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DYNAMICAL SYSTEMS AND TOPOLOGY OF MAGNETIC FIELDS IN CONDUCTING MEDIUM

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Abstract. We discuss application of contemporary methods of the theory of dynamical systems with regular and chaotic hyperbolic dynamics to investigation of topological structure of magnetic fields in conducting media. For substantial classes of magnetic fields, we consider well-known physical models allowing us to reduce investigation of such fields to study of vector fields and Morse–Smale diffeomorphisms as well as diffeomorphisms with nontrivial basic sets satisfying the A axiom introduced by Smale. For the point–charge magnetic field model, we consider the problem of the separator playing an important role in the reconnection processes and investigate relations between its singularities. We consider the class of magnetic fields in the solar corona and solve the problem of topological equivalency of fields in this class. We develop a topological modification of the Zeldovich funicular model of the nondissipative cinematic dynamo, constructing a hyperbolic diffeomorphism with chaotic dynamics that is conservative in the neighborhood of its transitive invariant set.

CONTENTS

1. Introduction

The topological structure of magnetic fields in conducting moving media is one of the most important problems of natural science. The main and most actual example of a strongly conducting moving medium is the plasma. The investigation of magnetic fields in conducting media forms a part of physics called "magnetohydrodynamics" (for its main definitions and notions, see [3, 14, 32, 59]). Its theory is based on the classical magnetic field equations and the hydrodynamic motion equations for the continuous medium (these equations are provided in Sec. 2). Mathematical physics has many famous theoretical methods for the investigation of this system of equations. In several recent papers, properties of solutions are studied by means of methods of the geometrical (qualitative) theory of dynamical systems based on classical works of Poincaré and Lyapunov. The present review is devoted to applications of methods of the geometrical theory of dynamical systems to the investigation of the topological structure of the magnetic field of a conducting medium.

Any motion of a well-conducting medium specifically effects its electromagnetic field: the electric field arising in it is rather rapidly graded by the arising currents. Therefore, the main point in the investigation of properties of well-conducting media is the investigation of the interaction between the medium and its magnetic field. The Alfven theorem on the "freezing" of magnetic lines of force into the moving ideally conducting medium (see [1, 3]) is quite important for theoretical investigations

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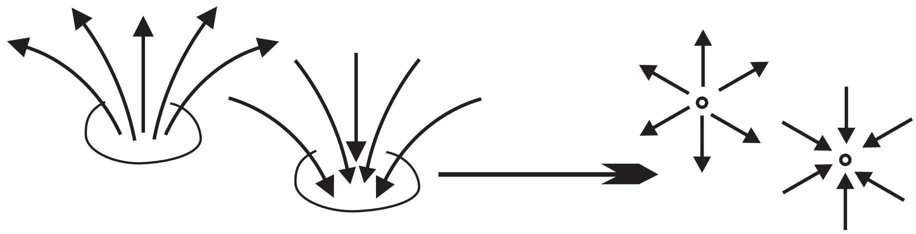

Fig. 1. The idealization of a magnetic charge.

and applications. According to this theorem, the movement of lines of force of the magnetic field is the same as in the case where they are "frozen" into the conducting medium (plasma). Hence, if the movements of an ideally conducting medium are simple and small with respect to time, then the topological structure of the magnetic field is not changed. After a sufficiently long interval of time, the freezing-in of lines of force may give rise to regions with conflicting parts of the magnetic field, i.e., points such that the boundaries of these regions are close to each other, but the magnetic fields of these regions have different directions near these points. This leads to the arising of new null points, i.e., points where the magnetic induction $\vec{\boldsymbol{B}}$ is equal to zero.

Moreover, in the photosphere and solar corona, which are examples of well-conducting media, local regions with an intensive magnetic field regularly arise so that they look like local sources or sinks of vector fields (from the global viewpoint). An idealized model with point positive and negative magnetic charges, which are retractions to points of such regions, occurs to be applied for the qualitative study of the topology of such a magnetic field.

No magnetic charges are found in nature, but the above *point–charge model* is successfully applied for the investigation of the structure of the magnetic field and its bifurcations for solar bursts with small numbers of retraction charges (see, e.g., [6–8, 11, 13, 19, 34–37, 43, 47, 51]). From the viewpoint of the theory of dynamical systems, the specified idealized model can be treated as a Morse–Smale vector field such that its sink and source equilibrium states correspond to idealized (retraction) charges, saddle equilibrium states correspond to null points of the magnetic field, one-dimensional separatrices of each saddle equilibrium state correspond to oppositely directed lines of force of the magnetic field, and its two-dimensional separatrix belongs to the boundary of the conflicting parts of the magnetic field.

The class of such dynamical systems (Morse–Smale vector fields and flows) is among the most studied classes in the contemporary theory of dynamical systems (see, e.g., [4, 21, 23, 55]). The appearance of such classes is related to the notion of coarse systems, which is very important for the theory of dynamical systems: in 1937, this notion is introduced by Andronov and Pontrjagin for two-dimensional surface vector fields in the plane. Coarse vector fields preserve the phase portrait under sufficiently small C^1-perturbations and are such that the homeomorphism transforming trajectories of the original vector field into trajectories of the perturbed field is close to the identical one. In 1959, Peixoto extended the coarseness notion to flows defined on closed surfaces and replaced the coarseness notion by the notion of the structural stability, omitting the requirement for the homeomorphism taking trajectories of the C^1-close vector field to trajectories of the original field to be close to the identical one. Similarly to the case where a vector field is defined in a bounded part of the plane, vector fields on surfaces form an open dense set in the space of all vector fields equipped with C^1-topology.

In 1960, Smale generalized the two-dimensional case by introducing the class of vector fields on manifolds of dimension exceeding 2 and such that their nonwandering set consists of a finite number of hyperbolic equilibrium states and closed trajectories such that the intersection of their stable and unstable manifolds is transversal. It turns out that such vector fields are structurally stable. They are called *Morse–Smale vector fields*. Note that these fields do not form an everywhere dense set in the set of all fields and do not coincide with the set of all structurally stable fields. Unlike structurally stable flows with a countable set of saddle periodic trajectories simulating processes with a chaotic behavior, Morse–Smale vector fields describe processes with regular dynamics.

The geometrical theory of Morse–Smale dynamical systems allows the authors of the present review to obtain new quantitative relations between the number of charges of different signs and the number of null points of the magnetic field. These relations are provided in Sec. 3.

Null points of the three-dimensional magnetic field are divided into two types as follows. From the viewpoint of the theory of Morse–Smale dynamical systems, a fixed null point is a saddle hyperbolic equilibrium state (a saddle) such that one of its separatrix is one-dimensional, while the other one is two-dimensional. We say that a null point is of *type one* (*two*) if the dimension of the unstable separatrix is equal to 1 (respectively, 2). In the theory of dynamical systems, this dimension is called the *Morse index* of the saddle equilibrium state.

A structurally stable three-dimensional vector field might admit intersections of two-dimensional stable and unstable separatrices of different saddle equilibrium states. Any trajectory belonging to such an intersection is called a *heteroclinic trajectory*. Such trajectories correspond to lines of force, connecting two null points of a magnetic field. In astrophysics, such magnetic lines are called *separators*. It is clear that separators are important for the topological structure of magnetic fields. At the moment, a lot is known about the existence or nonexistence of heteroclinic trajectories of Morse–Smale dynamical systems; this existence/nonexistence problem can be reduced to the existence/nonexistence problem for separators in magnetic fields.

In contemporary astrophysics, the nature of solar bursts is explained as follows: the energy liberation taking place at solar bursts is a result of the *magnetic reconnection*, i.e., bifurcations related to the arising and vanishing of separators of the magnetic field. This changes the topological structure of the magnetic field so that the new topological configuration possesses a lesser energy (see [48, 49, 53]). The energy excess liberated due to this bifurcation is spent for the intensive radiation of electromagnetic waves in various ranges of the spectrum, heating of the plasma, acceleration of charged particles to high values of the energy, etc. Thus, information on the amount and location of null points and separators of the magnetic field is quite important for the analysis of magnetic-reconnection processes. This is discussed in Secs. 3-4. Also, in Sec. 4, we provide necessary and sufficient conditions for the topological equivalence of a special class of magnetic fields in the solar corona.

In Secs. 3-4, the topological structure of magnetic fields at a fixed time is considered. Such a consideration does not take into account the influence of the motion of the conducting medium (plasma) on the change of the topology of the magnetic field. However, to explain the nature of important bifurcations of the topology of magnetic fields related to the birth and vanishing of separators, reconnections, etc., one has to take the motion of the medium into account. In Sec. 5, we propose the following approach. In the moving plasma, we select the investigated region V. Then we define the movement of this region to the three-dimensional closed manifold M_V in order that the obtained transformation of the manifold M_V be a Morse–Smale diffeomorphism. Though the information about the behavior of the magnetic field outside the region V is lost under such an approach, we succeed in obtaining conditions guaranteeing the existence (or absence) of separators of the original magnetic field in the region V. Actually, the obtained conditions follow from profound results on Morse–Smale diffeomorphism defined on closed three-dimensional manifolds (see [9, 21, 23, 27, 28]).

It is known that Earth and Sun both possess a self-magnetic field. Magnetic fields of other planets and stars are found by radioastronomy methods. It turns out that magnetic fields in planets, stars, galaxies, and intergalactic space are frequently primary factors of the dynamics of various astrophysical processes. The natural question about the genesis of these magnetic fields is within the theory called the *kinematic dynamo* theory (see [60]). The behavior of the magnetic field under a given flow of the conducting medium (see [32, 38, 39]) is an important problem of the said theory.

Various aspects of the kinematic dynamo theory are considered in [16, 17, 44] (see [12, 56] as well). These theoretical investigations face substantial mathematical difficulties because the problem is essentially nonlinear. An important part of the kinematic dynamo theory is the theory of fast kinematic dynamo investigating the existence of motion of the medium, causing the exponential growth of the so-called seed magnetic field (or magnetic energy) under a small magnetic diffusion (see [5, 40, 61]). It is generally agreed that the effect of fast kinematic dynamo causes the existence of magnetic fields in the cosmic scale. Up to now, no stable experiment providing an expected effect that could be treated as an exact analog of the natural one is obtained in the laboratory environment (see [57]).

The great theoretical difficulties of the fast kinematic dynamo problem motivate the development of various geometric and topological constructions of movements of the conducting medium, leading to multiple growth of the seed magnetic field. Roughly speaking, the main idea is to construct a conservative map that takes into account the freezing-in of the lines of force and leads to a multiple increase of the density of the lines of force of the magnetic field. The most known constructions are the Alfven decomposition of the magnetic tube and the so-called Zeldovich octuple (both constructions were proposed in 1970s and correspond to known constructions of the contemporary theory of dynamical systems). For example, the Alfven construction from [2] corresponds to the so-called *baker's transformation*, while the Zeldovich one corresponds to the *Smale map* for the constructing of a hyperbolic solenoid (see [4, 29, 30, 55]). The ideas of the Zeldovich construction (also called the *funicular dynamo*) underlies many constructions of three-dimensional models of a fast dynamo (see [5, 60]). From the viewpoint of the contemporary theory of dynamical systems, the Zeldovich construction is an Ω-stable map of the solid torus into itself introduced in [55]. The nonwandering set of this map is a topological solenoid and a stretched attractor (see [29, 30, 55]). In [5, Chap. V], it is noted that, from the viewpoint of the kinematic dynamo theory, a substantial disadvantage of this construction is that the proposed map is not conservative. In Sec. 6, we present a modification of the Zeldovich construction such that it is free from this disadvantage in a neighborhood of the nonwandering set.

2. Main Statements of Magnetohydrodynamics

Magnetohydrodynamics (MHD) studies the interaction between the electromagnetic field with a liquid or vapor moving conductor treated as continuous medium. MHD equations combine the Maxwell equations for the electromagnetic field and the usual hydrodynamic equations describing the movement of a continuous medium (plasma, liquid, or gas). The Maxwell equations are as follows:

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c_0} \cdot \vec{j} + \frac{\varepsilon_0}{c_0} \cdot \frac{\partial \vec{E}}{\partial t}, \tag{2.1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c_0} \cdot \frac{\partial \vec{H}}{\partial t}, \tag{2.2}$$

$$\vec{\nabla} \cdot \vec{H} = 0, \tag{2.3}$$

and

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{c_0} \cdot \rho_e, \tag{2.4}$$

where ρ_e is the electric-charge density, $\vec{j}$ is the current density, and c_0 is the electrodynamic constant (the light speed in vacuum). The Maxwell equations include the magnetic induction $\vec{B}$. However, for simplicity, we do not distinct the magnetic field $\vec{H}$ and the magnetic induction vector $\vec{B} = \mu\vec{H}$ in Eqs. (2.1)–(2.4) assuming that $\mu \approx 1$.

If $\vec{v}$ denotes the hydrodynamic field of velocities of the continuous medium, then the current density $\vec{j}$ is the sum of the so-called *convection current* $\rho_e\vec{v}$, *conductivity current* $\sigma\vec{E}$, and *induction current* arising in the case where an electrically conducting medium moves in the magnetic field $\vec{H}$ with velocity $\vec{v}$, where σ is the medium conductivity. Thus,

$$\vec{j} = \rho_e\vec{v} + \sigma\vec{E} + \frac{\sigma}{c_0} \cdot \vec{v} \times \vec{H}. \tag{2.5}$$

The two hydrodynamic equations are the *Euler hydrodynamic equation*

$$\varrho \cdot \frac{d\vec{v}}{dt} = -\vec{\nabla} P + \vec{F}_e + \eta \cdot \Delta \vec{v} \tag{2.6}$$

and the *equation of continuity*

$$\frac{\partial \varrho}{\partial t} = \vec{\nabla} \cdot (\varrho \vec{v}) = 0, \tag{2.7}$$

where ϱ is the density of the medium, P is the pressure, η is the viscosity, and $\vec{F}_e$ is the electromagnetic force. The more usual form of the left-hand side of Eq. (2.6) is as follows:

$$\varrho \cdot \frac{d\vec{v}}{dt} = \varrho \cdot \left[\frac{\partial \vec{v}}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) \vec{v}\right] = \varrho \cdot \left[\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}\right].$$

The classical MHD is divided into nonrelativistic and relativistic. In the sequel, it is assumed that we are within the framework of the nonrelativistic MHD, i.e., the considered velocities satisfy the condition $|\vec{v}| \ll c_0$. Therefore, the term $\frac{\varepsilon_0}{c_0} \cdot \frac{\partial \vec{E}}{\partial t}$ of Eq. (2.1) can be neglected. Exclude the electric intensity $\vec{E}$ and the density of the current $\vec{j}$ from the Maxwell equations. This yields the following induction equation, which is among the main MHD equations:

$$\frac{\partial \vec{H}}{\partial t} = rot \, \left[\vec{v}\vec{H}\right] + \nu \nabla^2 \vec{H}, \tag{2.8}$$

where ν is the magnetic viscosity inverse to the Reynolds magnetic number (see [5, 32]).

The theory of kinematic dynamo is mainly based on Eq. (2.8), where the medium velocity $\vec{v}$ is assumed to be a given function of the coordinates and time.

In many papers, exact solutions of MHD equations with various initial conditions are found under additional assumptions (see, e.g., [15]). Also, a comprehensive bibliography is provided in [52].

3. Model of One-Point Charges

In Morse–Smale vector fields simulating magnetic fields $\vec{H}$ generated by the set of so-called magnetic (retraction) charges, the equilibrium states are divided into two classes. The first class consists of all equilibrium states of the vector field corresponding to null points of the magnetic field. These points are responsible for the continuity of the vector field simulating the magnetic field with conflicting regions. The second class consists of all sink and source equilibrium states of the vector field. They correspond to positive and negative retraction charges.

Let p_0 be the equilibrium state of the vector field corresponding to a null point of a magnetic field $\vec{H}$, and λ_1, λ_2, and λ_3 be the eigenvalues of the Jacobi matrix of the system of equations defining the field $\vec{H}$ in a neighborhood of the point p_0. By assumption, the relation $\nabla \cdot \vec{H} = 0$ holds outside the union of sufficiently small neighborhoods of sinks and sources of the vector field $\vec{H}$ (see (2.3)). This implies the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since equilibrium states of Morse–Smale are hyperbolic, it follows that the real parts of all eigenvalues are different from zero. This implies that, from the viewpoint of the theory of dynamical systems, each null point of a magnetic field is a saddle equilibrium state with two one-dimensional separatrices and one two-dimensional separatrix. In physics, each one-dimensional separatrix is called a *spine*, while each one-dimensional one is called a *fan* (see [46, 48, 49] and Fig. 3(a)). If the magnetic line of force of a one-dimensional separatrix is directed from the null point, then all magnetic lines on the separatrix surface are directed to the null point, and vise versa.

Thus, the first class of equilibrium states of the vector field $\vec{H}$ consists of saddle equilibrium states. Recall that the *Morse index* of an equilibrium state p is the dimension $\dim W^u(p)$ of the unstable manifold $W^u(p)$ of the equilibrium state p, while the *topological index* is $(-1)^{\dim W^u(p)}$. If p_0 is a saddle equilibrium state, then only the following cases are possible (up to denotation of the eigenvalues):

(1) $\lambda_1 > 0$, $\operatorname{Re} \lambda_2$, and $\operatorname{Re} \lambda_3 < 0$;

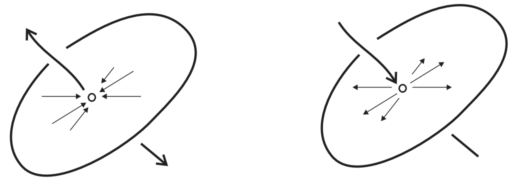

Fig. 2. Positive and negative null points.

(2) $\lambda_1 < 0$, $\operatorname{Re}\lambda_2$, and $\operatorname{Re}\lambda_3 > 0$.

In the first case, we say that the null point p_0 is *positive* because $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$. From the viewpoint of the theory of dynamical systems, the positive null point is a saddle equilibrium state, its Morse index is equal to 1, and its topological index is equal to -1. This equilibrium state has two one-dimensional unstable separatrices and one two-dimensional stable separatrix (see Fig. 2 (a)). In the second case, we say that the null point p_0 is *negative* because $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 < 0$. It is a saddle equilibrium state, its Morse index is equal to 2, its topological index is equal to 1, and it has two one-dimensional stable separatrices and one two-dimensional unstable separatrix (see Fig. 2 (b)).

To present the main results of this section, we refine the notion of positive and negative charges of a magnetic field $\vec{\boldsymbol{H}}$ (according to the point–charge model). We say that the charge of a magnetic field $\vec{\boldsymbol{H}}$ is *positive* if it is contained in an arbitrarily small ball such that the magnetic field at its boundary is directed outwards. The negative charge is defined in the same way. From the viewpoint of the Morse–Smale theory of dynamical systems, each positive charge is a source equilibrium state of the vector field, and each negative one is its sink equilibrium state.

In each publication applying the point–charge model known to the authors, the number of charges is finite and sufficiently small. For example, in [47, 50], groups of two and three charges are considered. In [6, 19, 36, 41, 58], groups of four charges are considered. In [35], a group of six charges is considered. In each paper, the coordinates of the charges are defined concretely. Also, it is assumed that the field of magnetic induction $\vec{\boldsymbol{B}}$ is potential and a particular relation is applied to compute it. Null points of the field $\vec{\boldsymbol{B}}$ and separators important for the description of the topological structure of the magnetic field are found by means of intermediate computations. To verify the existence of null points, the Euler–Poincaré relation is applied. Note that the main concern of the cited papers is the investigation of the restructuring of the magnetic field under variations of the location of charges.

We say that a magnetic field is *quasi-typical* if all its null points and charges (treated as equilibrium states of a vector field) are hyperbolic, two-dimensional separatrices of null points either intersect each other transversally or do not intersect each other at all, each one-dimensional separatrix has no intersections with two-dimensional ones, and no one-dimensional separatrices connecting null points exist (this refers to separatrices connecting a null point with itself as well). In the general case, one-dimensional separatrices either do not intersect each other or coincide with each other. In the sequel, only quasi-typical magnetic fields are considered.

Since the Sun radiates energy, the investigation of the so-called positively unbalanced groups of charges is important in practice. We say that a group $\mathcal{C}$ of charges is *positively unbalanced* if it is contained in a ball B such that the magnetic field at its boundary is directed outwards, the magnetic field is quasi-typical inside the ball, and there are no closed magnetic lines inside the ball. The said ball $B = B(\mathcal{C})$ is called the *source region of the group* $\mathcal{C}$. We assume that if a line of force does not form a one-dimensional separatrix of a null point and does not belong to its two-dimensional separatrix, then it either tends to a singularity of the magnetic field or leaves the source region. Negatively unbalanced groups of charges (and, respectively, sink regions of groups of charges) are defined in the same way. The investigation of negatively unbalanced groups is important as well because families of negatively unbalanced groups might lie inside a positively unbalanced group and bifurcations of the former ones might cause a burst (see [46]).

For each region containing an unbalanced group of charges, introduce the following notation: S^+ is the number of positive null points, S^- is the number of negative null points, N^+ is the number of positive charges, and N^- is the number of negative charges. The Euler–Poincaré relation implies that

$$1 + N^- - S^+ + S^- - N^+ = 0.$$

However, the next result shows that not all negative numbers S^+, S^-, N^+, and N^- satisfying this relation can be realized.

Recall that, following [46, 48, 49], we call each magnetic line connecting two null points a *separator*. From the viewpoint of the theory of dynamical systems, each separator is a heteroclinic trajectory belonging to the intersection of separatrix surfaces of different equilibrium states (see Fig. 3 (b)).

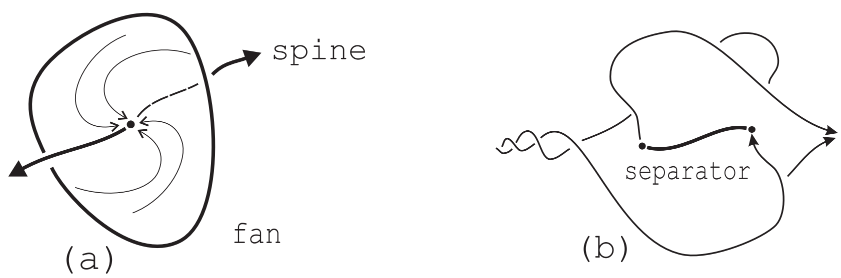

Fig. 3. The null point structure (a) and the heteroclinic separator (b).

Results of the present section are presented below and proved in [63].

Theorem 3.1. *Let a positively unbalanced group $\mathcal{C}$ contain N^+, $N^+ \geq 1$, positive charges and N^-, $N^- \geq 0$, negative charges. Then the source region $B(\mathcal{C})$ of this group contains at least $N^+ - 1$ negative null points and N^- positive null points,*

$$S^- \geq N^+ - 1, \qquad S^+ \geq N^-.$$

If the group $\mathcal{C}$ consists of N^+ positive charges, $N^+ \geq 2$, and $B(\mathcal{C})$ contains $N^+ - 1$ and only $N^+ - 1$ null points, then all null points are negative, there are no separators in $B(\mathcal{C})$, and the structure of the magnetic field in the region $B(\mathcal{C})$ is unique up to the topological equivalence.

Corollary 3.1. *Let $\mathcal{C}$ be a negatively unbalanced group containing N^- negative charges, $N^- \geq 2$. Then the sink region $B(\mathcal{C})$ of this group contains at least $N^- - 1$ positive null points. If the group $\mathcal{C}$ consists of N^- negative charges, $N^- \geq 2$, and $B(\mathcal{C})$ contains $N^- - 1$ and only $N^- - 1$ null points, then all these points are positive null points and $B(\mathcal{C})$ contains no separators.*

If we have the least possible number of null points of both signs (these least numbers are determined by Theorem 3.1), then the existence of separators is not guaranteed. However, the next theorem shows that the appearance of at least one "extra" null point implies the existence of at least one separator. This does not depend of the type of the extra null point (it might be either positive or negative). For definiteness, we present the said assertion for the case where the extra null point is negative.

Theorem 3.2. *Let a positively unbalanced group $\mathcal{C}$ contain N^+, $N^+ \geq 2$, positive charges and N^-, $N^- \geq 0$, negative charges. If $B(\mathcal{C})$ contains N^+ and only N^+ negative null points, then $B(\mathcal{C})$ contains at least one separator.*

The proofs of Theorems 3.1-3.2 are based on the fact that each quasi-typical magnetic field of positively unbalanced groups of charges can be extended as a Morse–Smale vector field on the three-dimensional sphere; then the technique developed in works about the classification of Morse–Smale dynamical systems on manifolds (see [21, 23–25, 28]) can be applied to such vector fields.

4. Topology of Magnetic Fields

According to one of the contemporary viewpoints, the magnetic field in the solar corona is generated by a large number of dipoles located inside the Sun (see, e.g., [48, 49]). These dipoles generate flow tubes of the magnetic field crossing the photosphere of the Sun and leaving into its corona. According to the point–charge model, points where flow tubes leave the photosphere and get back to it are interpreted as point sources and sinks (positive and negative charges) on the photosphere (see Fig. 4). For the model of a magnetic field **B** with point sources, we follow [7] to use the two-dimensional sphere $P = \{(x, y, z, w) \in \mathbb{S}^3 \mid w = 0\}$ in the three-dimensional sphere $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ as the photosphere and the region $\{(x, y, z, w) \in \mathbb{S}^3 \mid w > 0\}$ as the solar corona. Moreover, we assume that **B** is symmetrically extended to the region $\{(x, y, z, w) \in \mathbb{S}^3 \mid w < 0\}$ called the *corona mirror*; hence, it is defined on $M = \mathbb{S}^3 \setminus \bigcup_{i=1}^{k} q_i$, where $q_1, \dots, q_k$ are points of the photosphere where the charges are located.

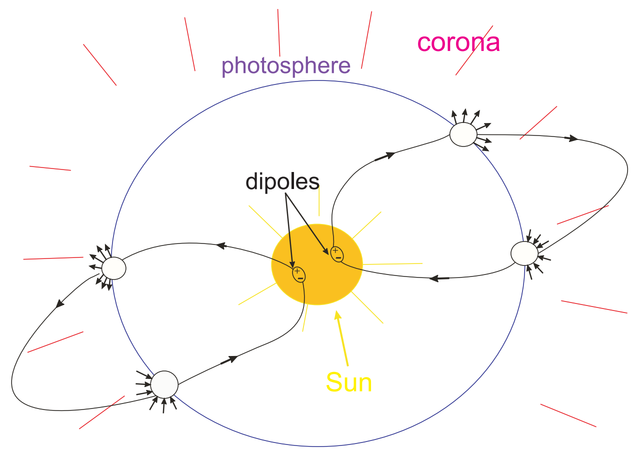

Fig. 4. Dipoles inside the Sun.

It is frequently assumed that the corona magnetic field is irrotational. Taking into account that only the field topology is studied in this section, we assume (for simplicity) that **B** is a potential field, i.e., $\mathbf{B} = -\nabla\Phi$, where Φ is a scalar potential. The next natural assumption is that the potential Φ is a Morse function. Recall that a C^2-function ϕ defined on an n-manifold is called a *Morse function* if for each of its critical point p there exists an open neighborhood V_p with a coordinate system $X = (x_1, \dots, x_n)$ and an integer λ_p from $[0, n]$, called the *index* of p, such that $\phi(x)|_{V_p} = \phi(p) - \sum_{i=1}^{\lambda_p} x_i^2 + \sum_{i=\lambda_p+1}^{n} x_i^2$.

Since $\nabla \cdot \mathbf{B} = 0$, it follows that the three eigenvalues λ_1, λ_2, and λ_3 of the zero of the magnetic field satisfy the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since **B** is a potential, it follows that all eigenvalues are real. Since Φ is a Morse function, it follows that each eigenvalue is different from 0. We say that a zero is *positive* (*negative*) if $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$ ($\lambda_1 \cdot \lambda_2 \cdot \lambda_3 < 0$, respectively). If a zero belongs to the photosphere, then we say that it is a *photospheric* zero. If the one-dimensional invariant manifold of a photospheric zero lies in the photosphere, then we say that it is a *horizontal* zero. Each photospheric zero with the vertically directed one-dimensional invariant manifold is called a *vertical* zero. Each zero located in the solar corona is called a *coronal zero* (see [8]).

If two-dimensional invariant manifolds of null points intersect each other, then they form a *separator* connecting two zeroes with opposite signs. Two-dimensional manifolds divide the corona into different regions called *domes*. The appearance and vanishing of separators changes the topology of the decomposition into domes. This situation, called a *separator reconnection*, is one of the main mechanisms of the energy redistribution in the solar corona (see [51]). The simplest reconnection is known as an *intersecting state* (see [7] and Fig. 5).

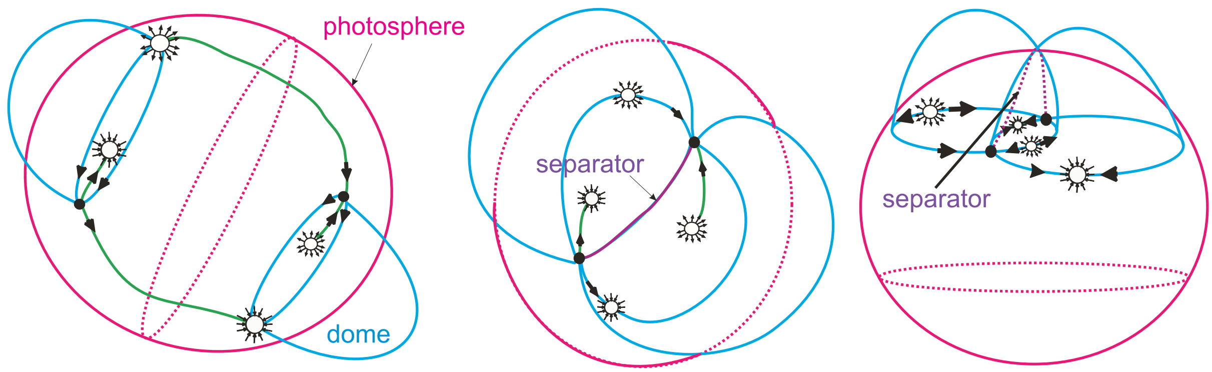

Fig. 5. The intersecting state.

A lot of papers (see [13, 34, 36, 37]) are devoted to the classification of configurations of magnetic-field domes arising from such point sources. It is natural to introduce the following definition from the classical work [45] (see [55] as well).

Definition 4.1. Coronal magnetic fields $\mathbf{B}$ and $\mathbf{B}'$ are *topologically equivalent* if there exists a homeomorphism $H : M \to M'$ taking magnetic lines of $\mathbf{B}$ to magnetic lines of $\mathbf{B}'$ and preserving the orientation on lines.

It is known (see, e.g., [18]) that the gradient vector field ξ generated by a Morse function ϕ possesses a so-called *self-indexing energy function* φ, i.e., a Morse function with the following properties:

(1) the set of critical points of ϕ and φ coincide with each other;
(2) if p is a critical point, then $\varphi(x) = \phi(x) + \text{const}$ provided that $x \in V_p$ and $\varphi(p) = \lambda_p$;
(3) $\xi(\varphi) < 0$ outside critical points.

Let $\mathcal{B}$ be the set of magnetic fields $\mathbf{B}$ possessing the energy function φ. In the present section, we investigate the dependence between the existence of separators and the type of zeroes of the magnetic field $\mathbf{B}$ from $\mathcal{B}$ and find the relation between the number of zeroes and number of charges. Also, we obtain a classification of magnetic fields from $\mathcal{B}$ up to the topological equivalence.

If p is a zero of a magnetic field $\mathbf{B}$, then F_p denotes its two-dimensional invariant manifold and S_p denotes its one-dimensional one. Assign $T_p = F_p \cap P$, i.e., T_p is the trace of the intersection of F_p with the photosphere P.

Results of the present section are presented below and proved in [26].

Theorem 4.1. *The following assertions hold for each magnetic field* $\mathbf{B}$:

(1) *if p is a zero of the magnetic field $\mathbf{B}$ and l_p is a connected component of the set $S_p \setminus p$, then there exists i from $\{1, \dots, k\}$ such that the set $cl\ l_p \setminus (l_p \cup p)$ consists of one and only one charge q_i;*
(2) *if p is a zero of the magnetic field $\mathbf{B}$, then the two-dimensional manifold F_p contains no separators if and only if there exists i from $\{1, \dots, k\}$ such that the set $cl\ F_p \setminus F_p$ consists of one and only one charge q_i;*
(3) *if there exist zeroes $p_1, \dots, p_n$ of the magnetic field $\mathbf{B}$ such that $\bigcup_{i=1}^{n} cl\ S_{p_i}$ is a simple closed curve, then the set F_{p_i} contains at least one separator for each $i \in \{1, \dots, n\}$.*

Theorem 4.2. *The following assertions hold for any arbitrary coronal magnetic field $\mathbf{B}$ from $\mathcal{B}$:*

(1) *the two-dimensional manifold F_p of each coronal and each vertical null point p contains at least one separator;*

(2) *the two-dimensional manifold F_p of any horizontal null point p contains no separators if and only if there exists $i \in \{1, \dots, k\}$ such that cl $T_p \setminus T_p$ consists of one and only one charge q_i.*

Let m denote the number of zeroes of a field $\mathbf{B}$ from $\mathcal{B}$. Assign $g = (m - k + 2)/2$.

Theorem 4.3. *The following assertions hold for any arbitrary coronal magnetic field $\mathbf{B}$ from $\mathcal{B}$:*

(1) *g is a nonnegative integer;*
(2) *the vector field $\mathbf{B}$ has at least $2g$ zeroes such that their two-dimensional invariant manifolds contain at least one separator;*
(3) *the level set $\Sigma = \varphi^{-1}(3/2)$ is an orientable surface of genus g.*

Suppose that $N_+(N_-)$ denotes the set of positive (negative) zeroes of a magnetic field $\mathbf{B}$. Assign $F_+ = \bigcup\limits_{p \in N_+} F_p$ $(F_- = \bigcup\limits_{p \in N_-} F_p)$.

Definition 4.2. We say that self-indexing energy functions φ and φ' of magnetic fields $\mathbf{B}$ and $\mathbf{B}'$ are *consistently equivalent* if there exists homeomorphism $H : M \to M'$ preserving the orientation and possessing the following properties:

(1) $\varphi' H = \varphi$;
(2) $H(\Sigma \cap F_+) = \Sigma' \cap F'_+$ and $H(\Sigma \cap F_-) = \Sigma' \cap F'_-$.

Theorem 4.4. *The magnetic fields $\mathbf{B}$ and $\mathbf{B}'$ are topologically equivalent if and only if their self-indexing energy functions φ and φ' are consistently equivalent.*

5. Existence Conditions for Separators in Moving Plasma

In Secs. 3-4, existence conditions for separators are provided. However, they do not take into account the plasma movement. In the present section, we provide conditions taking this movement into account.

In [22], the following approach to this problem is proposed. In the plasma, a three-dimensional body of a special kind is selected. A movement is considered such that all boundary components of the body are translated inside or outside and, once the movement is over, all boundary components are parallel to the original boundary components (strict definitions are provided below). Since the topological structure of the magnetic field is not changed during the movement, it is assumed (for simplicity) that the skeleton of the magnetic field inside the selected body is invariant with respect to the plasma movement. Note that no immovability of all skeleton points are required, but the movement inside the body leaves all the skeleton points on the skeleton. The only substantive restriction is as follows: all null points are hyperbolic not only for the field, but for the plasma movement simulated by a diffeomorphism with fixed hyperbolic points. It is important to note that, due to the Kupka–Smale theorem from the theory of dynamical systems, all periodic points (including fixed ones) of any typical diffeomorphism are hyperbolic (see [42]). Thus, one can assume that the proposed model describes the class of typical movements of plasma.

Let us pass to strict definitions.

Let M_p^2 be an orientable closed surface of a nonnegative genus p smoothly embedded into the Euclidean space $\mathbb{R}^3$. By virtue of the orientability, M_p^2 decomposes $\mathbb{R}^3$ into a bounded region (interior) and unbounded region (exterior). The union of the interior with the boundary M_p^2 is denoted by M_p^3 and is called a *body of genus p*, $p \geq 0$. The simplest example is the closed three-dimensional ball $M_0^3 \stackrel{\text{def}}{=} D^3$ bounded by the two-dimensional sphere S^2. The body $M_1^3 \stackrel{\text{def}}{=} P^3$ is a *solid torus*, i.e., the set $D^2 \times S^1$ homeomorphic to the product of the two-dimensional closed disk D^2 and the circle S^1.

We say that two smoothly embedded surfaces $M_{p_1}^2$ and $M_{p_2}^2$ are *parallel* if $p_1 = p_2 = p$ and they bound a region of $\mathbb{R}^3$ homeomorphic to $M_p^2 \times (0; 1)$. This implies that $M_{p_1}^2 \cap M_{p_2}^2 = \varnothing$.

Let a body M^3_p contain pairwise disjoint bodies $M^3_{p_1}, \ldots, M^3_{p_k}$ inside itself. Assign

$$M^3_p \setminus \left(\operatorname{int} M^3_{p_1} \cup \ldots \cup \operatorname{int} M^3_{p_k}\right) \stackrel{\text{def}}{=} M^3_{p(p_1,\ldots,p_k)}.$$

In particular, $M^3_{0(0)} = \mathcal{S}$ is the closed spherical layer, i.e., the set $\mathcal{S} = S^2 \times [-1;+1]$ homeomorphic to the product of the sphere S^2 and the closed segment $[-1;+1]$. It is clear that the topological type of the body $M^3_{p(p_1\cdots p_k)}$ depends on the embedding of $M^3_{p_1}, \ldots, M^3_{p_k}$ into M^3_p. For example, the so-called *thick surface*, i.e., the body homeomorphic to the product of the two-dimensional surface M^2_p of a positive genus p and the segment $[0;1]$ is a case of $M^3_{p(p)}$. Let $\mathcal{M}^3$ denote a body of type $M^3_{p(p,0,0)}$, which is a thick surface with two holes.

Consider a body $M^3_{p(p_1,\ldots,p_k)}$ smoothly embedded into the space $\mathbb{R}^3$. The body $M^3_{p(p_1,\ldots,p_k)}$ is a part of the plasma of an astrophysical object with a magnetic field $\vec{\boldsymbol{B}}$. Let $\vec{\boldsymbol{B}}_0$ denote the restriction of the field $\vec{\boldsymbol{B}}$ to $M^3_{p(p_1,\ldots,p_k)}$, i.e., $\vec{\boldsymbol{B}}_0 = \vec{\boldsymbol{B}}|_{M^3_{p(p_1,\ldots,p_k)}}$. Assume that the field $\vec{\boldsymbol{B}}_0$ is quasi-typical (see Sec. 3). This implies that $M^3_{p(p_1,\ldots,p_k)}$ contains only a finite set of null points. Further, assume the following assumptions:

(1) if separatrices of null points intersect each other, then they intersect each other transversally;
(2) if separatrices intersect components $M^2_{p_1}, \ldots, M^2_{p_k}$ of the body $M^3_{p(p_1,\ldots,p_k)}$, then they intersect them transversally.

A map

$$f_0 : M^3_{p(p_1,\ldots,p_k)} \to f_0\left(M^3_{p(p_1,\ldots,p_k)}\right) \subset \mathbb{R}^3$$

is called an (a-d)-*movement* if it satisfies the following conditions:

(a) f_0 is a diffeomorphism onto its image such that it preserves the orientation and its nonwandering set consists of fixed hyperbolic points coinciding with zeroes of the magnetic field $\vec{\boldsymbol{B}}_0$;
(b) boundary components of the body $f_0\left(M^3_{p(p_1,\ldots,p_k)}\right)$ are pairwise disjoint with boundary components of the body $M^3_{p(p_1,\ldots,p_k)}$;
(c) at least one boundary component $M^2_{p_i}$ is mapped inside $M^3_{p(p_1,\ldots,p_k)}$, and at least one boundary component $M^2_{p_j}$ is mapped outside $M^3_{p(p_1,\ldots,p_k)}$, i.e.,

$$f_0(M^2_{p_i}) \subset M^3_{p(p_1,\ldots,p_k)}, \quad f_0(M^2_{p_j}) \cap M^3_{p(p_1,\ldots,p_k)} = \varnothing;$$

(d) fans and spines are invariant with respect to f_0, while fixed points of the diffeomorphism f_0 have the same type as zeroes of the field $\vec{\boldsymbol{B}}_0$.

Note that no transversal intersections of lines of force of the magnetic field $\vec{\boldsymbol{B}}_0$ with boundary components are required. Therefore, in general, intersections of fans and spines with boundary components of the body $M^3_{p(p_1\cdots p_k)}$ over several connected components are possible. From the viewpoint of physics, the proposed model means that the plasma movement is considered during the time interval such that singular points with fans and spines are preserved within this interval. It follows from the provided properties that if separators exist, then they are invariant with respect to f_0 and their number is not changed during the observed time interval (this is still valid for the case where this number is equal to zero).

For a closed spherical layer $\mathcal{S}$, Conditions (a)–(d) mean the following. Assume that a sphere $S^2 \times \{-1\} = S_{int}$ (called the *internal sphere*) bounds a ball B^3 in $\mathbb{R}^3$ such that B^3 does not contain $\mathcal{S}$. The sphere $S^2 \times \{+1\} = S_{ext}$ is called the *external sphere*. Then components of intersections of spines and fans with the spheres S_{int} and S_{ext} are points and (closed or nonclosed) curves. Without loss of generality, one can assume that Condition (d) has the following form: $f_0(S_{int}) \subset \mathcal{S}$ and $f_0(S_{ext}) \subset \mathbb{R}^3 \setminus \left(\mathcal{S} \cup B^3\right)$ such that $f_0(S_{int})$ decomposes $\mathcal{S}$ into two spherical rings (see Fig. 6). Now, we present the initial two results of [22] for (a-d)-movements of a plasma spherical layer.

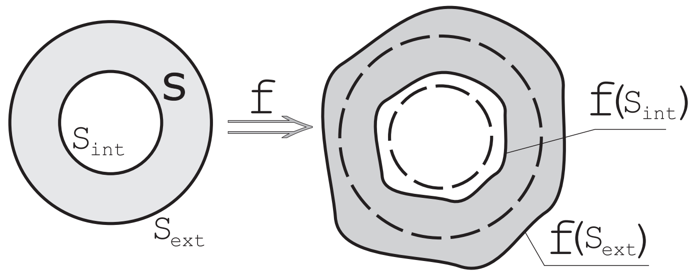

Fig. 6. The movement of the spherical layer $\mathcal{S}$.

Their proofs are based on results on the topological interdependence between the dynamics of Morse–Smale diffeomorphisms and the topology of enveloping manifolds (see, e.g., [23]). It is possible to apply these results because the movements introduced above can be extended as Morse–Smale diffeomorphisms defined on closed three-dimensional manifolds with a known topological structure.

Theorem 5.1. *Let $f_0 : \mathcal{S} \to f_0(\mathcal{S}) \subset \mathbb{R}^3$ be an (a-d)-movement of a spherical layer $\mathcal{S}$ of a plasma with a magnetic field $\vec{\boldsymbol{B}}_0$. Assume that $\vec{\boldsymbol{B}}_0$ has null points in $\mathcal{S}$. Then their number is even (and, therefore, there are at least two null points). Their separatrix surfaces intersect each other, and the number of heteroclinic operators is positive and finite.*

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *be satisfied, and the spine and fans of null points of the magnetic field $\vec{\boldsymbol{B}}_0$ have no intersections in $\mathcal{S}$. Then the separatrix surface of each null point of $\mathcal{S}$ contains at least one heteroclinic operator.*

The thick surface $\mathcal{M}^3$ with two holes has four boundary components, which are the two 2-spheres S_1 and S_2 and two two-dimensional surfaces T_1 and T_2 of a positive genus p. For the movement of the body $\mathcal{M}^3$, realize Condition (d) as follows.

(d) One sphere (for definiteness, the sphere S_1) is mapped inside the body $\mathcal{M}^3$, while the other one is mapped outside. One surface (for definiteness, the surface T_1) is mapped inside $\mathcal{M}^3$, while the other one is mapped outside. The restriction $f_0|_{T_i} : T_i \to f_0(T_i)$ is homotopically trivial for each $i = 1, 2$.

Let us clarify the notion of the homotopic triviality. Unlike the sphere having (from the homotopic viewpoint) only one class of homeomorphisms preserving the orientation, each surface of nonzero genus has a countable family of such classes. For each $i = 1, 2$, the surfaces T_i and $f_0(T_i)$ are parallel to each other. Therefore, one can assume a natural isomorphism between generators of their fundamental groups. The homotopic triviality means that the restrictions $f_0|_{T_i}$ are homotopically identical. The following assertions are valid for (*a*–*d*)-movements of the body $\mathcal{M}^3$.

Theorem 5.3. *Let $f_0 : \mathcal{M}^3 \to f_0(\mathcal{M}^3) \subset \mathbb{R}^3$ be an (a-d)-movement of a body $\mathcal{M}^3$ belonging to a plasma region with a magnetic field $\vec{\boldsymbol{B}}_0$. Then the field $\vec{\boldsymbol{B}}_0$ has at least two null points in $\mathcal{M}^3$ such that their separatrix surfaces intersect each other, and the number of heteroclinic separators is positive and finite.*

Theorem 5.4. *Let the assumptions of Theorem* 5.3 *be satisfied, and spine and fans of null points of the magnetic field $\vec{\boldsymbol{B}}_0$ have no intersections in $\mathcal{M}^3$. Then the separatrix surface of each null point of $\mathcal{M}^3$ contains at least one heteroclinic operator.*

In [20], an existence condition for separators close to Theorem 5.4 is provided.

6. Funicular-Type Model of Kinematic Dynamo

In the present section, we modify the Zeldovich construction finding a diffeomorphism and a vector field such that the latter exponentially increases under the action of the former.

On the Cartesian plane $\mathbb{R}^2$, consider the disk $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and the map $w : D^2 \to \mathbb{R}^2$ forming the *Smale horseshoe* (see [54, 55]). Namely, the map w is the composition of a contraction along the Ox axis, extension along the Oy axis, inflection (regardless its direction) of the obtained ellipse, and a translation such that the intersection $D^2 \cap w(D^2)$ is the union of two disjoint bands symmetric with respect to the Oy axis.

It is known from [4, 54] that w can be extended as a map of the whole plane $\mathbb{R}^2$ such that it is the identical map outside of a neighborhood of the disk D^2. It is clear that, using the contraction and extension, one can ensure that the Jacobian $J(w)$ of the map w onto D^2 is equal to $1/2$. In the sequel, we assume these conditions are satisfied.

Let $sh_0 : \mathbb{R}^2 \to \mathbb{R}^2$ denote the translation $(x;y) \longrightarrow \left(x+\dfrac{1}{2};y\right)$ along the Ox axis. Let $S_0 : \mathbb{R}^2 \to \mathbb{R}^2$ denote the central symmetry with respect to the origin $(0;0)$, $S_0(x;y) = (-x;-y)$. Again, using the contraction, extension, and inflection, one can ensure the fulfillment of the following conditions:

(1) the intersection $D^2 \cap sh_0 \circ w(D^2)$ consists of two disjoint bands;

(2) $w(D^2) \cap \big(S_0 \circ w(D^2)\big) = \varnothing$.

The first condition means that the map $sh_0 \circ w = w_0$ forms a Smale horseshoe. The second one means that the horseshoe $w(D^2)$ does not intersect its image with respect to the central symmetry S_0. Note that $S_0 \circ w(D^2)$ forms a horseshoe configuration as well.

Let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ denote the counterclockwise rotation

$$\bar{x} = x\cos\pi t - y\sin\pi t, \quad \bar{y} = x\sin\pi t + y\cos\pi t$$

of the plane $\mathbb{R}^2$ to the angle πt. Assign

$$w_t = R_{2t} \circ w_0 \circ R_{-t} : D^2 \to \mathbb{R}^2.$$

This map can be interpreted as follows: a horseshoe is formed in the direction orthogonal to the line $y = \tan\pi t \cdot x$; then R_t is counterclockwise rotated to the angle πt.

Let $S^1 = [0;1]/(0{\sim}1)$ be a circle endowed with the natural parametrization $[0;1] \to [0;1]/(0{\sim}1) = S^1$. The map $E_2 : S^1 \to S^1$ of kind $t \to 2t \bmod 1$ is an extending endomorphism of a circle of power 2. In the space $\mathbb{R}^3$, consider the embedded solid torus $S^1 \times D^2 \subset \mathbb{R}^3$ and the map $F : S^1 \times D^2 \to \mathbb{R}^3$ acting as follows:

$$(t;(x;y)) \longmapsto (E_2(t); w_t(x;y)), \qquad t \in S^1, \quad (x;y) \in D^2.$$

Assign $D_t = \{t\} \times D^2 \subset S^1 \times D^2$ and $\mathbb{R}^2_t = \{t\} \times \mathbb{R}^2$. By virtue of the definition of the map F, we have

$$F(D_t) \subset \mathbb{R}^2_{E_2(t)} = \mathbb{R}^2_{2t \bmod 1}.$$

The map $F : S^1 \times D^2 \to F(S^1 \times D^2)$ is a diffeomorphism onto its image.

Since the Jacobian $J(w)$ of the map w is equal to $\dfrac{1}{2}$ on D^2, it follows that the Jacobian of the map F is equal to $J(F) = J(w) \cdot DE_2 = \dfrac{1}{2} \cdot 2 = 1$. Therefore, F is a conservative diffeomorphism onto its image. In the standard way, complete the space $\mathbb{R}^3$ by the point at infinity $\{\infty\}$ so that the union $\mathbb{R}^3 \cup \{\infty\}$ is identified with the three-dimensional sphere S^3.

From the technique developed in [10, 62], it follows that f is extended as a diffeomorphism $f : S^3 \to S^3$ of the sphere possessing the conservativity property in a neighborhood of the solid torus $S^1 \times D^2$.

The solid torus $S^1 \times D^2$ embedded into S^3 is called the *base solid torus* and is denoted by $\mathcal{B}$. Assign

$$\Omega = \bigcap_{n=-\infty}^{\infty} f(\mathcal{B}).$$

The set Ω is invariant with respect to f (see [4]) and is not empty because it contains an invariant nontrivial (zero-dimensional) set Ω_0 of the Smale horseshoe in $D_0 = \{0\} \times D^2 \subset \mathcal{B}$ (see [4, 54, 55]). Let $\mathit{Diff}^1(S^3)$ denote the space of diffeomorphisms of the 3-sphere S^3 endowed with the C^1-topology.

The set Ω is hyperbolic and the restriction $f|_\Omega$ of the diffeomorphism f on Ω has a positive (topological) entropy. Moreover, the space $\mathit{Diff}^1(S^3)$ contains a neighborhood $U(f)$ of the diffeomorphism f such that each diffeomorphism g from $U(f)$ has a hyperbolic invariant set Ω_g that is a subset of $\mathcal{B}$, the diffeomorphisms $f|_\Omega$ and $g|_{\Omega_g}$ are adjoint, and the restriction $g|_{\Omega_g}$ has a positive entropy.

Now, on $S^1 \times D^2$, consider the magnetic field $\vec{\boldsymbol{B}}$ formed by unit vectors tangent to the curves $S^1 \times \{z\}$, $z \in D^2$. We assume that the curves $S^1 \times \{z\}$ are oriented towards the direction of growth of the parameter. It is clear that $\vec{\boldsymbol{B}}$ can be extended to the whole sphere S^3 as a unit (and, therefore, divergence-free) vector field. We assume that the diffusion of $\vec{\boldsymbol{B}}$ is equal to zero (i.e., the magnetic energy is not scattered). Since the curves $S^1 \times \{z\}$ are extended two times under the action of f, it follows that f takes the field $\vec{\boldsymbol{B}}$ to the field $f_*(\vec{\boldsymbol{B}})$ possessing the following property: there exists a constant λ such that $\lambda > 1$ and the length of vectors of the field $f_*(\vec{\boldsymbol{B}})$ exceeds the length of vectors of the field $\vec{\boldsymbol{B}}$ at least λ times. The same property holds for the lengths of vectors of the field $f_*^{n+1}(\vec{\boldsymbol{B}})$ with respect to the field $f_*^n(\vec{\boldsymbol{B}})$. If the energy dissipation is not taken into account, then this implies that the energy of the vector field $f_*^n(\vec{\boldsymbol{B}})$ increases exponentially with the positive exponent $\log \lambda$. Thus, the diffeomorphism $f : S^3 \to S^3$ is a fast nondissipative kinematic dynamo with respect to the magnetic field $\vec{\boldsymbol{B}}$.

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