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NONLOCAL TURBULENT DIFFUSION MODELS

V. V. Uchaikin UDC 517.958

Abstract. A brief review of the emergence and development of the nonlocal approach to the problem of turbulent diffusion with a discussion of the physical reasons of the nonlocality is given. The main attention is paid to fractional differential operators. In concluding the paper, the author's original results on applications to the diffusion of cosmic rays in the interstellar galactic medium are presented.

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1. Introduction. In this paper, we will focus on the nonlocal method of describing the transport process in turbulent media, here referred to as "turbulent diffusion" for brevity.

It must be said, first of all, that nonlocality due to integral operators is not always *fatal*; it can often be overcome by identical transformations, just as conversely, equations containing local differential operators can be converted to nonlocal type. Take, for example, the Navier—Stokes equation for an incompressible fluid:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{1}{\rho}\operatorname{grad} p + \nu\Delta\boldsymbol{v}. \tag{1}$$

Taking the divergence of both sides of the equation and considering that $\operatorname{div}\boldsymbol{v} = 0$, we obtain the nonuniform Poisson equation

$$\operatorname{div}\big((\boldsymbol{v}\nabla)\boldsymbol{v}\big) = -\frac{1}{\rho}\Delta p,$$

with the solution

$$p(\boldsymbol{x}, t) = \frac{\rho}{4\pi}\int \frac{\big\{\operatorname{div}\big((\boldsymbol{v}\nabla)\boldsymbol{v}\big)\big\}_{\boldsymbol{x}'}}{|\boldsymbol{x} - \boldsymbol{x}'|}d\boldsymbol{x}'. \tag{2}$$

Combining equations (1)-(2) into one, we see that it is not only nonlinear, but also nonlocal. However, this nonlocality is removable by performing these actions in reverse order. We will consider further only situations with nonremovable nonlocality.

Historically, the development of nonlocal methodology began within the framework of a linear integral model, and, in general, this continues up to now. In 1893 Dugem believed that the stress at each point of a solid should, in principle, depend on the state of the whole of this body, and not only on the local deformation at this point. The nonlocal approach was used by Rayleigh in optimizing sliding bearings (1918), by Oseen in modeling liquid crystals (1933), by Chandrasekhar in radiation transfer problems (1950), and by Hodgkin in describing the propagation of electric waves in neural networks (1964). Rogula proposed a nonlocal form of a constitutive law for elastic materials (1965). Early work on the nonlocal theory of elasticity was motivated mainly by the idea of homogenizing the atomic structure of crystal lattices in order to more adequately describe phenomena occurring on scales comparable to interatomic distances. It was found that the nonlocal model of the continuum well approximates the dispersion of short waves and better describes the interactions between crystal defects (vacancies, interstices, dislocations).

We now turn to the problem of turbulent diffusion, which, in fact, is the subject of this work.

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2. Local model of Brownian diffusion. The concept of a diffusion (moreover, of a general *random*) process is associated primarily with the image of Brownian motion. This is the movement of the smallest particles under the influence of continuously continuing impacts of the molecules of the medium, making random thermal motion. The equation of this process derived by A. Einstein belongs to the *statistical ensemble of independent particles*; it is based on the Markov model of the random wals of a Brownian particle and is written not for the coordinates of this particle itself, but for the probability distribution, which behaves similarly to a liquid spreading in a basic medium. This distribution is characterized by density, $f(\boldsymbol{x}, t)$, normalized to one,

$$\int f(\boldsymbol{x}, t) d\boldsymbol{x} = 1,$$

and satisfying the equation

$$\frac{\partial f}{\partial t} = D\Delta f(\boldsymbol{x}, t),$$

where D is a continuous coefficient of diffusion, and

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the Laplace operator. If at a certain (*initial*) moment of time t_0 the position $\boldsymbol{x}_0$ of the particle is known, the distribution of f is called *propagator* and is denoted by $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$. In the case of a homogeneous and stationary environment, the propagator is invariant with respect to space-time shifts,

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0),$$

and satisfies the same diffusion equation

$$\frac{\partial G}{\partial t} = D\Delta G(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) \tag{3}$$

with the initial condition

$$G(\boldsymbol{x} - \boldsymbol{x}_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0).$$

Using the Fourier transform over the space variable

$$\tilde{G}(\boldsymbol{k}, t - t_0) = \int e^{i\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}_0)} G(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) d\boldsymbol{x}$$

it is reduced to an ordinary differential equation for the propagator transformant

$$\frac{\partial \tilde{G}(\boldsymbol{k}, t)}{\partial t} = -Dk^2 \tilde{G}(\boldsymbol{k}, t) \tag{4}$$

with the initial condition $\tilde{G}(\boldsymbol{k}, 0) = 1$.

Let us discuss some aspects of the process under consideration. There is no particle in the equation (3): it is written for a probability density, which from a dynamic point of view can be interpreted as the density of some incompressible (due to the law of probability preservation) fluid. Probabilistic fluid (or perhaps more figuratively, *liquid probability*?) flows according to a rather peculiar law: a "droplet" of a probability that arises at some point, a moment later takes all the space, ignoring the postulate of its creator about the relativistic limitations of the velocities of all bodies in the universe. At the same time, the elements of this fluid (which are called in hydrodynamics liquid particles) do not interact at all, freely penetrating each other and not leaving any memory of this event in each other. This is easily explained by the fact that such a liquid exists only in the imagination; in fact, there is only one particle, which at a given moment can be only at one point, there are no other particles of *this liquid* in space, and it only interacts with atoms of the medium, the thermal motion of which determines the motion of this particle. It is assumed that each of the atoms colliding with this particle carries away a part of the momentum that it inherits, and does not return to the particle any more;

therefore, the particle all the time collides with new atoms. It is this assumption (often treated as *absence of the process memory*) that makes it possible to use the Markov process model.

An alternative description of the motion of an individual particle is made using the Langevin equation. Unlike the Einstein equation, whose solution is a deterministic function, the Langevin equation is written for the random implementation of the process and is associated with the idea of the *statistical ensemble of trajectories*, and not just with the positions of the particle at a given time. N. Wiener's study of the trajectories corresponding to the Einstein equation found that these trajectories, being continuous, are nowhere differentiable, and the speed as the limit of the ratio of the increment of coordinates to the corresponding time interval does not exist. We emphasize that it is impossible to determine not only the magnitude, but also the direction of speed. These trajectories are *self-similar*: on a single time interval such a trajectory contains as many turning points as on doubled, and this is possible only with infinite density of these points on the time axis. This picture is clearly incompatible with what we see by throwing a handful of seeds into a turbulent flow: each of these "test particles" (*tracer*) moves along its own trajectory, and, if necessary, it is easy to determine its speed using conventional measuring equipment.

3. Local model of turbulent diffusion. In a nonstationary and inhomogeneous medium with a given velocity field $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}, t)$ and a *molecular* diffusion coefficient $D = D_{\mathrm{m}}(\boldsymbol{x}, t)$ the diffusion propagator satisfies the advection-diffusion equation

$$\frac{\partial G}{\partial t} + \boldsymbol{U}\nabla G = \nabla\big[D_{\mathrm{m}}\nabla G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\big]. \tag{5}$$

The turbulent character gives this nonstationary and inhomogeneous medium an additional quality: the fields $\boldsymbol{U}$ and D_{m}, and with them the propagator G themselves become random. Immediately there arises a whole "bouquet" of tasks: determining the average characteristics of the motion of the tracer, its fluctuations, correlations, probabilities of emissions (large deviations), etc. The mathematical apparatus of the theory is enriched: a field of application opens up for spectral theory, the theory of non-Markov processes and non-Gaussian fields, nonlocal (including fractional-differential) operators. The most important task is to establish (select) a statistical ensemble, over which the averaging will be performed. The next most important task is the selection of the procedure for decoupling correlations, i.e. transformation of, say, ensemble-averaged equation (5)

$$\frac{\partial\langle G\rangle}{\partial t} + \langle\boldsymbol{U}\nabla G\rangle = \nabla\langle\big[D_{\mathrm{m}}(\boldsymbol{x}, t)\nabla G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\big]\rangle,$$

to the equation for the average propagator $\langle G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\rangle \equiv g(\boldsymbol{x}-\boldsymbol{x}_0, t-t_0)$. However, the assumptions about the smallness of fluctuations and their space-time small-scale make it possible to circumvent these difficulties and obtain for turbulent diffusion in a uniform stationary isotropic (*average*) random environment the equation

$$\frac{\partial g}{\partial t} = D\Delta g(\boldsymbol{x}, t), \tag{6}$$

where D is the total (molecular and vortex) diffusion coefficient. We emphasize that with the above restrictions in the asymptotics of large distances and times, the turbulent diffusion *on average* is described by the same local equation as the molecular one, but with a greater numerical value of the diffusion coefficient.

4. Spatial nonlocality. The nonlocal nature of the turbulent diffusion equation can be traced from the long-standing work [4], which is based on the results of W. Heisenberg and A. N. Kolmogorov on the theory of turbulence. In this paper, by the known transformations of a hydrodynamic turbulent system, the author reduced the equation for the Fourier transform of the average tracer propagator (labeled molecules) to the form

$$\frac{\partial^2\tilde{g}(\boldsymbol{k}, t)}{\partial t^2} = [k^2\nu(k)]^2\tilde{g}(\boldsymbol{k}, t),$$

where the turbulent viscosity $\nu(k)$ is related to the spectral density of turbulent dissipation per unit time $F(k)$ by the Heisenberg formula (see [6]):

$$\nu(k) = \kappa \int_k^\infty \sqrt{\frac{F(q)}{q^3}} dq; \tag{7}$$

here κ is an arbitrary number near 1. The family of solutions for (7) contains the solutions of the equation

$$\frac{\partial \tilde{g}(\boldsymbol{k}, t)}{\partial t} = -k^2 \nu(k) \tilde{g}(\boldsymbol{k}, t). \tag{8}$$

If we ignore the dependence of $\nu(k)$ on k, putting $\nu(k) = \nu_0$ in the whole range of wavenumbers, then we arrive at the usual diffusion equation

$$\frac{\partial g(\boldsymbol{x}, t)}{\partial t} = \nu_0 \Delta g(\boldsymbol{x}, t).$$

In fact, the value of ν is not constant; its dependence on k is determined by the form of the spectral function F in the formula (7). According to the 5/3 *Kolmogorov–Obukhov law*, the latter is represented in the power form:

$$F(q) = q^{-5/3} \psi(q), \tag{9}$$

where $\psi(q)$ is a window function (filter), which rapidly decreases outside the inertia region. This fact is the most important argument against the local model of turbulent diffusion.

Using formulas (7) and (9) and taking the inverse Fourier transform, let us write the spacial analog of equation (8) in the form

$$\frac{\partial g(\boldsymbol{x}, t)}{\partial t} = \hat{N} g(\boldsymbol{x}, t), \tag{10}$$

where

$$\hat{N} g(\boldsymbol{x}, t) \equiv \frac{1}{(2\pi)^3} \int d\boldsymbol{k} e^{-i\boldsymbol{k}\boldsymbol{x}} \left[-\kappa k^2 \int_k^\infty q^{-7/3} \psi(q) dq \right] \int d\boldsymbol{x} e^{i\boldsymbol{k}\boldsymbol{x}} g(\boldsymbol{x}, t)$$

is an explicitly nonlocal operator.

If we now take $\psi(q)$ as a constant equal to, say, ψ_0 for all $q > 0$ (note that this is not the same thing as taking the constant $\nu(k)$), then (10) takes the form of an evolution equation with a Laplacian in a fractional degree:

$$\frac{\partial g(\boldsymbol{x}, t)}{\partial t} = -\kappa (-\Delta)^{\alpha/2} g(\boldsymbol{x}, t), \tag{11}$$

where $\alpha = 2/3$, but in the general case $\alpha \in (0, 2]$. The explicit form of the fractional Laplacian is

$$(-\Delta)^{\alpha/2} f(\boldsymbol{x}) = C_\alpha \text{p.v.} \int \frac{f(\boldsymbol{x}) - f(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^{3+\alpha}} d\boldsymbol{x}',$$

where p.v. is the symbol of the principal value of the integral, and

$$C_\alpha = \frac{2^{\alpha-1} \alpha \Gamma((3+\alpha)/2)}{\pi^{3/2} \Gamma(1 - \alpha/2)}$$

a norming constant. The inverse operator

$$(-\Delta)^{-\alpha/2} f(\boldsymbol{x}) = C_{-\alpha} \int \frac{f(\boldsymbol{x}') d\boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^{3-\alpha}}$$

is called the *Riesz potential*.

The solution of the equation (11) with the initial condition $g(\boldsymbol{x},0)=\delta(\boldsymbol{x}-\boldsymbol{x}_0)$ is expressed in terms of the three-dimensional density of the isotropic stable distribution (the Levi–Feldheim distribution) $p_3(\boldsymbol{x};\alpha)$ by the equality

$$g(\boldsymbol{x},t)=(\kappa t)^{-3/\alpha}p_3\Big((\kappa t)^{-1/\alpha}\boldsymbol{x};\alpha\Big),\quad \alpha\in(0,2].$$

For $\alpha=2$ this distribution coincides with the three-dimensional normal distribution, for $\alpha<2$ it is narrower than the normal in the middle part and heavier at the periphery, and as a result, the second moment turns into infinity. To characterize the propagation of a diffusion packet with time, in this case one should choose another measure of its width, for example, the width at half-height or the radius of the sphere containing the given probability. Being proportional to each other, these measures grow with time in proportion to $t^{1/\alpha}$, which agrees with the Richardson law for $\alpha=2/3$.

However, it should be noted that ignoring the deviation of the actual behavior of the fluctuation spectrum from a power law outside the inertial interval (as a result of which the fractional degree of the Laplace operator appears) somewhat discredits this equation, prompting us to look for a more adequate representation of the nonlocal version of the theory of turbulent diffusion. One of the directions of this search is the adaptation of fractional operators to the conditions of boundedness of wave numbers by introducing a damping factor into their core, limiting the influence of a power factor.

In addition, there are still at least two problems associated with the development of the theory of turbulent diffusion: the problem of collective motions and the return to the model of the concept of velocity.

5. Relative diffusion. The simplest and most studied example of a collective effect is the problem of the relative motion of a pair of tracers that were at the initial moment in a neighborhood of each other. The specificity of the turbulent diffusion of a pair of particles is due to the effect on this pair of vortices of different sizes that exist in a turbulent medium. The distance between them can significantly change in a short time only under the action of a vortex, the dimensions of which are comparable with this distance. The farther from each other are these particles, the larger is the size of the vortices that distribute them from each other, and the faster the increase in the distance l between them. The experiment shows that $\langle l^2(t)\rangle\propto t^3$. In the framework of the classical (local) diffusion theory, this effect can be achieved by introducing the dependence of the relative diffusion coefficient either on distance or on time. The first approach was implemented by L. Richardson (see [10]), obtaining $D_l(r)\propto r^{4/3}$ and different from the normal distribution density of the distance $\propto e^{-9r^{2/3}/4t}$; the second was used in the work of G. Batchelor [1], who presented the coefficient of relative diffusion proportional to t^2, which corresponds to an increase in the width of the distribution of the random distance $\Delta_l(t)\propto t^{3/2}$. Richardson explained the dependence of $D_l(r)$ by the multiscale vortices in a turbulent medium: with increasing distance, larger vortices are involved in the process of relative motion, and the particle speeds increase. In the framework of the Kolmogorov concept of turbulence, the exponent 4/3 in the formula for the coefficient of relative diffusion is a direct consequence of the dimension: it suffices to assume that both the initial and final distances are much smaller than the typical size L of the largest vortices and is much larger than the Kolmogorov length $\eta=(\nu^3/\epsilon)^{1/4}$, where ν is the viscosity and ϵ is the average dissipation rate of the turbulent component of the kinetic energy (see [9]). Finally, A. S. Monin derived an equation for relative turbulent diffusion with a "diffusion coefficient" that does not depend on coordinates or time, but gives the same propagation law for a diffusion package (see [7]). This was achieved at the cost of introducing the fractional Laplacian $\Delta^{1/3}$, although due to the divergence of the second moment, it was necessary to change the measure of the width of the diffusion package. The very solution, i.e. the probability distribution density for the vector of the relative position of a pair of particles turned out to be the *isotropic stable Levy–Feldheim density*, whose width Δ increases in proportion to $t^{1/\alpha}$, and this is fundamentally important. When $\alpha=2$, the graph of the $\Delta(t)\propto\sqrt{t}$ curve initially abruptly goes up, then the curve becomes flatter, and the velocity corresponding to its values decreases (the divergence of particles slows down). This happens

in the case of *local diffusion*, a type of which is molecular diffusion (Brownian motion). The molecules that are close at the initial moment quickly diverge, and then the process of divergence (increasing the distance between them) slows down more and more. But the diffusion is nonlocal, $\alpha = 2/3$, the graph of the function $\Delta(t) \propto t^{3/2}$ initially lies on the x-axis, and only after some time noticeably detaches from it and with acceleration begins to grow. This means that the behavior of the pair of tracers in this case is opposite: at first, the particles remain close to each other for some time (although they may move with the fluid element into which they fell) and only after some time they begin to diverge from ever increasing (in average) relative speed.

The visual *kinematic* interpretation of the nonlocality of turbulent diffusion due to the vortex nature of turbulent flows was given by Schönfeld (see [13]). Defending his point of view, he examined a flat vortex of characteristic size ϱ centered on the point (x, y) of the coordinate plane, generating at the origin of coordinates a velocity w directed at an angle φ to the x axis. The contribution of this vortex to the change in impurity concentration

$$n(x, y) \mapsto n\big(x - \varrho\cos\varphi, y - \varrho\sin\varphi\big)$$

he estimated through the x-component of the diffusion flow

$$\delta j_x = n\big(x - \varrho\cos\varphi, y - \varrho\sin\varphi\big) w\cos\varphi$$

and summed it over all the distances ϱ from the axes of the vortices, characterized by the axisymmetric (independent of the angle φ) distribution $W d\varrho/\varrho$:

$$j_x = \frac{1}{2\pi}\int\limits_0^\infty \frac{d\varrho}{\varrho} W(\rho) n\big(x - \varrho w\cos\chi, y - \varrho\sin\chi\big)\cos\chi.$$

The Fourier transform over both spatial variables with the subsequent substitution of the continuity equation into the Fourier transform

$$\frac{\partial\widetilde{n}}{\partial t} = ik_x\widetilde{j}_x + ik_y\widetilde{j}_y$$

gives

$$\frac{\partial\widetilde{n}}{\partial t} + k^2\nu(k)\widetilde{n}(\boldsymbol{k}, t) = 0. \tag{12}$$

As before, we arrive at the Fourier transform of the nonlocal diffusion equation (8).

Chavanis, returning in [5] to Onsager's long-standing idea of representing the turbulent medium using the microcanonical vortex ensemble (see [8]) and using the mathematical apparatus developed by Chandrasekar to analyze the Poisson star distribution model, developed a statistical mechanics of two-dimensional systems of plane vortices with parallel axes, the random positions of which form a homogeneous (on average) Poisson ensemble with a density of n. We will consider here only one moment connected with the definition of the propagation law for a diffusion package in a turbulent medium. Chavanis found an equilibrium distribution $W(\boldsymbol{V}|N)$ of the total velocity

$$\boldsymbol{V}(\boldsymbol{x}) = -\sum_{i=1}^{N}\frac{\gamma_i}{2\pi}\left[\boldsymbol{z}, \frac{\boldsymbol{x}_i - \boldsymbol{x}}{|\boldsymbol{x}_i - \boldsymbol{x}|^2}\right],$$

created in the center of the region containing N vortices, as well as the characteristic duration of fluctuations of a given velocity value $T(V|N)$. Averaging $T(V|N)$ over speed, he obtained (in the main approximation over $\ln N$)

$$t_N = \int\limits_0^\infty T(V|N) W(\boldsymbol{V}|N) 2\pi V\, dV \propto \frac{1}{n\gamma\sqrt{\ln N}}.$$

Elementary calculations lead to the formula for the *vortex* diffusion coefficient

$$D_N = \frac{1}{4}\int T(V|N)W(\boldsymbol{V}|N)V^2\,d^2\boldsymbol{V} \propto \gamma\sqrt{\ln N}.$$

As $N \to \infty$, it is natural to consider $D_N t_N$ as Dt, thus we obtain[1]

$$\langle r^2\rangle = Dt = \frac{1}{n}.$$

Referring to direct numerical simulation and experiments, Chavanis gives the formula for the decay (relaxation) of the turbulent vortices in the form $n \propto t^{-\nu}$, where $\nu \geq 1$. As a result, the formula for the diffusion coefficient takes the form $D \propto t^{\nu-1}$. Obviously, for $\nu = 1$, the result agrees with the local model, and for $\nu = 3/2$ with the numerical simulation data. The experiments also give values close to the theoretical: 1.3–1.4.

6. Nonlocality in time. As is known, hydrodynamic equations can be derived from the Boltzmann-type gas-dynamic equations describing the random walks of particles along trajectories consisting of straight-line segments connecting the collision points. At the same time, piecewise-linear gas-dynamic trajectories are crushed, turning into smooth hydrodynamic trajectories. For computational purposes, you can use the inverse procedure: to make the transition from a hydrodynamic model in which a liquid particle continuously changes its speed of movement, to a gas-dynamic one, in which the speed of a particle (molecule) changes only at the collision points. Generally speaking, this transformation can be considered as a variant of the finite difference method with randomized time nodes. The fact that a particle has a constant velocity between collisions introduces correlations between successive displacements, which is not the case in the Brownian (local) model, but these correlations are clearly distinguishable by direct observations of turbulent diffusion. The paths themselves are also considered independent in the case of molecular (local) diffusion distributed exponentially.

From the one-dimensional linearized Boltzmann equation, the following relation between current and concentration is easily derived:

$$\frac{\partial j}{\partial t} + 2\mu j(x,t) = F(x,t),$$

where the right-hand side has the form

$$F(x,t) = -v^2\frac{\partial n}{\partial x}.$$

Assuming that the right-hand side is known and supplementing the equation with the initial condition $j(x,0) = 0$, we solve it with respect to $j(x,t)$:

$$j(x,t) = \int\limits_0^t e^{-2\mu(t-t')}F(x,t')dt'.$$

Using this transformation, Bourret presented the result in the form

$$j(x,t) = -\frac{\partial}{\partial x}\int\limits_0^t R^0(t')n(x,t-t')dt'$$

(see [3]) and substituted the exponential autocorrelation function $R^0(t) = v^2e^{-2\mu t}$ by the more general $R(t)$:

$$j(x,t) = -\frac{\partial}{\partial x}\int\limits_0^t R(t')n(x,t-t')dt'. \tag{13}$$

[1]Recall that n is the concentration of the axes of vortices represented by points in a two-dimensional space; its dimension is inverse to the area dimension.

As a result, an eliminable (because of its exponential nature) nonlocality in time turned into a noneliminable, and the process itself into a non-Markov one:

$$\frac{\partial n}{\partial t} = \frac{\partial^2}{\partial x^2} \int\limits_0^t R(t')n(x, t-t')dt'.$$

Supplementing this equation by its three-dimensional analog, Bourret wrote the result in the form

$$\frac{\partial n(\boldsymbol{r}, t)}{\partial t} = \frac{\partial^2}{\partial x^i \partial x^j} \int\limits_0^t R_{ij}(t')n(\boldsymbol{r}, t-t')dt',$$

where $R_{ij}(t') = \langle v_i(t)v_j(t+t')\rangle$ is the autocorrelational tensor. In support of this approach, the author referred to the opinions of Batchelor and Townsend (see [2]), who argued that a satisfactory theory of turbulent diffusion should be written in the form of an integral equation. Despite this, Bourret remained dissatisfied with his result. He wrote: "The absence of any spatial correlation term in our equation is conspicuous; consequently, use of the new formulation is probably justified only in application to regimes in which the spatial coherence is negligible in comparison with time coherence. In particular, application to plasmas with collective excitations may be precluded for these reason."

7. Nonlocality in space and time. Due to the special nature of Brownian (in the Wiener sense) trajectories, which manifests itself in nondifferentiability (absence of smoothness), they cannot be adequately represented either analytically or numerically, unless you restrict yourself to a finite set of points in time. It can be both deterministic and random choice. Combining in the latter case the successive positions of the particle in straight-line segments, we obtain some approximation of the Brownian trajectory. If these segments (free paths) are considered to be distributed exponentially, and their directions are isotropic, then we obtain the transport model described by the linear Boltzmann equation underlying the kinetic theory of nuclear reactors. In the limit of small runs (or the same, large times) the Boltzmann equation turns into a diffusion one with the coefficient

$$D = \frac{lv}{3},$$

where l is the mean free path as the only reminder of the kinetic origin of the diffusion equation (the equation itself does not contain any signs of the run-collision scheme). If we do not go to the limit, we get a model with continuous trajectories represented by broken lines consisting of independent straight-line segments of a random variable interpreted as free paths of particles between collisions. In the case of short (for example, exponential) runs, such a scheme imitates Brownian motion, but with a power distribution of ranges, long ranges characteristic of turbulent diffusion are observed. Such a model of turbulent diffusion was proposed in [14]. At the same time, R. Kraichnan introduced the space-time nonlocality into the diffusion model, but N. Romanov extended this model to diffusion in an anisotropic medium (see [11]), and gave the diffusion equation in the form

$$\frac{\partial\langle C(\boldsymbol{x}, t)\rangle}{\partial t} + \frac{\partial\langle u_i C(\boldsymbol{x}, t)\rangle}{\partial x_i} = \frac{\partial}{\partial x_i} \int\limits_0^t dt' \int\limits_{R^3} d\boldsymbol{x}' D_{ij}(\boldsymbol{x}, t \leftarrow \boldsymbol{x}', t') \frac{\partial\langle C(\boldsymbol{x}', t')\rangle}{\partial x'_j}.$$

If we add the capture model to the considered model (with some probability as a result of another collision the particle finds itself in a trap and stays there at random time, after which it continues to wander; an analogue of such a process takes place in nuclear reactors), we obtain a temporary nonlocality, which is eliminable if the time distribution of waiting in traps is exponential. Assuming a nonexponential nature of runs and rest times, we come to a group of nonlocal models called *continuous time random walk*, abbreviated as CTRW. The time interval between successive collisions is made up of time in the trap and the time of motion. Neglecting the latter (i.e., assuming the speed is infinite),

we obtain a jump process, which is often denoted by the same term, but to avoid uncertainty, we will use Yermin *random flights with continuous time* (RFCT). If runs (and often times in traps as well) are distributed according to the law with a heavy tail of a power type, then this process is called *Levy flights*. If power asymptotics have both distributions, then the equation for the propagator contains fractional derivatives both in coordinates (fractional Laplacian) and in time:

$$\frac{\partial^\beta g}{\partial t^\beta} + D_\alpha(-\Delta)^{\alpha/2} g(\boldsymbol{r},t) = \delta(\boldsymbol{r})\delta_\beta(t), \quad \alpha \in (0,2], \quad \beta \in (0,1] \tag{14}$$

(this equation is derived in the book [19], its one-dimensional analogue was obtained earlier in [12]). Here $\delta_\beta(t)$ is the Riemann—Liouville derivative of fractional order β with respect to time of the unit step function $1_+(t)$. Note that the dimension of each term in this equation is $L^{-3}T^{-\beta}$, and the coefficient D_α, which stands in the place of the diffusion coefficient, has the dimension $L^\alpha T^{-\beta}$ and has this form strictly speaking, only for $\alpha = 2$, $\beta = 1$. The solutions of these equations are expressed in terms of probability densities belonging to the class *fraction-stable distributions* (see [15]).

The disadvantage of these RFCT equations is the spasmodic view of the trajectories of the processes described by them, which is not very characteristic of representing the motion of an impurity in a turbulent flow. More suitable for this purpose are the equations derived in [16], in which the sum of partial derivatives of fractional orders in time and coordinates is replaced by the total derivative of fractional order in time, followed by averaging over directions. In natural variables, the corresponding equations are as follows:

$$\left\langle \left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla \right)^\alpha \right\rangle g(\boldsymbol{x},t) = S_\alpha(\boldsymbol{x},t), \qquad 0 < \alpha < 1;$$

$$\left[\frac{\partial}{\partial t} - \frac{A_\alpha}{v^{\alpha-1}\langle R\rangle} \left\langle \left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla \right)^\alpha \right\rangle \right] g(\boldsymbol{x},t) = \frac{1}{\langle R\rangle} S_\alpha(\boldsymbol{x},t), \quad 1 < \alpha < 2;$$

$$\left(\frac{\partial}{\partial t} - \frac{v\langle R^2\rangle}{6\langle R\rangle}\Delta \right) g(\boldsymbol{x},t) = \frac{1}{\langle R\rangle} S_2(\boldsymbol{x},t), \qquad \alpha > 2.$$

Here R denotes a random run, angle brackets mean averaging over the direction of velocity (its value is assumed to be constant), and S_α is the density of sources corresponding to a given propagator. The fractional power of the operator of the total time derivative is determined by the ratio

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla \right)^\alpha N(\boldsymbol{r},t) = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla \right) \left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla \right)^{\alpha-1} N(\boldsymbol{r},t)$$

$$= \left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla \right) \int\limits_0^t \frac{N(\boldsymbol{r} - \boldsymbol{v}(t-\tau),\tau)}{\Gamma(1-\alpha)(t-\tau)^\alpha} d\tau.$$

The reader can find details related to the derivation and application of these equations in the reviews [17, 18].

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V. V. Uchaikin
Ulyanovsk State University, Ulyanovsk, Moscow
E-mail: vuchaikin@gmail.com