q)$ with a variable exponent as the set of real-valued functions v measurable on Q and such that

$$\rho_{p(\cdot),Q}(v) = \int\limits_Q |v(\mathrm{x})|^{p(\mathrm{x})} d\mathrm{x} < \infty.$$

The Luxembourg norm in the space $L_{p(\cdot)}(Q)$ is defined by the relation

$$\|v\|_{L_{p(\cdot)}(Q)} = \|v\|_{p(\cdot),Q} = \inf \left\{ k > 0 \ \middle| \ \rho_{p(\cdot),Q}(v/k) \leq 1 \right\}.$$

In the sequel, we use the notation $\|v\|_{p(\cdot),\Omega} = \|v\|_{p(\cdot)}$ and $\rho_{p(\cdot),\Omega}(v) = \rho_{p(\cdot)}(v)$. The norm of the space $L_p(Q)$ is denoted by $\|v\|_{p,Q}$ while $\|v\|_{p,\Omega} = \|v\|_p$. The space $L_{p(\cdot)}(Q)$ is a separable reflexive Banach space (see [17]).

For each $u \in L_{p'(\cdot)}(Q)$ and each $v \in L_{p(\cdot)}(Q)$, the following Hölder inequality holds:

$$\left| \int\limits_Q u(\mathrm{x})v(\mathrm{x})d\mathrm{x} \right| \leq 2\|u\|_{p'(\cdot),Q}\|v\|_{p(\cdot),Q}. \tag{1.7}$$

Also, the following relations hold (see [17]):

$$\|v\|^{p^-}_{p(\cdot),Q} - 1 \leq \rho_{p(\cdot),Q}(v) \leq \|v\|^{p^+}_{p(\cdot),Q} + 1, \tag{1.8}$$

$$\left(\rho_{p(\cdot),Q}(v) - 1\right)^{1/p^+} \leq \|v\|_{p(\cdot),Q} \leq \left(\rho_{p(\cdot),Q}(v) + 1\right)^{1/p^-}. \tag{1.9}$$

Introduce the vector $\overrightarrow{\mathrm{p}}(\cdot) = (p_1(\cdot), p_2(\cdot), \ldots, p_n(\cdot)) \in (C^+(\overline{Q}))^n$. Introduce

$$p_+(\mathrm{x}) = \max_{i=\overline{1,n}} p_i(\mathrm{x}), \quad \mathrm{x} \in Q.$$

The anisotropic Sobolev space $\mathring{H}^1_{\overrightarrow{\mathrm{p}}(\cdot)}(Q)$ with variable exponents we define as the completion of the space $C_0^\infty(Q)$ with respect to the norm

$$\|v\|_{\mathring{H}^1_{\overrightarrow{\mathrm{p}}(\cdot)}(Q)} = \sum_{i=1}^n \|v_{x_i}\|_{p_i(\cdot),Q}.$$

The space $\mathring{H}^1_{\overrightarrow{\mathrm{p}}(\cdot)}(Q)$ is a reflexive Banach space (see [19]).

Let

$$\overline{p}(\mathrm{x}) = n\left(\sum_{i=1}^n 1/p_i(\mathrm{x})\right)^{-1}, \quad p_*(\mathrm{x}) = \begin{cases} \dfrac{n\overline{p}(\mathrm{x})}{n-\overline{p}(\mathrm{x})}, & \overline{p}(\mathrm{x}) < n, \\ +\infty, & \overline{p}(\mathrm{x}) \geq n, \end{cases} \quad \text{and} \quad p_\infty(\mathrm{x}) = \max\{p_*(\mathrm{x}), p_+(\mathrm{x})\}.$$

The following embedding theorem for the space $\mathring{H}^1_{\overrightarrow{\mathrm{p}}(\cdot)}(Q)$ is proved in [19, Theorem 2.5].

Lemma 1.1. *Let Q be a bounded domain and $\overrightarrow{\mathrm{p}}(\cdot) = (p_1(\cdot), p_2(\cdot), \dots, p_n(\cdot)) \in (C^+(\overline{Q}))^n$. If $q \in C^+(\overline{Q})$ and*

$$q(\mathrm{x}) < p_\infty(\mathrm{x}) \quad \forall\, \mathrm{x} \in Q, \tag{1.10}$$

then the continuous compact embedding $\mathring{H}^1_{\overrightarrow{\mathrm{p}}(\cdot)}(Q) \hookrightarrow L_{q(\cdot)}(Q)$ holds.

Remark 1.1. If no additional assumptions are imposed on the exponent $p(\mathrm{x})$, then it is not guaranteed that smooth functions are dense in $W^1_{p(\cdot)}(Q)$ (see [30]). However, if the modulus of continuity of the exponent $p(\mathrm{x})$ satisfies the logarithmic condition, then smooth functions are dense in the space $W^1_{p(\cdot)}(Q)$ and, defining the Sobolev space $\mathring{H}^1_{p(\cdot)}(Q)$ with a variable exponent as the completion of the space $C_0^\infty(Q)$ with respect to the norm $\|\nabla \cdot\|_{p(\cdot),Q}$, one causes no contradiction.

2. Assumptions and Results

Let $\overrightarrow{\mathbf{p}}(\cdot) = (p_0(\cdot), p_1(\cdot), \dots, p_n(\cdot)) \in (C^+(\overline{\Omega}))^{n+1}$ and

$$p_+(\mathrm{x}) \leq p_0(\mathrm{x}) < p_*(\mathrm{x}), \quad \mathrm{x} \in \Omega. \tag{2.1}$$

It is assumed that the functions $a(\mathrm{x}, s_0)$ and $a_i(\mathrm{x}, \mathrm{s})$, $i = 1, \dots, n$, contained in (1) are measurable with respect to $\mathrm{x} \in \Omega$ provided that $s_0 \in \mathbb{R}$ and $\mathrm{s} = (s_1, \dots, s_n) \in \mathbb{R}^n$ and are continuous with respect to $s_0 \in \mathbb{R}$ and $\mathrm{s} \in \mathbb{R}^n$ for almost all $\mathrm{x} \in \Omega$. It is assumed that the function $a(\mathrm{x}, s_0)$ does not decrease with respect to $s_0 \in \mathbb{R}$. Also, we assume that there exist positive numbers $\widehat{a}$ and $\overline{a}$ and nonnegative measurable functions $\Phi_i \in L_{p_i'(\cdot)}(\Omega)$, $i = 1, \dots, n$, such that the inequalities

$$|a_i(\mathrm{x}, \mathrm{s})| \leq \widehat{a}(\mathrm{P}(\mathrm{x}, \mathrm{s}))^{1/p_i'(\mathrm{x})} + \Phi_i(\mathrm{x}), \quad i = 1, \dots, n, \tag{2.2}$$

$$(\mathrm{a}(\mathrm{x}, \mathrm{s}) - \mathrm{a}(\mathrm{x}, \mathrm{t})) \cdot (\mathrm{s} - \mathrm{t}) > 0, \quad \mathrm{s} \neq \mathrm{t}, \tag{2.3}$$

$$\mathrm{a}(\mathrm{x}, \mathrm{s}) \cdot \mathrm{s} \geq \overline{a}\mathrm{P}(\mathrm{x}, \mathrm{s}) \tag{2.4}$$

hold for a. e. $\mathrm{x} \in \Omega$ and for all $\mathrm{s}, \mathrm{t} \in \mathbb{R}^n$, where $\mathrm{P}(\mathrm{x}, \mathrm{s}) = \sum\limits_{i=1}^n |s_i|^{p_i(\mathrm{x})}$, $\mathrm{s} \cdot \mathrm{t}$ denotes the scalar product of $\mathrm{s} = (s_1, \dots, s_n)$ and $\mathrm{t} = (t_1, \dots, t_n)$ from $\mathbb{R}^n$, and $\mathrm{a}(\mathrm{x}, \mathrm{s}) = (a_1(\mathrm{x}, \mathrm{s}), \dots, a_n(\mathrm{x}, \mathrm{s}))$. Also, we use the notation $\mathrm{P}'(\mathrm{x}, \mathrm{s}) = \sum\limits_{i=1}^n |s_i|^{p_i'(\mathrm{x})}$ and $\mathbf{P}(\mathrm{x}, \mathrm{s}_0, \mathrm{s}) = \mathrm{P}(\mathrm{x}, \mathrm{s}) + |s_0|^{p_0(\mathrm{x})}$.

Applying (1.6), we derive the following estimates from inequalities (2.2):

$$|a_i(\mathrm{x}, \mathrm{s})|^{p_i'(\mathrm{x})} \leq \widehat{A}\mathrm{P}(\mathrm{x}, \mathrm{s}) + \Psi_i(\mathrm{x}), \quad i = 1, \dots, n, \tag{2.2'}$$

Here Ψ_i are nonnegative measurable functions from $L_1(\Omega)$, $i = 1, \dots, n$.

The following additional conditions are used in the existence theorem. Assign $a(\mathrm{x}, s_0) = a(\mathrm{x}, 0) + b(\mathrm{x}, s_0)$ and assume that

$$a(\mathrm{x}, 0) \in L_1(\Omega) \tag{2.5}$$

and

$$\sup_{|s_0|\leq k} |b(\mathrm{x}, s_0)| = G_k(\mathrm{x}) \in L_{1,\mathrm{loc}}(\Omega). \tag{2.6}$$

The function $b(\mathrm{x}, s_0)$ is a Caratheodory nondecreasing function with respect to $s_0 \in \mathbb{R}$ and $b(\mathrm{x}, 0) = 0$ for a. e. $\mathrm{x} \in \Omega$. Therefore, the following inequality holds for a. e. $\mathrm{x} \in \Omega$ and $s_0 \in \mathbb{R}$:

$$b(\mathrm{x}, s_0)s_0 \geq 0. \tag{2.7}$$

Let $\mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega)$ denote the space $L_{p_1(\cdot)}(\Omega) \times \ldots \times L_{p_n(\cdot)}(\Omega)$ with the norm

$$\|\mathrm{v}\|_{\mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega)} = \|\mathrm{v}\|_{\overrightarrow{\mathrm{p}}(\cdot)} = \|v_1\|_{p_1(\cdot)} + \ldots + \|v_n\|_{p_n(\cdot)}, \quad \mathrm{v} = (v_1, \ldots, v_n) \in \mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega).$$

Let $\mathbf{L}_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ denote the space $L_{p_0(\cdot)}(\Omega) \times \mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega)$ with the norm

$$\|\mathbf{v}\|_{\mathbf{L}_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)} = \|v_0\|_{p_0(\cdot)} + \|\mathrm{v}\|_{\overrightarrow{\mathrm{p}}(\cdot)}, \quad \mathbf{v} = (v_0, v_1, \ldots, v_n) \in \mathbf{L}_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega).$$

Define the Sobolev space $\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ with variable exponents as the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)} = \|v\|_{p_0(\cdot)} + \|v\|_{\mathring{H}^1_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega)}.$$

Define the function

$$T_k(r) = \begin{cases} k & \text{for } r > k, \\ r & \text{for } |r| \leq k, \\ -k & \text{for } r < -k. \end{cases}$$

Introduce the notation $\langle u \rangle = \int\limits_\Omega u d\mathrm{x}$.

Definition 2.1. A measurable function $u : \Omega \to \mathbb{R}$ is called an *entropy solution* of problem (1)-(2) if:

(1) $A(\mathrm{x}) = a(\mathrm{x}, u) \in L_1(\Omega)$;
(2) $T_k(u) \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ for all positive k;
(3) the inequality

$$\langle (a(\mathrm{x}, u) + |u|^{p_0(\mathrm{x})-2}u)T_k(u - \xi)\rangle + \langle \mathrm{a}(\mathrm{x}, \nabla u) \cdot \nabla T_k(u - \xi)\rangle \leq 0 \tag{2.8}$$

holds for each positive k and each $\xi(\mathrm{x}) \in C_0^1(\Omega)$.

Theorem 2.1. *Let conditions* (2.1)–(2.4) *hold and u^1 and u^2 be entropy solutions of problem* (1)-(2). *Then $u^1 = u^2$ in* Ω.

Theorem 2.2. *Let conditions* (2.1)–(2.7) *hold. Then there exists an entropy solution of problem* (1)-(2).

3. Preliminaries

In the sequel, all constants are assumed to be positive.

Let $\chi_G(\mathrm{x})$ be a characteristic function of a set G. From the second assumption of the definition of the entropy solution, it follows that

$$\nabla T_k(u) = \chi_{\{\Omega:|u|<k\}} \nabla u \in \mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega) \tag{3.1}$$

for every positive k, whence, applying (2.2′), we prove that

$$\chi_{\{\Omega:|u|<k\}} \mathrm{a}(\mathrm{x}, \nabla u) \in \mathrm{L}_{\overrightarrow{\mathrm{p}}'(\cdot)}(\Omega) \tag{3.2}$$

for each positive k.

Lemma 3.1. *If u is an entropy solution of problem* (1)-(2), *then the inequality*

$$\int\limits_{\{\Omega:|u|<k\}} \mathbf{P}(\mathrm{x},u,\nabla u)d\mathrm{x} + k\int\limits_{\{\Omega:|u|\geq k\}} |u|^{p_0(\mathrm{x})-1}d\mathrm{x} \leq C_1k \tag{3.3}$$

holds for each positive k.

Proof. Due to inequality (2.8) and the first assumption of the definition of entropy solutions, the following inequality holds provided that $\xi = 0$:

$$\int\limits_{\Omega} |u|^{p_0(\mathrm{x})-2}uT_k(u)d\mathrm{x} + \int\limits_{\{\Omega:|u|<k\}} \mathrm{a}(\mathrm{x},\nabla u)\cdot\nabla ud\mathrm{x} \leq -\int\limits_{\Omega} a(\mathrm{x},u)T_k(u)d\mathrm{x} \leq k\|A\|_1.$$

Applying inequality (2.4), we establish the inequality

$$k\int\limits_{\{\Omega:|u|\geq k\}} |u|^{p_0(\mathrm{x})-1}d\mathrm{x} + \int\limits_{\{\Omega:|u|<k\}} |u|^{p_0(\mathrm{x})}d\mathrm{x} + \overline{a}\int\limits_{\{\Omega:|u|<k\}} \mathrm{P}(\mathrm{x},\nabla u)d\mathrm{x} \leq k\|A\|_1.$$

This yields (3.3). $\square$

Lemma 3.2. *Let a measurable function $v:\Omega\to\mathbb{R}$ be such that if $k>0$, then $T_kv \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ and*

$$\int\limits_{\{\Omega:|v|\geq k\}} |v|^{p_0(\mathrm{x})-1}d\mathrm{x} \leq C_2. \tag{3.4}$$

Then

$$\operatorname{meas}\{\Omega:|v|\geq k\}\to 0, \quad k\to\infty, \tag{3.5}$$

and

$$|v|^{p_0(\mathrm{x})-1}\in L_1(\Omega). \tag{3.6}$$

Proof. The inclusion (3.6) is an obvious corollary from (3.4). Moreover, inequality (3.4) implies that

$$k^{p_0^- -1}\operatorname{meas}\{\Omega:|v|\geq k\}\leq C_2, \quad k\geq 1,$$

which yields (3.5). $\square$

Remark 3.1. If u is an entropy solution of problem (1)-(2), then Lemmas 3.1-3.2 imply that

$$\operatorname{meas}\{\Omega:|u|\geq k\}\to 0, \quad k\to\infty, \tag{3.7}$$

and

$$|u|^{p_0(\mathrm{x})-1}\in L_1(\Omega). \tag{3.8}$$

Lemma 3.3. *Let a measurable function $v:\Omega\to\mathbb{R}$ be such that if $k>0$, then $T_kv \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ and*

$$\int\limits_{\{\Omega:|v|<k\}} \mathrm{P}(\mathrm{x},\nabla v)d\mathrm{x} + k\int\limits_{\{\Omega:|v|\geq k\}} |v|^{p_0(\mathrm{x})-1}d\mathrm{x} \leq C_3k. \tag{3.9}$$

Then

$$\operatorname{meas}\{\Omega:\mathrm{P}(\mathrm{x},\nabla v)\geq h\}\to 0, \quad h\to\infty. \tag{3.10}$$

Proof. Assign $\Phi(k,h)=\operatorname{meas}\{\Omega:|v|\geq k,\ \mathrm{P}(\mathrm{x},\nabla v)\geq h\},\ k,h>0$. It is proved above (see (3.5)) that

$$\Phi(k,0)\to 0, \quad k\to\infty.$$

Since $h\to\Phi(k,h)$ is a nonincreasing function, it follows that

$$\Phi(0,h)\leq\frac{1}{h}\int\limits_0^h\Phi(0,\varrho)d\varrho\leq\Phi(k,0)+\frac{1}{h}\int\limits_0^h(\Phi(0,\varrho)-\Phi(k,\varrho))d\varrho \tag{3.11}$$

for all positive k and h.

Note that

$$\Phi(0,\varrho)-\Phi(k,\varrho)=\operatorname{meas}\{\Omega:|v|<k,\ \mathrm{P}(\mathrm{x},\nabla v)\geq\varrho\}.$$

Therefore, (3.9) implies that

$$\int\limits_0^\infty(\Phi(0,\varrho)-\Phi(k,\varrho))d\varrho=\int\limits_{\{\Omega:|v|<k\}}\mathrm{P}(\mathrm{x},\nabla v)d\mathrm{x}\leq C_3k.$$

Now, from (3.11), we obtain the inequality

$$\Phi(0,h)\leq\Phi(k,0)+C_3k/h.$$

Select k to satisfy the inequality $\Phi(k,0)<\varepsilon$. Then select h to satisfy the inequality $\Phi(0,h)<2\varepsilon$. This proves (3.10). □

Lemma 3.4. *Let $p\in C^+(\overline{\Omega})$ and $v^m(\mathrm{x})$, $m\in\mathbb{N}$, and v be functions from $L_{p(\cdot)}(\Omega)$ such that the sequence $\{v^m\}_{m\in\mathbb{N}}$ is bounded in $L_{p(\cdot)}(\Omega)$ and*

$$v^m\to v,\quad m\to\infty,\quad a.\,e.\ in\quad\Omega.$$

Then $v^m\rightharpoonup v$ weakly in $L_{p(\cdot)}(\Omega)$ as $m\to\infty$.

For the case of bounded domains, Lemma 3.4 is proved in [4]. That proof is valid for the case of unbounded domains as well.

Lemma 3.5. *If u is an entropy solution of problem (1)-(2), then inequality (2.8) holds for every function $\xi\in\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)\cap L_\infty(\Omega)$.*

Proof. By the definition of the space $\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)\cap L_\infty(\Omega)$, there exists a sequence of functions $\xi^m\in C_0^\infty(\Omega)$ such that it is bounded in $L_\infty(\Omega)$, $\nabla\xi^m\to\nabla\xi$ in $\mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega)$ as $m\to\infty$, and $\xi^m\to\xi$ in $L_{p_0(\cdot)}(\Omega)$ as $m\to\infty$. This implies that $\xi^m\to\xi$ and $\nabla\xi^m\to\nabla\xi$ in $L_{1,\mathrm{loc}}(\Omega)$ as $m\to\infty$; hence, there exists a subsequence (we preserve the same notation for it) such that $\xi^m\to\xi$ and $\nabla\xi^m\to\nabla\xi$ a. e. in Ω. Then the following limit relations hold for every positive k:

$$T_k(u-\xi^m)\to T_k(u-\xi)\quad\text{and}\quad\nabla T_k(u-\xi^m)\to\nabla T_k(u-\xi),\quad m\to\infty,\quad\text{a. e.}\quad\text{in}\quad\Omega.\tag{3.12}$$

Let $\widehat{k}=k+\sup\limits_{m\in\mathbb{N}}(\|\xi^m\|_\infty,\|\xi\|_\infty)$. Then

$$|\nabla T_k(u-\xi^m)|\leq|\nabla T_{\widehat{k}}(u)|+|\nabla\xi^m|,\quad\mathrm{x}\in\Omega,\quad m\in\mathbb{N}.$$

Since the converging sequence $\{\nabla\xi^m\}$ is bounded in $\mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega)$, it follows from (3.1) that the norms $\|\nabla T_k(u-\xi^m)\|_{\overrightarrow{\mathrm{p}}(\cdot)}$, $m\in\mathbb{N}$, are bounded. Applying (3.12) and using Lemma 3.4, we obtain the following convergence for each positive k:

$$\nabla T_k(u-\xi^m)\rightharpoonup\nabla T_k(u-\xi),\quad m\to\infty,\quad\text{in}\quad\mathrm{L}_{\overrightarrow{\mathrm{p}}(\cdot)}(\Omega).\tag{3.13}$$

Now, let us pass to the limit as $m\to\infty$ in the inequality

$$\int\limits_\Omega(a(\mathrm{x},u)+|u|^{p_0(\mathrm{x})-2}u)T_k(u-\xi^m)d\mathrm{x}+\int\limits_\Omega\mathrm{a}(\mathrm{x},\nabla u)\cdot\nabla T_k(u-\xi^m)d\mathrm{x}\leq0.$$

Since $a(\mathrm{x},u)$ and $|u|^{p_0(\mathrm{x})-2}u$ belong to $L_1(\Omega)$ (see Definition 2.1 and (3.8)), it follows from the Lebesgue theorem that, applying (3.12) to the first term of the latter inequality, one can pass to the limit as $m\to\infty$. Since $\mathrm{a}(\mathrm{x},\nabla u)\chi_{\{\Omega:|u|<\widehat{k}\}}\in\mathrm{L}_{\overrightarrow{\mathrm{p}}'(\cdot)}(\Omega)$ (see (3.2)), it follows from (3.13) that the second term of the latter inequality has a limit as $k\to\infty$ as well. □

Remark 3.2. In the sequel, to avoid a cumbersome reasoning, instead of assertions like "from the sequence $\{u^m\}$, one can select a subsequence (we preserve the same notation for it) converging a. e. in Ω as $m \to \infty$," we say "the sequence u^m *selectively converges* a. e. in Ω as $m \to \infty$." Respectively, we use the term "*selectively weakly converges*," etc.

Lemma 3.6 (see [12, Lemma 2]). *Let* $(X, \mathcal{T}, \mathrm{meas})$ *be a measurable space such that* $\mathrm{meas}(X) < \infty$. *Let* $\gamma : X \to [0, +\infty]$ *be a measurable function such that* $\mathrm{meas}\{\mathrm{x} \in X : \gamma(\mathrm{x}) = 0\} = 0$. *Then for any positive* ε *there exists a positive* δ *such that the inequality*

$$\int\limits_Q \gamma(\mathrm{x}) d\mathrm{x} < \delta$$

implies the inequality $\mathrm{meas}\, Q < \varepsilon$.

4. Uniqueness of Solution

Consider the function $T_{k,h}(r) = T_k(r - T_h(r))$ under the assumption that k and h are positive. It is obvious that

$$T_{k,h}(r) = \begin{cases} 0 & \text{for } |r| < h, \\ r - h \operatorname{sign} r & \text{for } h \le |r| < k + h, \\ k \operatorname{sign} r & \text{for } |r| \ge k + h. \end{cases}$$

Let u be an entropy solution of problem (1)-(2). Fix positive k and h and assign $\xi = T_h(u)$ in (2.8). Then $\xi \in L_\infty(\Omega) \cap \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$. We have

$$\int\limits_\Omega |u|^{p_0(\mathrm{x})-2} u T_{k,h}(u) d\mathrm{x} + \int\limits_\Omega \mathrm{a} \cdot \nabla T_{k,h}(u) d\mathrm{x}$$

$$= k \int\limits_{\{\Omega : |u| \ge k+h\}} |u|^{p_0(\mathrm{x})-1} d\mathrm{x} + \int\limits_{\{\Omega : h \le |u| < k+h\}} |u|^{p_0(\mathrm{x})-2} u T_{k,h}(u) + \int\limits_{\{\Omega : h \le |u| < k+h\}} \mathrm{a} \cdot \nabla u d\mathrm{x}$$

$$\le -\int\limits_\Omega a(\mathrm{x}, u) T_{k,h}(u) d\mathrm{x} \le k \int\limits_{\{\Omega : h \le |u|\}} |A| d\mathrm{x}.$$

Applying (2.4), we deduce the inequality

$$k \int\limits_{\{\Omega : |u| \ge k+h\}} |u|^{p_0(\mathrm{x})-1} d\mathrm{x} + \overline{a} \int\limits_{\{\Omega : h \le |u| < k+h\}} \mathrm{P}(\mathrm{x}, \nabla u) d\mathrm{x} \le k \int\limits_{\{\Omega : h \le |u|\}} |A| d\mathrm{x}. \tag{4.1}$$

Since $A \in L_1(\Omega)$, it follows from (3.7) that the right-hand side of (4.1) tends to zero as $h \to \infty$.

Proof of Theorem 2.1. Let u^1 and u^2 be entropy solutions of problem (1)-(2). In inequality (2.8) for u^1, assign $\xi = T_h(u^2)$. In inequality (2.8) for u^2, assign $\xi = T_h(u^1),\ h > k$. Summing these integral inequalities, we see that

$$I(h, k) = \int\limits_{\Omega^1(h,k)} \mathrm{A}^1 \cdot \nabla(u^1 - T_h(u^2)) d\mathrm{x} + \int\limits_{\Omega^2(h,k)} \mathrm{A}^2 \cdot \nabla(u^2 - T_h(u^1)) d\mathrm{x} \tag{4.2}$$

$$\le -\int\limits_{\Omega(h,k)} (A^1 + |u^1|^{p_0(\mathrm{x})-2} u^1) T_k(u^1 - T_h(u^2)) d\mathrm{x} - \int\limits_{\Omega(h,k)} (A^2 + |u^2|^{p_0(\mathrm{x})-2} u^2) T_k(u^2 - T_h(u^1)) d\mathrm{x} = J(h, k),$$

where $\mathrm{A}^i(\mathrm{x}) = \mathrm{a}(\mathrm{x}, \nabla u^i),\ A^i(\mathrm{x}) = a(\mathrm{x}, u^i)$, and $\Omega^i(k, h) = \{\mathrm{x} \in \Omega :\ |u^i - T_h(u^{3-i})| < k\},\ i = 1, 2.$

Represent the sets $\Omega^1(h,k)$ and $\Omega^2(h,k)$ by unions of disjoint subsets: $\Omega^1(h,k) = \Omega^{12}(h,k) \cup \Omega^1_1(h,k) \cup \Omega^1_2(h,k)$, $\Omega^2(h,k) = \Omega^{12}(h,k) \cup \Omega^2_1(h,k) \cup \Omega^2_2(h,k)$, where

$$\Omega^{12}(h,k) = \{\mathrm{x} \in \Omega : \ |u^1 - u^2| < k, \ |u^1| < h, \ |u^2| < h\},$$

$$\Omega^i_{3-i}(h,k) = \{\mathrm{x} \in \Omega : \ |u^i - h \operatorname{sign} u^{3-i}| < k, \ |u^{3-i}| \geq h\}, \quad i = 1, 2,$$

$$\Omega^i_i(h,k) = \{\mathrm{x} \in \Omega : \ |u^i - u^{3-i}| < k, \ |u^i| \geq h, \ |u^{3-i}| < h\}, \quad i = 1, 2.$$

On the left-hand side of (4.2), the integrals of the functions $\mathrm{A}^i \cdot \nabla(u^i - T_h(u^{3-i}))$, $i = 1, 2$, over the set $\Omega^{12}(h,k)$ take the form

$$\int\limits_{\Omega^{12}(h,k)} (\mathrm{A}^1 - \mathrm{A}^2) \cdot \nabla(u^1 - u^2) d\mathrm{x} = I^{12}(h,k). \tag{4.3}$$

The integrals of the functions $\mathrm{A}^i \cdot \nabla(u^i - T_h u^{3-i})$ over the sets $\Omega^i_{3-i}(h,k)$, $i = 1, 2$, respectively, are nonnegative due to (2.4):

$$\int\limits_{\Omega^1_2(h,k)} \mathrm{A}^1 \cdot \nabla u^1 d\mathrm{x} + \int\limits_{\Omega^2_1(h,k)} \mathrm{A}^2 \cdot \nabla u^2 d\mathrm{x} \geq 0. \tag{4.4}$$

Finally, using (2.4), we obtain that

$$\begin{aligned} &\int\limits_{\Omega^1_1(h,k)} \mathrm{A}^1 \cdot \nabla(u^1 - u^2) d\mathrm{x} + \int\limits_{\Omega^2_2(h,k)} \mathrm{A}^2 \cdot \nabla(u^2 - u^1) d\mathrm{x} \\ \geq -&\int\limits_{\Omega^1_1(h,k)} \mathrm{A}^1 \cdot \nabla u^2 d\mathrm{x} - \int\limits_{\Omega^2_2(h,k)} \mathrm{A}^2 \cdot \nabla u^1 d\mathrm{x} = -I^1_1(h,k) - I^2_2(h,k). \end{aligned} \tag{4.5}$$

Combining (4.3)–(4.5), we establish the estimate

$$I(h,k) \geq I^{12}(h,k) - I^3(h,k) \quad \text{and} \quad I^3(h,k) = I^1_1(h,k) + I^2_2(h,k).$$

Let us show that $I^3(h,k) \to 0$ as $h \to \infty$. Using (1.5), we obtain the following estimate of the integral:

$$|I^1_1(h,k)| \leq \|\chi_{\{\Omega : h \leq |u^1| < h+k\}} \mathrm{P}'(\mathrm{x}, \mathrm{A}^1)\|_1 + \|\chi_{\{\Omega : h-k \leq |u^2| < h\}} \mathrm{P}(\mathrm{x}, \nabla u^2)\|_1.$$

Applying (4.1), (3.2), and (3.7), we find that $I^1_1(h,k) \to 0$ as $h \to \infty$. The integral $I^2_2(h,k)$ is estimated in the same way.

The following representation is obvious: $\Omega = \widetilde{\Omega}^{12}(h) \cup \widetilde{\Omega}^1(h) \cup \widetilde{\Omega}^2(h)$, where

$$\widetilde{\Omega}^{12}(h) = \{\mathrm{x} \in \Omega : \ |u^1| < h, \ |u^2| < h\}, \quad \widetilde{\Omega}^i(h) = \{\mathrm{x} \in \Omega : \ |u^i| \geq h\}, \quad i = 1, 2.$$

Due to the nondecreasing of the functions $a(\mathrm{x}, s_0)$ and $|s_0|^{p_0(\mathrm{x})-2} s_0$ with respect to s_0, the integrals of the functions $-(A^i + |u^i|^{p_0(\mathrm{x})-2} u^i) T_k(u^i - T_h(u^{3-i}))$, $i = 1, 2$, over the set $\widetilde{\Omega}^{12}(h)$, which are contained on the right-hand side of inequality (4.2), satisfy the relations

$$J^{12}(h) = -\int\limits_{\widetilde{\Omega}^{12}(h)} (a(\mathrm{x}, u^1) - a(\mathrm{x}, u^2) + |u^1|^{p_0(\mathrm{x})-2} u^1 - |u^2|^{p_0(\mathrm{x})-2} u^2) T_k(u^1 - u^2) d\mathrm{x} \leq 0.$$

For the integrals of these functions over the set $\widetilde{\Omega}^1(h)$, we obtain the estimate

$$|J^1(h)| \leq k \int\limits_{\widetilde{\Omega}^1(h)} (|A^1| + |A^2| + |u^1|^{p_0(\mathrm{x})-1} + |u^2|^{p_0(\mathrm{x})-1}) d\mathrm{x}. \tag{4.6}$$

A similar estimate holds for the integrals of these functions over the set $\widetilde{\Omega}_2(h)$:

$$|J^2(h)| \leq k \int\limits_{\widetilde{\Omega}^2(h)} (|A^1| + |A^2| + |u^1|^{p_0(\mathrm{x})-1} + |u^2|^{p_0(\mathrm{x})-1})d\mathrm{x}. \tag{4.7}$$

Since A^1 and A^2 belong to $L_1(\Omega)$, $|u^1|^{p_0(\mathrm{x})-1}$ and $|u^2|^{p_0(\mathrm{x})-1}$ belong to $L_1(\Omega)$, and the measure of the sets $\widetilde{\Omega}^1(h)$ and $\widetilde{\Omega}^2(h)$ tends to zero as $h \to \infty$ (see (3.7)), it follows from estimates (4.6)-(4.7) that $\lim\limits_{h\to\infty}(|J^1(h)| + |J^2(h)|) = 0$.

Thus, the limit passage in (4.2) yields the relation

$$\lim_{h\to\infty} I^{12}(h,k) = \lim_{h\to\infty} \int\limits_{\Omega^{12}(h,k)} (\mathrm{A}^1 - \mathrm{A}^2)\cdot\nabla(u^1 - u^2)d\mathrm{x} \leq 0.$$

The set $\Omega^{12}(h,k)$ converges to $\widehat{\Omega}^{12}(k) = \{\mathrm{x} \in \Omega \mid |u^1 - u^2| \leq k\}$ as $h \to \infty$. Therefore, the following inequality holds for each positive k:

$$\lim_{h\to\infty} I^{12}(h) = \int\limits_{\widehat{\Omega}^{12}(k)} (\mathrm{a}(\mathrm{x},\nabla u^1) - \mathrm{a}(\mathrm{x},\nabla u^2)\cdot\nabla(u^1 - u^2)d\mathrm{x} \leq 0.$$

This contradicts condition (2.3). Hence, $\nabla(u^1 - u^2) = 0$ a. e. in $\widehat{\Omega}^{12}(k)$ for each positive k. This implies that $u^1 = u^2$ a. e. in Ω. ☐

5. Existence of Solution

Consider the equation

$$\sum_{i=1}^{n}(a_i(\mathrm{x},\nabla u))_{x_i} - a_0(\mathrm{x},u) = 0, \quad \mathrm{x} \in \Omega. \tag{5.1}$$

Let there exist positive numbers $\widehat{a}$ and $\overline{a}$ and measurable nonnegative functions $\phi \in L_1(\Omega)$ and $\Phi_i \in L_{p_i'(\cdot)}(\Omega)$, $i = 0,1,\ldots,n$, such that the following inequalities hold for a. e. $\mathrm{x} \in \Omega$ and every $\mathbf{s} = (s_0,\mathrm{s}) \in \mathbb{R}^{n+1}$:

$$|a_0(\mathrm{x},s_0)| \leq \widehat{a}|s_0|^{p_0(\mathrm{x})-1} + \Phi_0(\mathrm{x}), \quad |a_i(\mathrm{x},\mathrm{s})| \leq \widehat{a}(\mathrm{P}(\mathrm{x},\mathrm{s}))^{1/p_i'(\mathrm{x})} + \Phi_i(\mathrm{x}), \quad i = 1,\ldots,n, \tag{5.2}$$

$$a_0(\mathrm{x},s_0)s_0 + \sum_{i=1}^{n} a_i(\mathrm{x},\mathrm{s})s_i \geq \overline{a}\mathbf{P}(\mathrm{x},s_0,\mathrm{s}) - \phi(\mathrm{x}). \tag{5.3}$$

Definition 5.1. A function $u \in \mathring{W}^1_{\vec{\mathbf{p}}(\cdot)(\Omega)}$ is called a *generalized solution* of problem (5.1), (2) if it satisfies the integral identity

$$\langle a_0(\mathrm{x},u)v\rangle + \langle \mathrm{a}(\mathrm{x},\nabla u)\cdot\nabla v\rangle = 0 \tag{5.4}$$

for any function $v \in \mathring{W}^1_{\vec{\mathbf{p}}(\cdot)}(\Omega)$.

In [26], the following assertion is proved.

Theorem 5.1. *If conditions* (2.3), (5.2), (5.3), *and* (2.1) *are satisfied, then there exists a generalized solution of problem* (5.1), (2).

The proof of Theorem 2.2 is based on Theorem 5.1.

Proof of Theorem 2.2.

Step 1. Select a sequence of functions $A^m(\mathrm{x}) \in C_0^\infty(\Omega)$ to satisfy the limit relations

$$A^m(\mathrm{x}) \to A^0(\mathrm{x}) = a(\mathrm{x},0), \quad m \to \infty, \quad \text{in} \quad L_1(\Omega) \tag{5.5}$$

and the inequalities

$$\|A^m\|_1 \le \|A^0\|_1, \quad m \in \mathbb{N}. \tag{5.6}$$

Consider the equation

$$\sum_{i=1}^{n}(a_i(\mathrm{x}, \nabla u))_{x_i} = a_0^m(\mathrm{x}, u), \quad m \in \mathbb{N}, \tag{5.7}$$

where $a_0^m(\mathrm{x}, s_0) = A^m(\mathrm{x}) + b^m(\mathrm{x}, s_0) + |s_0|^{p_0(\mathrm{x})-2}s_0$, $b^m(\mathrm{x}, s_0) = T_m(b(\mathrm{x}, s_0))\chi_{\Omega(m)}$, and $\Omega(m) = \{\mathrm{x} \in \Omega : |\mathrm{x}| < m\}$. It is obvious that

$$|b^m(\mathrm{x}, s_0)| \le |b(\mathrm{x}, s_0)|, \quad \mathrm{x} \in \Omega, \quad s_0 \in \mathbb{R}. \tag{5.8}$$

Applying (2.7), we establish the inequality

$$b^m(\mathrm{x}, s_0)s_0 \ge 0, \quad \mathrm{x} \in \Omega, \quad s_0 \in \mathbb{R}. \tag{5.9}$$

Any generalized solution of problem (5.7), (2) is a function $u^m \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ satisfying the integral identity

$$\langle (A^m(\mathrm{x}) + T_m(b(\mathrm{x}, u^m))\chi_{\Omega(m)} + |u^m|^{p_0(\mathrm{x})-2}u^m)v\rangle + \langle \mathrm{a}(\mathrm{x}, \nabla u^m) \cdot \nabla v\rangle = 0, \quad m \in \mathbb{N}, \tag{5.10}$$

for any function $v \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$.

Verify conditions (5.2)-(5.3) for the functions $\mathrm{a}(\mathrm{x}, \mathrm{s})$ and $a_0^m(\mathrm{x}, s_0)$. It is obvious that

$$|b^m(\mathrm{x}, s_0)| = |T_m(b(\mathrm{x}, s_0))|\chi_{\Omega(m)} \le m\chi_{\Omega(m)} \in L_{p_0'(\cdot)}(\Omega).$$

Therefore,

$$|a_0^m(\mathrm{x}, s_0)| \le |A^m(\mathrm{x})| + |b^m(\mathrm{x}, s_0)| + |s_0|^{p_0(\mathrm{x})-1} \le |s_0|^{p_0(\mathrm{x})-1} + \Phi_0^m(\mathrm{x}), \quad \Phi_0^m \in L_{p_0'(\cdot)}(\Omega). \tag{5.11}$$

Inequalities (5.2) follow from (2.2) and (5.11).

Further, applying (1.5) and (5.9), we conclude that

$$a_0^m(\mathrm{x}, s_0)s_0 = (A^m(\mathrm{x}) + b^m(\mathrm{x}, s_0) + |s_0|^{p_0(\mathrm{x})-2}s_0)s_0 \ge |s_0|^{p_0(\mathrm{x})} - \varepsilon|s_0|^{p_0(\mathrm{x})} - C(\varepsilon)|A^m|^{p_0'(\mathrm{x})}.$$

Then, selecting $\varepsilon < 1$, we obtain the inequality

$$a_0^m(\mathrm{x}, s_0)s_0 \ge (1-\varepsilon)|s_0|^{p_0(\mathrm{x})} - \phi_0^m(\mathrm{x}), \quad \phi_0^m(\mathrm{x}) \in L_1(\Omega). \tag{5.12}$$

Combining (2.4) and (5.12), we establish inequality (5.3).

Due to Theorem 5.1, for any $m \in \mathbb{N}$ there exists a generalized solution $u^m \in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ of problem (5.7), (2). The uniqueness of the solution of problem (5.7), (2) follows from the strong monotonicity condition posed by (2.3) and the nondecreasing of the function $a(\mathrm{x}, s_0)$ with respect to $s_0 \in \mathbb{R}$.

Step 2. Assign $v = T_{k,h}(u^m)$ in (5.10). Then, taking into account (5.9), we have the inequality

$$\begin{gathered}\int\limits_{\{\Omega:h\le|u^m|<k+h\}} \mathrm{a}(\mathrm{x}, \nabla u^m) \cdot \nabla u^m d\mathrm{x} + k \int\limits_{\{\Omega:|u^m|\ge k+h\}} \left(|b^m(\mathrm{x}, u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) d\mathrm{x} \\ + \int\limits_{\{\Omega:h\le|u^m|<k+h\}} \left(b^m(\mathrm{x}, u^m) + |u^m|^{p_0(\mathrm{x})-2}u^m\right)(u^m - h \,\mathrm{sign}\, u^m) d\mathrm{x} \le k \int\limits_{\{\Omega:|u^m|\ge h\}} |A^m| d\mathrm{x}.\end{gathered} \tag{5.13}$$

Due to (5.9), the inequality $(b^m(\mathrm{x}, u^m) + |u^m|^{p_0(\mathrm{x})-2}u^m)(u^m - h \,\mathrm{sign}\, u^m) \ge 0$ holds provided that $h \le |u^m|$. Taking this into account, from (5.13), we deduce the inequality

$$\begin{gathered}\int\limits_{\{\Omega:h\le|u^m|<k+h\}} \mathrm{a}(\mathrm{x}, \nabla u^m) \cdot \nabla u^m d\mathrm{x} \\ + k \int\limits_{\{\Omega:|u^m|\ge k+h\}} \left(|b^m(\mathrm{x}, u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) d\mathrm{x} \le k \int\limits_{\{\Omega:|u^m|\ge h\}} |A^m| d\mathrm{x}.\end{gathered}$$

Using (2.4) and (5.6), we reduce the last inequality to the form

$$\overline{a} \int\limits_{\{\Omega: h \le |u^m| < k+h\}} \mathrm{P}(\mathrm{x}, \nabla u^m) d\mathrm{x} + k \int\limits_{\{\Omega: |u^m| \ge k+h\}} \left(|b^m(\mathrm{x}, u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) d\mathrm{x} \tag{5.14}$$

$$\le k \int\limits_{\{\Omega: |u^m| \ge h\}} |A^m| d\mathrm{x} \le k\|A^0\|_1, \quad m \in \mathbb{N}.$$

Now, take $T_k(u^m)$ for the test function in (5.10). Applying (5.6), we establish the inequality

$$\int\limits_{\{\Omega: |u^m| < k\}} \mathrm{a}(\mathrm{x}, \nabla u^m) \cdot \nabla u^m d\mathrm{x} + k \int\limits_{\{\Omega: |u^m| \ge k\}} \left(|b^m(\mathrm{x}, u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) dx$$

$$+ \int\limits_{\{\Omega: |u^m| < k\}} |u^m|^{p_0(\mathrm{x})} d\mathrm{x} \le k\|A^m\|_1 \le k\|A^0\|_1.$$

Then, using inequality (2.4), we conclude that

$$\int\limits_{\{\Omega: |u^m| < k\}} \mathrm{P}(\mathrm{x}, \nabla u^m) d\mathrm{x} + k \int\limits_{\{\Omega: |u^m| \ge k\}} \left(|b^m(\mathrm{x}, u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) d\mathrm{x}$$

$$+ \int\limits_{\{\Omega: |u^m| < k\}} |u^m|^{p_0(\mathrm{x})} d\mathrm{x} \le kC_1, \quad m \in \mathbb{N}. \tag{5.15}$$

From estimate (5.15), we have

$$\int\limits_{\Omega} |T_k(u^m)|^{p_0(\mathrm{x})} d\mathrm{x} = \int\limits_{\{\Omega: |u^m| < k\}} |u^m|^{p_0(\mathrm{x})} d\mathrm{x} + \int\limits_{\{\Omega: |u^m| \ge k\}} k^{p_0(\mathrm{x})} d\mathrm{x}$$

$$\le \int\limits_{\{\Omega: |u^m| < k\}} |u^m|^{p_0(\mathrm{x})} d\mathrm{x} + k \int\limits_{\{\Omega: |u^m| \ge k\}} |u^m|^{p_0(\mathrm{x})-1} d\mathrm{x} \le kC_1, \quad m \in \mathbb{N}. \tag{5.16}$$

Also, (5.15) implies the estimate

$$\int\limits_{\{\Omega: |u^m| < k\}} \mathrm{P}(\mathrm{x}, \nabla u^m) d\mathrm{x} = \int\limits_{\Omega} \mathrm{P}(\mathrm{x}, \nabla T_k(u^m)) d\mathrm{x} \le C_1 k, \quad m \in \mathbb{N}. \tag{5.17}$$

Since the positive k is selected arbitrarily, it follows from inequality (5.15) that

$$\|b^m(\mathrm{x}, u^m)\|_1 + \||u^m|^{p_0(\mathrm{x})-1}\|_1 \le C_1, \quad m \in \mathbb{N}. \tag{5.18}$$

Finally, using (5.8) and (2.6), we establish the relation

$$\sup_{|u^m| \le k} (|b^m(\mathrm{x}, u^m)| + |u^m|^{p_0(\mathrm{x})-1}) \le \sup_{|u^m| \le k} |b(\mathrm{x}, u^m)| + k^{p_0^+ - 1} + 1 \tag{5.19}$$

$$= G_k(\mathrm{x}) + k^{p_0^+ - 1} + 1 \in L_{1,\mathrm{loc}}(\Omega), \quad m \in \mathbb{N}.$$

Step 3. Due to Lemma 3.2, from (5.15), we deduce that

$$\mathrm{meas}\,(\Omega : |u^m| \ge h) \to 0, \quad h \to \infty, \quad \text{uniformly with respect to } m \text{ from } \mathbb{N}. \tag{5.20}$$

Let us prove the limit relation

$$u^m \to u, \quad m \to \infty, \quad \text{a.e. in} \quad \Omega. \tag{5.21}$$

Let $\eta_R(r) = \min(1, \max(0, R+1-r))$. Applying (1.6), from estimate (5.17), we deduce the inequalities

$$\int\limits_{\Omega} \mathrm{P}(\mathrm{x}, \nabla(\eta_R(|\mathrm{x}|)T_k(u^m)))d\mathrm{x} \le C_2 \int\limits_{\{\Omega:|u^m|<k\}} \mathrm{P}(\mathrm{x}, \nabla u^m)d\mathrm{x} + C_2 \int\limits_{\Omega} \mathrm{P}(\mathrm{x}, T_k(u^m)\nabla\eta_R(|\mathrm{x}|))d\mathrm{x}$$
$$\le C_3(k, R), \quad m \in \mathbb{N}.$$

This implies the boundedness of the sequence $\{\eta_R(|\mathrm{x}|)T_k(u^m)\}_{m\in\mathbb{N}}$ in $\mathring{H}^1_{\vec{\mathbf{p}}(\cdot)}(\Omega(R+1))$ for every fixed positive k and R. By Lemma 1.1, the space $\mathring{H}^1_{\vec{\mathbf{p}}(\cdot)}(\Omega(R+1))$ is compactly embedded into the space $L_{p_0(\cdot)}(\Omega(R+1))$ because condition (2.1) is satisfied. Thus, for every fixed positive k and R, the selective convergence $\eta_R(|\mathrm{x}|)T_k(u^m) \to v_k$ in $L_{p_0(\cdot)}(\Omega(R+1))$ as $m \to \infty$ is established. This implies the convergence $T_k(u^m) \to v_k$ in $L_{p_0(\cdot)}(\Omega(R))$ as well as the selective convergence $T_k(u^m) \to v_k$ a. e. in $\Omega(R)$ as $m \to \infty$, $k \in \mathbb{N}$. Using the diagonal process, we show the existence of a measurable function $u : \Omega \to \mathbb{R}$ such that $v_k = T_k(u)$ and $u^m \to u$ a. e. in $\Omega(R)$ for any positive R. This implies the convergence (5.21).

From the convergence of $u^m \to u$ a. e. in $\Omega(R)$, the convergence with respect to measure follows for any positive R. Hence, it implies that the sequence $\{u^m\}$ is fundamental with respect to measure, i.e.,

$$\operatorname{meas}\{\Omega(R) : |u^m - u^l| \ge \nu\} \to 0 \quad \text{as} \quad m, l \to \infty \quad \text{for each positive } \nu. \tag{5.22}$$

Step 4. From (5.17) and (2.2′), we have the following estimate for any positive k:

$$\|\mathrm{P}'(\mathrm{x}, \mathrm{a}(\mathrm{x}, \nabla u^m))\chi_{\{\Omega:|u^m|<k\}}\|_1 \le C_4(k), \quad m \in \mathbb{N}. \tag{5.23}$$

Due to Lemma 3.3, from inequality (5.15), we have the following convergence:

$$\operatorname{meas}\{\Omega : \mathrm{P}(\mathrm{x}, \nabla u^m) \ge h\} \to 0 \quad \text{as} \quad h \to \infty \quad \text{uniformly with respect to } m \in \mathbb{N}. \tag{5.24}$$

First, we establish the convergence

$$\nabla u^m \to \nabla u, \quad m \to \infty, \quad \text{locally with repect to measure.} \tag{5.25}$$

For positive ν, θ, h, and R, consider the set

$$E_{\nu,\theta,h}(R) = \{\Omega(R) : |u^l - u^m| < \nu,\ \mathrm{P}(\mathrm{x}, \nabla u^l) \le h,\ \mathrm{P}(\mathrm{x}, \nabla u^m) \le h,\ |u^l| < h,\ |u^m| < h,$$
$$|\nabla(u^l - u^m)| \ge \theta\}.$$

Since the inclusion

$$\{\Omega(R) : |\nabla(u^l - u^m)| \ge \theta\} \subset \{\Omega : \mathrm{P}(\mathrm{x}, \nabla u^l) > h\} \cup \{\Omega : \mathrm{P}(\mathrm{x}, \nabla u^m) > h\}$$
$$\cup\{\Omega(R) : |u^l - u^m| \ge \nu\} \cup \{\Omega : |u^l| \ge h\} \cup \{\Omega : |u^m| \ge h\} \cup E_{\nu,\theta,h}(R)$$

holds, it follows from (5.20) and (5.24) that one can select h to satisfy the inequalities

$$\operatorname{meas}\{\Omega(R) : |\nabla(u^l - u^m)| \ge \theta\} \tag{5.26}$$
$$< 4\varepsilon + \operatorname{meas} E_{\nu,\theta,h}(R) + \operatorname{meas}\{\Omega(R) : |u^l - u^m| \ge \nu\}, \quad m, l \in \mathbb{N}.$$

By the monotonicity condition (2.3) and the well-known fact that a function continuous on a compact set attains its infimum, there exists $\gamma(\mathrm{x})$ positive a. e. in Ω and such that the inequality

$$(\mathrm{a}(\mathrm{x}, \mathrm{s}) - \mathrm{a}(\mathrm{x}, \mathrm{t})) \cdot (\mathrm{s} - \mathrm{t}) \ge \gamma(\mathrm{x}) \tag{5.27}$$

holds provided that $\mathrm{P}(\mathrm{x}, \mathrm{s}) \le h$, $\mathrm{P}(\mathrm{x}, \mathrm{t}) \le h$, and $|\mathrm{s} - \mathrm{t}| \ge \theta$.

Introduce the notation $A_0^m(\mathrm{x}) = a_0^m(\mathrm{x}, u^m) = A^m(\mathrm{x}) + b^m(\mathrm{x}, u^m) + |u^m|^{p_0(\mathrm{x})-2}u^m$. From (5.6) and (5.18), it follows that the sequence $\{A_0^m\}_{m\in\mathbb{N}}$ is bounded in $L_1(\Omega)$. Substitute u^m and u^l into (5.10) and subtract the latter from the former. We obtain that

$$\int\limits_{\Omega} \Big(\mathrm{a}(\mathrm{x}, \nabla u^m) - \mathrm{a}(\mathrm{x}, \nabla u^l)\Big) \cdot \nabla v d\mathrm{x} + \int\limits_{\Omega} (A_0^m - A_0^l)v d\mathrm{x} = 0.$$

Substituting the test function $v = \eta_R(|\mathrm{x}|)\eta_h(|u^l|)\eta_h(|u^m|)T_\nu(u^m - u^l)$, we establish the relation

$$\int\limits_\Omega \Big(\mathrm{a}(\mathrm{x}, \nabla u^m) - \mathrm{a}(\mathrm{x}, \nabla u^l)\Big) \cdot \nabla(\eta_R(|\mathrm{x}|)\eta_h(|u^l|)\eta_h(|u^m|)T_\nu(u^m - u^l))d\mathrm{x}$$

$$= -\int\limits_\Omega (A_0^m - A_0^l)\eta_R(|\mathrm{x}|)\eta_h(|u^l|)\eta_h(|u^m|)T_\nu(u^m - u^l)d\mathrm{x} \le C_5\nu, \quad m, l \in \mathbb{N}. \tag{5.28}$$

Further, applying (5.27), we deduce the inequality

$$\int\limits_{E_{\nu,\theta,h}(R)} \gamma(\mathrm{x})d\mathrm{x} \le \int\limits_{E_{\nu,\theta,h}(R)} \Big(\mathrm{a}(\mathrm{x}, \nabla u^m) - \mathrm{a}(\mathrm{x}, \nabla u^l)\Big) \cdot \nabla(u^m - u^l)d\mathrm{x}$$

$$\le \int\limits_{\{\Omega:|u^m - u^l|<\nu\}} \eta_R(|\mathrm{x}|)\eta_h(|u^l|)\eta_h(|u^m|)(\mathrm{a}(\mathrm{x}, \nabla u^m) - \mathrm{a}(\mathrm{x}, \nabla u^l))\nabla(u^m - u^l)d\mathrm{x}. \tag{5.29}$$

Combining (5.28) and (5.29) and applying (1.5), (5.17), and (5.23), we see that

$$\int\limits_{E_{\nu,\theta,h}(R)} \gamma(\mathrm{x})d\mathrm{x} \le \sum_{i=1}^n \int\limits_{\{\Omega:|u^m|<h+1,|\mathrm{x}|<R+1\}} |a_i(\mathrm{x}, \nabla u^m)||T_\nu(u^m - u^l)|d\mathrm{x}$$

$$+ \sum_{i=1}^n \int\limits_{\{\Omega:|u^l|<h+1,|\mathrm{x}|<R+1\}} |a_i(\mathrm{x}, \nabla u^l)||T_\nu(u^m - u^l)|d\mathrm{x}$$

$$+ \sum_{i=1}^n \int\limits_{\{\Omega:h<|u^l|<h+1,|u^m|<h+1\}} (|a_i(\mathrm{x}, \nabla u^m)| + |a_i(\mathrm{x}, \nabla u^l)|)|u^l_{x_i}||T_\nu(u^m - u^l)|d\mathrm{x}$$

$$+ \sum_{i=1}^n \int\limits_{\{\Omega:h<|u^m|<h+1,|u^l|<h+1\}} (|a_i(\mathrm{x}, \nabla u^m)| + |a_i(\mathrm{x}, \nabla u^l)|)|u^m_{x_i}||T_\nu(u^m - u^l)|d\mathrm{x} + C_5\nu \tag{5.30}$$

$$\le \nu(3\|\mathrm{P}'(\mathrm{x}, \mathrm{a}(\mathrm{x}, \nabla u^m))\chi_{\{\Omega:|u^m|<h+1\}}\|_1 + 3\|\mathrm{P}'(\mathrm{x}, \mathrm{a}(\mathrm{x}, \nabla u^l))\chi_{\{\Omega:|u^l|<h+1\}}\|_1$$

$$+ 2\|\mathrm{P}(\mathrm{x}, \nabla u^m)\chi_{\{\Omega:|u^m|<h+1\}}\|_1 + 2\|\mathrm{P}(\mathrm{x}, \nabla u^l)\chi_{\{\Omega:|u^l|<h+1\}}\|_1 + C_6(R)) \le C_7(R, h)\nu.$$

Take an arbitrary positive δ and fix R and h. Then, selecting ν from (5.30), we establish the inequality

$$\int\limits_{E_{\nu,\theta,h}(R)} \gamma(\mathrm{x})d\mathrm{x} < \delta.$$

Applying Lemma 3.6, for any positive ε, we deduce the inequality

$$\operatorname{meas} E_{\nu,\theta,h}(R) < \varepsilon. \tag{5.31}$$

Due to (5.22), one can select $m_0(\nu, R, \varepsilon)$ to satisfy the inequality

$$\operatorname{meas}\{\Omega(R) : |u^l - u^m| \ge \nu\} < \varepsilon, \quad m, l \ge m_0. \tag{5.32}$$

Thus, combining (5.26), (5.31), and (5.32), we deduce the following inequality for any positive θ:

$$\operatorname{meas}\{\Omega(R) : |\nabla(u^l - u^m)| \ge \theta\} < 6\varepsilon, \quad m, l \ge m_0.$$

This implies that the sequence $\{\nabla u^m\}_{m\in\mathbb{N}}$ is fundamental with respect to measure on the set $\Omega(R)$ for any positive R. This implies convergence (5.25) as well as the selective convergence

$$\nabla u^m \to \nabla u, \quad m \to \infty, \quad \text{a.e. in} \quad \Omega. \tag{5.33}$$

Step 5. Let us prove that

$$|u^m|^{p_0(\mathrm{x})-2}u^m \to |u|^{p_0(\mathrm{x})-2}u \quad \text{and} \quad b^m(\mathrm{x},u^m) \to b(\mathrm{x},u), \quad m\to\infty, \quad \text{in} \quad L_{1,\mathrm{loc}}(\Omega) \tag{5.34}$$

and

$$|u^m|^{p_0(\mathrm{x})-2}u^m \to |u|^{p_0(\mathrm{x})-2}u \quad \text{and} \quad b^m(\mathrm{x},u^m) \to b(\mathrm{x},u), \quad m\to\infty, \quad \text{a.e. in} \quad \Omega. \tag{5.35}$$

Assigning $k=h$ in (5.14), we obtain the inequality

$$\int\limits_{\{\Omega:|u^m|\geq 2h\}} \left(|b^m(\mathrm{x},u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) d\mathrm{x} \leq \int\limits_{\{\Omega:|u^m|\geq h\}} |A^m - A^0| d\mathrm{x} + \int\limits_{\{\Omega:|u^m|\geq h\}} |A^0| d\mathrm{x}, \quad m\in\mathbb{N}.$$

Fix an arbitrary positive ε. Using the inclusion $A^0\in L_1(\Omega)$, convergence (5.5), and the absolute convergence of the integrals on the right-hand side of the latter inequality, we take into account (5.20) to find sufficiently large h such that

$$\int\limits_{\{\Omega:|u^m|\geq 2h\}} \left(|b^m(\mathrm{x},u^m)| + |u^m|^{p_0(\mathrm{x})-1}\right) d\mathrm{x} < \varepsilon, \quad m\in\mathbb{N}. \tag{5.36}$$

From the continuity of $b(\mathrm{x},s_0)$ with respect to s_0 and convergence (5.21), it follows that for any fixed h, we have the limit relations

$$\chi_{\{\Omega:|u^m|<2h\}}|u^m|^{p_0(\mathrm{x})-2}u^m \to \chi_{\{\Omega:|u|\leq 2h\}}|u|^{p_0(\mathrm{x})-2}u, \quad m\to\infty, \quad \text{a.e. in} \quad \Omega,$$

$$\chi_{\{\Omega:|u^m|<2h\}}b^m(\mathrm{x},u^m) \to \chi_{\{\Omega:|u|\leq 2h\}}b(\mathrm{x},u), \quad m\to\infty, \quad \text{a.e. in} \quad \Omega.$$

Let K be an arbitrary compact subset of Ω. According to (5.19), applying the Lebesgue theorem, we prove that

$$\chi_{\{\Omega:|u^m|<2k\}}|u^m|^{p_0(\mathrm{x})-2}u^m \to \chi_{\{\Omega:|u|\leq 2k\}}|u|^{p_0(\mathrm{x})-2}u, \quad m\to\infty, \quad \text{in} \quad L_1(K),$$

$$\chi_{\{\Omega:|u^m|<2k\}}b^m(\mathrm{x},u^m) \to \chi_{\{\Omega:|u|\leq 2k\}}b(\mathrm{x},u), \quad m\to\infty, \quad \text{in} \quad L_1(K).$$

Taking into account (5.36), this implies (5.34).

From (5.18), (5.35), and the Fatou theorem, we conclude that $b(\mathrm{x},u)$ and $|u|^{p_0(\mathrm{x})-2}u$ belong to $L_1(\Omega)$. Hence, (2.5) yields the validity of the first assumption of Definition 2.1.

Step 6. Let us prove that $T_k(u)\in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ for any positive k. Combining (5.16), (5.17), and (1.9), for any fixed positive k, we deduce the estimate

$$\|T_k u^m\|_{\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)} = \sum_{i=0}^{n} \|D_{x_i}T_k(u^m)\|_{p_i(\cdot)} \leq \sum_{i=0}^{n}\left(1+\int\limits_{\Omega} |D_{x_i}T_k(u^m))_{x_i}|^{p_i(\mathrm{x})}d\mathrm{x}\right)^{1/p_i^-} \leq C_8(k), \quad m\in\mathbb{N}.$$

The reflexivity of the space $\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ allows one to select a subsequence such that $T_k u^m \rightharpoonup v,\ m\to\infty$, in $\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ and $v\in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$. The continuity of the natural mapping $\mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)\to \mathbf{L}_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$ implies the weak convergence

$$T_k(u^m)\rightharpoonup v, \quad m\to\infty, \quad \text{in} \quad L_{p_0(\cdot)}(\Omega). \tag{5.37}$$

Using convergence (5.21) and applying Lemma 3.4, we have the following weak convergence for any fixed positive k:

$$T_k(u^m)\rightharpoonup T_k(u), \quad m\to\infty, \quad \text{in} \quad L_{p_0(\cdot)}(\Omega). \tag{5.38}$$

Relations (5.37)-(5.38) imply that $v=T_k u\in \mathring{W}^1_{\overrightarrow{\mathbf{p}}(\cdot)}(\Omega)$.

Step 7. To prove (2.8), substitute the test function $v = T_k(u^m - \xi),\ \xi \in C_0^1(\Omega)$, into identity (5.10). We obtain the relation

$$\int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla u^m) \cdot \nabla T_k(u^m - \xi) d\mathrm{x} + \int\limits_{\Omega} \Big(b^m(\mathrm{x}, u^m) + |u^m|^{p_0(\mathrm{x})-2} u^m + A^m \Big) T_k(u^m - \xi) d\mathrm{x} = I^m + J^m = 0.$$

Assign $M = k + \|\xi\|_\infty$. If $|u^m| \ge M$, then $|u^m - \xi| \ge |u^m| - \|\xi\|_\infty \ge k$. Therefore, $\{\Omega : |u^m - \xi| < k\} \subseteq \{\Omega : |u^m| < M\}$, which means that

$$I^m = \int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla u^m) \cdot \nabla T_k(u^m - \xi) d\mathrm{x} \tag{5.39}$$

$$= \int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla T_M(u^m)) \cdot (\nabla T_M(u^m) - \nabla \xi) \chi_{\{\Omega : |u^m - \xi| < k\}} d\mathrm{x} = I_1^m - I_2^m.$$

From (5.21), (5.33), and the continuity of the function $\mathrm{a}(\mathrm{x}, \mathrm{s})$ with respect to s, we have the limit relation

$$\mathrm{a}(\mathrm{x}, \nabla T_M(u^m)) \cdot \nabla T_M(u^m) \chi_{\{\Omega : |u^m - \xi| < k\}} \to \mathrm{a}(\mathrm{x}, \nabla T_M(u)) \cdot \nabla T_M(u) \chi_{\{\Omega : |u - \xi| \le k\}}, \quad m \to \infty, \quad \text{a.e. in } \Omega.$$

Applying (5.17), (5.23), and (1.5), we establish the estimate

$$I_1^m = \int\limits_{\{\Omega :\, |u^m - \xi| < k\}} \mathrm{a}(\mathrm{x}, \nabla T_M(u^m)) \cdot \nabla T_M(u^m) d\mathrm{x} \le C_9(k), \quad m \in \mathbb{N}.$$

Then the Fatou lemma yields the inequality

$$\int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla T_M(u)) \cdot \nabla T_M(u) \chi_{\{\Omega : |u - \xi| \le k\}} d\mathrm{x} \le \lim_{m \to \infty} \inf I_1^m. \tag{5.40}$$

Estimate (5.23) implies the boundedness of the sequence of norms:

$$\|\mathrm{P}'(\mathrm{x}, \mathrm{a}(\mathrm{x}, \nabla T_M(u^m)) \chi_{\{\Omega : |u^m - \xi| < k\}}\|_1 \le \|\mathrm{P}'(\mathrm{x}, \mathrm{a}(\mathrm{x}, \nabla u^m)) \chi_{\{\Omega : |u^m| < M\}}\|_1 \le C_{10}(k), \quad m \in \mathbb{N}.$$

Applying Lemma 3.4, we establish the following weak convergence:

$$\mathrm{a}(\mathrm{x}, \nabla T_M(u^m)) \chi_{\{\Omega : |u^m - \xi| < k\}} \rightharpoonup \mathrm{a}(\mathrm{x}, \nabla T_M(u)) \chi_{\{\Omega : |u - \xi| \le k\}}, \quad m \to \infty, \quad \text{in} \quad \mathrm{L}_{\vec{\mathrm{p}}'(\cdot)}(\Omega).$$

Passing to the limit in I_2^m, we have the relation

$$\lim_{m \to \infty} I_2^m = \int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla T_M(u)) \cdot \nabla \xi \chi_{\{\Omega : |u - \xi| \le k\}} d\mathrm{x}. \tag{5.41}$$

Combining (5.39)–(5.41), we establish the relation

$$\lim_{m \to \infty} \inf I^m \ge \int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla T_M(u)) \cdot (\nabla T_M(u) - \nabla \xi) \chi_{\{\Omega : |u - \xi| \le k\}} d\mathrm{x}$$

$$= \int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla u) \cdot \nabla (u - \xi) \chi_{\{\Omega : |u - \xi| \le k\}} d\mathrm{x} = \int\limits_{\Omega} \mathrm{a}(\mathrm{x}, \nabla u) \cdot \nabla T_k(u - \xi) d\mathrm{x}. \tag{5.42}$$

Since

$$T_k(u^m - \xi) \to T_k(u - \xi), \quad m \to \infty, \quad \text{a.e. in} \quad \Omega$$

and

$$|v T_k(u^m - \xi)| \le k|v| \in L_1(\Omega), \quad \forall\, v \in L_1(\Omega), \quad m \in \mathbb{N},$$

it follows from the Lebesgue theorem that

$$T_k(u^m - \xi) \overset{*}{\rightharpoonup} T_k(u - \xi), \quad m \to \infty, \quad \text{in} \quad L_\infty(\Omega). \tag{5.43}$$

The integral J^m is decomposed into two terms as well. The first integral

$$J_1^m = \int\limits_{\Omega} \left(b^m(\mathrm{x}, u^m) + |u^m|^{p_0(\mathrm{x})-2} u^m \right) T_k(u^m - \xi) d\mathrm{x}$$

is estimated as follows. Consider the increasing sequence $\{K^l\}$ of compact subsets Ω such that $\bigcup\limits_{l=1}^{\infty} K^l = \Omega$. Let $\operatorname{supp} \xi \subset K^l$, $l \geq l_0$, $v^m = u^m - \xi$, $v = u - \xi$, $c^m(\mathrm{x}, u^m) = b^m(\mathrm{x}, u^m) + |u^m|^{p_0(\mathrm{x})-2} u^m$, and $c(\mathrm{x}, u) = b(\mathrm{x}, u) + |u|^{p_0(\mathrm{x})-2} u$. Then, taking into account (5.9), we have the following relation valid under the assumption that $l \geq l_0$:

$$J_1^m = \int\limits_{\Omega \setminus K^l} c^m(\mathrm{x}, u^m) T_k(u^m) d\mathrm{x} + \int\limits_{K^l} c^m(\mathrm{x}, u^m) T_k(v^m) d\mathrm{x} \geq \int\limits_{K^l} c^m(\mathrm{x}, u^m) T_k(v^m) d\mathrm{x} = \overline{J}_1^{lm}.$$

Applying (5.34) and (5.43), pass to the limit as $m \to \infty$. Then pass to the limit as $l \to \infty$. We obtain that

$$\int\limits_{\Omega} (b(\mathrm{x}, u) + |u|^{p_0(\mathrm{x})-2} u) T_k(u - \xi) d\mathrm{x} = \lim_{l \to \infty} \lim_{m \to \infty} \overline{J}_1^{lm} \leq \lim_{m \to \infty} \inf J_1^m. \tag{5.44}$$

Using (5.5) and (5.43), we pass to the limit as $m \to \infty$ in the second integral and obtain that

$$J_2^m = \int\limits_{\Omega} A^m T_k(u^m - \xi) d\mathrm{x} \to \int\limits_{\Omega} A^0 T_k(u - \xi) d\mathrm{x}. \tag{5.45}$$

Combining (5.42) and (5.44)-(5.45), we deduce (2.8).

□

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L. M. Kozhevnikova
Sterlitamak Branch of Bashkir State University, Sterlitamak, Russia;
Elabuga Branch of Kazan Federal University, Elabuga, Russia
E-mail: kosul@mail.ru