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UNBLOCKED IMPUTATIONS OF FUZZY GAMES. II: NONEMPTINESS OF CORES OF TWO MARKET GAMES

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We consider a generalization of the Scarf theorem on the nonemptiness of game cores to the case of fuzzy cooperative games without side payments. We consider applications of the generalized Scarf theorem to finding the nonemptiness conditions for fuzzy cores of two market games generated by the model of pure economic exchange and the spatial regional model. Bibliography: 15 *titles.*

We continue the study of [1] devoted to a generalization of the well-known Scarf theorem on the nonemptiness of game cores [2]–[4] to the case of fuzzy cooperative games without side payments. Compared with the traditional games, we have more blocking possibilities due to fuzzy coalitions. In this paper, we consider applications of the generalized variant [1] of the Scarf theorem to finding the nonemptiness condition for fuzzy cores for two classes of economic models: the classical exchange models [3]–[5] and spatial interregional economic models describing important spatial regional systems [6]. In each case, we consider the class of cooperative games in the strategic form and, based on the close connections between such games and generalized fuzzy games, find the existence conditions for unblocked allocations.

The paper is organized as follows. In Section 1, we describe the economic models under consideration. In particular, we introduce some notions used in the theory of generalized cooperative games without side payments and formulate the generalized theorem proved in [1] concerning the nonemptiness of cores of such games. In Section 2, we reduce the existence problem for unblocked allocations of games in the strategic form to a similar problem for generalized games without side payments. Finally, in Section 3, we describe rather simple conditions for the fuzzy cores in the classical exchange model to be nonempty. A similar result is obtained in Section 4 for the spatial interregional economic model introduced in [6].

1 The Main Notions and Generalized Scarf Theorem

We recall some notions used in the theory of cooperative games (cf. [7]–[10] for details).

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For a natural number $n \geqslant 2$ we put $N = \{1, \dots, n\}$ and denote by $\sigma_0 = 2^N$ the collection of subsets of N. In the traditional game-theoretic terminology [7], elements of N are called *players* and elements of σ_0 are referred to as *coalitions*. Sometimes, it is convenient to identify coalitions $S \subseteq N$ with the corresponding vertices of the unit n-dimensional cube $I^n = \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^N \mid \tau_i \in [0,1], i \in N\}$ (here and below, $\mathbb{R}$ denotes the set of real numbers). With each coalition $S \in \sigma_0$ we associate its indicator function e^S defined by

$$(e^S)_i = \begin{cases} 1, & i \in S, \\ 0, & i \in N \setminus S. \end{cases}$$

We recall (cf., for example, [5]) that, in addition to the vertices, an important role in the description of the Walrasian allocations of pure exchange models is also played by other elements of the hypercube I^n. We set $\sigma_F = I^n \setminus \{0\}$ and call elements of σ_F *fuzzy coalitions* (cf. [8]). The quantity of a component τ_i of a fuzzy coalition $\tau = (\tau_1, \dots, \tau_n)$ is interpreted as the level of participation of player i in coordinating the efforts of the participants of the grand coalition N. For a fuzzy coalition $\tau \in \sigma_F$ we denote by $N(\tau)$ the *support* of τ: $N(\tau) = \{i \in N \mid \tau_i > 0\}$.

Following [1], we list some facts concerning fuzzy cooperative n-person games.

Definition 1. A *generalized fuzzy cooperative n-person game* is a set-valued mapping $\tau \mapsto G(\tau)$, $\tau \in \sigma_F$, sending a coalition $\tau \in \sigma_F$ to a subset $G(\tau)$ of the space $\mathbb{R}^{N(\tau)}$. Elements of $G(\tau)$ are called *payoff vectors of the coalition τ*, and the payoff vectors of the coalition e^N are also referred to as the *payoff vectors of the game G*. We define G at zero by setting $G(0) = \varnothing$.

We distinguish those coalitions of the game G that provide a "nontrivial" cooperation effect. By an *efficient set* of the game G we mean the set $e(G)$ of all fuzzy coalitions τ such that $G(\tau) \neq \varnothing$, i.e., $e(G) := \{\tau \in \sigma_F \mid G(\tau) \neq \varnothing\}$. Elements of the set $e(G)$ are called *blocking coalitions*. In what follows, we use the following assumption.

Assumption 1. The sets $G(e^{\{1\}}), \dots, G(e^{\{n\}})$ and $G(e^N)$ are nonempty and closed. In particular, the efficiency set of the game G contains all one-element coalitions and the coalition of all participants: $\{e^{\{1\}}, \dots, e^{\{n\}}, e^N\} \subseteq e(G)$.

Definition 2. A generalized fuzzy cooperative game G is called *regular* if it satisfies Assumption 1.

We recall [1] the notion of an F-balanced cover which plays a key role in this paper. Here, not only usual coalitions, but also fuzzy coalitions can be taken for covering elements, unlike the classical situation, where all covering elements are standard coalitions [10]. A finite family of fuzzy coalitions $\{\tau^k\}_{k \in K}$ is called an *F-balanced cover* of the set N if there exist nonnegative numbers $\{\lambda_k\}_{k \in K}$ such that $\sum\limits_{k \in K} \lambda_k \tau^k = e^N$. By analogy with the classical case, λ_k are called *weights* of the fuzzy coalitions τ^k. We introduce an analog of the balanced vector [1, 4] for the generalized game G (in what follows, $u_S \in \mathbb{R}^S$ denotes the restriction of a vector $u = (u_1, \dots, u_n) \in \mathbb{R}^N$ onto $S \subseteq N : (u_S)_i = u_i, \quad i \in S$). Let G be a generalized cooperative n-person game. A vector $u \in \mathbb{R}^N$ is *G-balanced* if there is an F-balanced cover $\{\tau^k\}_{k \in K}$ of the set N such that $u_{N(\tau^k)} \in G(\tau^k)$ for all $k \in K$.

Definition 3. A game G is *F-balanced* if any G-balanced vector belongs to $G(N)$.

Definition 4. We say that a coalition $\tau \in e(G)$ *blocks* a payoff vector $u = (u_1, \dots, u_n) \in G(N)$ if there exists a vector $v = (v_i)_{i \in N(\tau)} \in G(\tau)$ such that $v_i > u_i$ for all $i \in N(\tau)$. The set of all payoff vectors in $G(N)$ that are not blocked by any coalition $\tau \in e(G)$ is denoted by $C(G)$ and called the *core* of the game G.

We recall (cf., for example, [2, 4]) that the notion of blocking in the classical cooperative game is defined in the same way as in Definition 4 (for blocking coalitions in σ_0). Unlike the classical situation, we introduce the notion of blocking not only for elements of a finite set of $2^n - 1$ standard coalitions, but also for all coalitions in $e(G)$.

We recall some assumptions of the classical Scarf theorem [2] that will be used to generalize this theorem. We set

$$u_i^0 = u_i^G := \sup \ \{u_i \in \mathbb{R} \mid u_i \in G(e^i)\}, \quad i \in N. \tag{1}$$

It is clear that the necessary condition for the core $C(G)$ to be nonempty is that for each player $i \in N$ the maximal guaranteed payoff u_i^G of this player is finite (otherwise, the one-element coalition e^i can block any vector in $G(N)$). Therefore, throughout the paper, we assume that the following condition is satisfied.

Assumption 2. u_i^G *are finite for all participants of the game* G.

As in the classical Scarf theorem, we introduce the set of individually rational payoff vectors of the grand coalition N:

$$\widehat{G}(N) := \{u \in G(N) \mid u \geqslant u^G\},$$

where $u^G = (u_1^G, \ldots, u_n^G)$ is a vector in $\mathbb{R}^n$ with components defined by (1). Hereinafter, we identify the notation N and e^N for the sake of brevity. Moreover, we use the standard abbreviations: for vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ in $\mathbb{R}^m$ we set $x \geqslant y$ $(x \gg y)$ $\Leftrightarrow x_k \geqslant y_k$ $(x_k > y_k)$, $k = 1, \ldots, m$. The term *individually rational* will be applied to all remaining coalitions τ: the vector $u \in G(\tau)$ is called the *individually rational payoff vector* of the coalition τ if $u \geqslant u^G_{N(\tau)}$, where $u^G_{N(\tau)}$ is the restriction of u^G onto $N(\tau)$: $u^G_{N(\tau)} = (u_i^G)_{i \in N(\tau)}$. The set of individually rational payoff vectors of a coalition τ is denoted by $\widehat{G}(\tau)$: $\widehat{G}(\tau) = \{u \in G(\tau) \mid u \geqslant u^G_{N(\tau)}\}$.

We introduce a useful analog of the notion used in the classical theory of cooperative game; namely, the set of imputations of the game without side payments [9]. By *imputations* of a generalized game G we mean elements of the set $I(G)$ of collectively rational payoff vectors in $\widehat{G}(e_N)$, i.e.,

$$I(G) := \bigl\{u \in \widehat{G}(N) \bigm| \text{there are no } v \in G(N) \text{ such that } v \gg u\bigr\}.$$

Remark 1. It is clear that the set $I(G)$ of imputations of a generalized game G, as in the case of usual games, is a collection of all payoff vectors of the grand coalition e^N blocked by neither one-element coalitions $e^{\{1\}}, \ldots, e^{\{n\}}$ nor the coalition of all players e^N. Hence $C(G) \subseteq I(G)$ for any regular game G. Therefore, in what follows, elements of the core $C(G)$ of a regular game G are called, *unblocked imputations*.

We also recall that the set $X \subseteq \mathbb{R}^m$ is said to be *comprehensive from below* if, together with each element $x \in X$, the set X contains any element $y \leqslant x$.

Definition 5. We say that a game G is *comprehensive from below* if all sets $G(\tau)$, $\tau \in e(G)$, are comprehensive from below.

As established in [1], the Scarf theorem on the nonemptiness of the core of classical cooperative games [2] admits the following generalization.

Theorem 1 (cf. [1])**.** *If a regular generalized fuzzy cooperative game G is comprehensive from below and F-balanced, all sets $G(\tau)$ are closed, and the set $\widehat{G}(N)$ is bounded from above, then the core of G is not empty.*

This generalization of the Scarf theorem will be used to find the nonemptiness conditions for fuzzy cores in pure exchange models and spatial interregional economic systems.

2 Cores of Cooperative Games in Strategic Form. Reduction to Generalized Cooperative Games

In the further analysis of fuzzy cores of particular economic systems, an important role is played by the "reduction" result established below, which concerns the nonemptiness of cores of the so-called cooperative games in the strategic form [8]. According to this result, the existence problem for unblocked allocations of complicated economic models is reduced to a similar problem for simpler generalized games without side payments (which allows us to use the generalized Scarf theorem formulated in Section 1).

We introduce necessary definitions. As above, we set $N := \{1, \dots, n\}$ and denote by σ_F the collection of fuzzy coalitions obtained from elements of the set N. For every $i \in N$ we fix a nonempty set X_i and a function $u_i : X_i \to \mathbb{R}$. Assume that for each coalition $\tau \in \sigma_F$ we are given some subset $X(\tau)$ of the set $\prod\limits_{i \in N(\tau)} X_i$. We recall that $N(\tau)$ denotes the support of a fuzzy coalition $\tau = (\tau_1, \dots, \tau_n)$: $N(\tau) = \{i \in N \mid \tau_i > 0\}$.

Definition 6. The system $\Gamma = \langle N, \{X_i, u_i\}_{i \in N}, \{X(\tau)\}_{\tau \in \sigma_F} \rangle$ is called a *generalized cooperative n-person game in the strategic form*, where N is the set of participants of the game Γ, X_i is the set of individual strategies used by player $i \in N$, $u_i : X_i \to \mathbb{R}$ is the payoff function of player $i \in N$, and $X(\tau) \subseteq \prod\limits_{i \in N(\tau)} X_i$ is the set of collective strategies of the coalition $\tau \in \sigma_F$.

Definition 7. A fuzzy coalition τ *blocks* a strategy $x = (x_i)_{i \in N} \in X(N)$ of the grand coalition N if there exists a coalition strategy $(\widetilde{x}_i)_{i \in N(\tau)} \in X(\tau)$ such that $u_i(\widetilde{x}_i) > u_i(x_i)$ for $i \in N(\tau)$. The set of collective strategies $\overline{x}$ in $X(N)$, that are not blocked by any coalition in σ_F is denoted by $C_F(\Gamma)$ and called the *core* of the game Γ.

To analyze the nonemptiness conditions for the core $C_F(\Gamma)$, it is more convenient to consider the generalized cooperative game G^F_Γ associated with Γ and defined by

$$G^F_\Gamma(\tau) = \bigl\{ v \in \mathbb{R}^{N(\tau)} \bigm| \exists (x_i)_{i \in N(\tau)} \in X(\tau) \; [v_i \leqslant u_i(x_i),\ i \in N(\tau)] \bigr\}, \quad \tau \in \sigma_F.$$

For the core of the game Γ and the associated generalized cooperative game G^F_Γ the following simple, but important fact holds.

Proposition 1. *The core $C_F(\Gamma)$ of the game Γ is nonempty if and only if the core $C(G^F_\Gamma)$ of the corresponding generalized cooperative game G^F_Γ is nonempty.*

Proof. The relation

$$C_F(\Gamma) \neq \varnothing \iff C(G^F_\Gamma) \neq \varnothing \tag{2}$$

is obtain from the implications below.

1) $C_F(\Gamma) \neq \varnothing \Rightarrow C(G_\Gamma^F) \neq \varnothing$. Indeed, let a collective strategy $\overline{x} = (\overline{x}_i)_{i\in N}$ belong to the core $C_F(\Gamma)$. We show that the imputation $\overline{u} = (u_i(\overline{x_i}))_{i\in N}$ of the coalition N belongs to the core $C(G_\Gamma^F)$ of the game G_Γ^F. Assume the contrary. Then there exists a coalition $\tau \in \sigma_F$, a strategy $(x_i)_{i\in N(\tau)} \in X(\tau)$, and a vector $v \in \mathbb{R}^{N(\tau)}$ such that $v \leqslant u_i(x_i)_{i\in N(\tau)}$; moreover $\overline{u}_{N(\tau)} \ll v$. By the last two inequalities, for the collective strategy $x = (x_i)_{i\in N(\tau)}$ of the coalition τ we have $u_i(x_i) > u_i(\overline{x}_i)$, $i \in N(\tau)$, which contradicts the assumption that the strategy $\overline{x}$ is unblocked.

2) $C(G_\Gamma^F) \neq \varnothing \Rightarrow C_F(\Gamma) \neq \varnothing$. Indeed, let $\overline{u}$ be an arbitrary imputation in the core $C(G_\Gamma^F)$. According to the definition of the game G_Γ^F, there exists a collective strategy $\overline{x} = (\overline{x_i})_{i\in N}$ of the coalition N such that $\overline{u} \leqslant (u_i(\overline{x}_i))_{i\in N}$. Let us show that $\overline{x}$ belongs to the core $C_F(\Gamma)$. Assume the contrary. Then there exists a coalition τ and its collective strategy $(x_i)_{i\in N(\tau)} \in X(\tau)$ such that $u_i(x_i) > u_i(\overline{x}_i)$ for all $i \in N(\tau)$. Setting $v = (u_i(x_i))_{i\in N(\tau)}$, we see that $v_i > u_i(\overline{x}_i)$ for all $i \in N(\tau)$. However, by the construction of v, we have $v \in G_\Gamma^F(\tau)$ and $v \gg \overline{u}_{N(\tau)}$, which contradicts the assumption $\overline{u} \in C(G_\Gamma^F)$.

Thus, we have proved (2), which completes the proof of Proposition 1. $\square$

3 Unblocked Allocations in Economic Exchange Model

We use Theorem 1 (the generalized Scarf theorem) to establish a rather unexpected result: the standard nonemptiness conditions for a usual core of the economic exchange model turn out to be sufficient for the nonemptiness of a much narrower fuzzy core of this model. The exchange model under consideration can be formally described by

$$\mathscr{E} = \langle N, \{X_i, u_i, w^i\}_{i\in N}\rangle,$$

where $N = \{1, \ldots, n\}$ is the set of participants, $X_i \subseteq \mathbb{R}^l$ are the consumption sets, $w^i \in \mathbb{R}^l$ are the endowments, and $u_i : X_i \to \mathbb{R}$ are the utility functions. The natural number l denotes the number of goods exchanged by the participants (economic agents) in the model. As was already mentioned, the individual initial endowments are determined by the vectors w^i, $i \in N$ (a more detailed description of parameters of the model $\mathscr{E}$ can be found, for example, in [3, 5, 4]).

Strategic possibilities $X(\tau) = X_{\mathscr{E}}(\tau)$ of a fuzzy coalition $\tau = (\tau_1, \ldots, \tau_n)$ in this model can be described as follows (cf., for example, [5, 8]):

$$X_{\mathscr{E}}(\tau) = \Big\{(x^i)_{i\in N(\tau)} \in \prod_{i\in N(\tau)} X_i \mid \sum_{i\in N(\tau)} \tau_i x^i = \sum_{i\in N(\tau)} \tau_i w^i\Big\}, \quad \tau \in \sigma_F. \tag{3}$$

In particular, the strategic possibilities of the grand coalition N are determined as the set $X(N) = X_{\mathscr{E}}(N)$ of all possible allocations of the total initial endowment of the exchange participants

$$X_{\mathscr{E}}(N) = \Big\{(x^i)_{i\in N} \in \prod_{i\in N} X_i \mid \sum_{i\in N} x^i = \sum_{i\in N} w^i\Big\},$$

and the strategic possibilities of one-element coalitions $\{i\}$ are exhausted by one-element sets $\{w^i\}$ (if w^i belongs to the consumption set X_i; otherwise, $X_{\mathscr{E}}(e^i) = \varnothing$). We note that we identify the notation $\{i\}$, $e^{\{i\}}$, and e^i for one-element coalitions.

We recall the definition of F-blocking (by fuzzy coalitions) in the model $\mathscr{E}$. It is a generalization of the standard notion of blocking coming back to Edgeworth (cf. also [8, 5]). The sets $X_{\mathscr{E}}(\tau)$ considered below are defined by formula (3).

Definition 8. We say that a coalition $\tau \in \sigma_F$ *blocks* an allocation $x = (x^i)_{i\in N} \in X_{\mathscr{E}}(N)$ if there exists $\widetilde{x} = (\widetilde{x}^i)_{i\in N(\tau)} \in X_{\mathscr{E}}(\tau)$ such that $u_i(\widetilde{x}^i) > u_i(x^i)$ for every $i \in N(\tau)$. The set of allocations in $X_{\mathscr{E}}(N)$ that are not blocked by any coalitions in σ_F is denoted by $C_F(\mathscr{E})$ and called the *fuzzy core* of the model $\mathscr{E}$.

Remark 2. Recall (cf., for example, [5]) that the classical notion of blocking can be obtained from F-blocking by restriction to the standard coalitions constituting the family σ_0 : a coalition $S \subseteq N$ blocks an allocation $x = (x^i)_{i\in N} \in X(N)$ if there exists $\widetilde{x} = (\widetilde{x}^i)_{i\in S} \in X(S)$ such that $u_i(\widetilde{x}^i) > u_i(x^i)$ for any $i \in S$, where

$$X(S) = \Big\{(x^i)_{i\in S} \in \prod_{i\in S} X_i \mid \sum_{i\in S} x^i = \sum_{i\in S} w^i\Big\}, \quad S \subseteq N.$$

The set of allocations in $X(N)$ that are not blocked by any coalitions $S \in \sigma_0$ is denoted by $C(\mathscr{E})$ and called the (*standard*) *core* of the model $\mathscr{E}$.

We note that the classical characteristic function $G_{\mathscr{E}}$ of the exchange model $\mathscr{E}$ is defined by

$$G_{\mathscr{E}}(S) = \{v \in \mathbb{R}^S \bigm| \exists (x^i)_{i\in S} \in X(S)[v_i \leqslant u_i(x^i),\ i \in S], \quad S \subseteq N.$$

Finally, we introduce the notion of a generalized cooperative game $G^F_{\mathscr{E}}$ associated with the economic exchange model $\mathscr{E}$ and characterizing the utility levels achievable by the efforts of some fuzzy coalition τ.

Definition 9. A generalized cooperative game $G^F_{\mathscr{E}}$ associated with the model $\mathscr{E}$ is defined by the formula

$$G^F_{\mathscr{E}}(\tau) = \Big\{v \in \mathbb{R}^{N(\tau)} \mid \exists \big(x^i\big)_{i\in N(\tau)} \in X_{\mathscr{E}}(\tau)\left[v_i \leqslant u_i\big(x^i\big),\ i \in N(\tau)\right]\Big\}, \quad \tau \in \sigma_F.$$

To distinguish the extension of the classical characteristic function $G_{\mathscr{E}}$ of the market game $\mathscr{E}$ from the vertices e^S of the unit cube I^n to the entire cube I^n, we use the subscript F, i.e., the extension is denoted by $G^F_{\mathscr{E}}$. As usual, we set $G^F_{\mathscr{E}}(0) = \varnothing$.

Remark 3. By the definition of $G^F_{\mathscr{E}}$, if the initial endowment w^i of each participant $i \in N$ belongs to the corresponding consumption set X_i, then $G^F_{\mathscr{E}}(i) = G^F_{\mathscr{E}}(e^i) = \big\{v_i \in \mathbb{R} \mid v_i \leqslant u_i(w^i)\big\}$, $i \in N$, which means that $u^0_i = \sup\big\{v \in G^F_{\mathscr{E}}(i)\big\} = u_i(w^i)$ for all $i \in N$.

We indicate a condition for a generalized cooperative game associated with the exchange model $\mathscr{E}$ to be F-balanced.

We recall that u_i is *quasiconcave* if $u_i(tx^i + (1-t)y^i) \geqslant \min\ \{u_i(x^i), u_i(y^i)\}$ for all $t \in [0, 1]$ and $x^i, y^i \in X_i$.

Proposition 2. *If the consumption sets X_i of the exchange model $\mathscr{E}$ are convex and the utility functions u_i are quasiconvcave, then the game $G^F_{\mathscr{E}}$ is F-balanced.*

Proof. Let $\{\tau^k\}_{k\in K}$ be a finite family of coalitions in σ_F which forms a balanced cover of N with weights λ_k, $k \in K$. We consider an arbitrary vector $v \in \mathbb{R}^n$ such that $v_{N_k} \in G^F_{\mathscr{E}}(\tau^k)$ for all $k \in K$, where $N_k = N(\tau^k)$, $k \in K$. By the definition of $G^F_{\mathscr{E}}$, for every $k \in K$ there is an allocation $(x^{k,i})_{i\in N_k} \in X(\tau^k)$ such that

$$v_i \leqslant u_i(x^{k,i}), \quad i \in N_k. \tag{4}$$

We set $\mu_{ki} = \lambda_k \tau_i^k$, $i \in N_k$, $k \in K$, and define the allocation $\overline{x} = (\overline{x}^i)_{i \in N}$ by

$$\overline{x}^i = \sum_{k \in K_i} \mu_{ki} x^{k,i}, \quad i \in N, \tag{5}$$

where $K_i = \{k \in K \mid i \in N_k\}$, $i \in N$. Taking into account that

$$\sum_{k \in K} \lambda_k \tau^k = e^N$$

and using the definition of μ_{ki}, we find that $\mu_{ki} \geqslant 0$ for $k \in K$ and $i \in N$; moreover,

$$\sum_{k \in K_i} \mu_{ki} = 1 \quad \forall i \in N.$$

Since the sets X_i are convex, from the inclusions $x^{k,i} \in X_i$ and formula (5) defining $\overline{x}^i$ we find that $\overline{x}^i \in X_i$ for every $i \in N$. Using again the fact that $\overline{x}^i$ are convex combinations of elements of X_i and taking into account the quasiconcavity of u_i and the inequalities (4), we get $u_i(\overline{x}^i) \geqslant v_i$ for every $i \in N$. To complete the proof of the inclusion $v \in G_{\mathscr{E}}^F(N)$, it remains to show that

$$\sum_{i \in N} \overline{x}^i = \sum_{i \in N} w^i$$

for $\overline{x} = (\overline{x}^i)_{i \in N}$. Making elementary transformations and taking into account (5), we find

$$\sum_{i \in N} \overline{x}^i = \sum_{i \in N} \sum_{k \in K_i} \mu_{ki} x^{k,i} = \sum_{k \in K} \sum_{i \in N_k} \mu_{ki} x^{k,i} = \sum_{k \in K} \lambda_k \Big(\sum_{i \in N_k} \tau_i^k x^{k,i} \Big).$$

But the last sum takes the form

$$\sum_{k \in K} \lambda_k \Big(\sum_{i \in N_k} \tau_i^k w^i \Big)$$

in view of the relation

$$\sum_{i \in N_k} \tau_i^k x^{k,i} = \sum_{i \in N_k} \tau_i^k w^i.$$

Exchanging the summation order, we get

$$\sum_{i \in N} \Big(\sum_{k \in K_i} \mu_{ki} \Big) w^i = \sum_{i \in N} w^i,$$

which is required. □

Proposition 3. *If the consumption sets X_i of the model $\mathscr{E}$ are closed bounded from below and include the initial endowments of the participants ($w^i \in X_i$, $i \in N$), whereas the utility functions u_i are continuous for every $i \in N$, then the sets $G_{\mathscr{E}}^F(\tau)$ are nonempty, closed, and comprehensive from below for all coalitions $\tau \in \sigma_F$; moreover, the set $\widehat{G}_{\mathscr{E}}^F(N)$ is nonempty and bounded from above.*

Proof. It is clear that all sets $X(\tau) = X_{\mathscr{E}}(\tau)$ are not empty since the allocations $(w^i)_{i \in N(\tau)}$ belong to $X(\tau)$ for all $\tau \in \sigma_F$ (which is a direct consequence of formula (3) defining the sets $X(\tau)$ and the assumption $w^i \in X_i$, $i \in N$). It is easy to verify that if the sets X_i are bounded from below, then all sets $X(\tau)$ are bounded: namely, for every $\tau \in \sigma_F$ there exists a positive number L_τ such that for any allocation $x = (x^i)_{i \in N(\tau)} \in X(\tau)$ the inequality $\|x^i\|_\infty \leqslant L_\tau$ holds for all $i \in N(\tau)$ (here, as usual, $\|x\|_\infty := \max\{|x_r| \mid r = 1, \dots, m\}$ for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$). Indeed, by assumption, for every $i \in N$ there exists a vector $a^i \in \mathbb{R}^l$ such that $y^i \geqslant a^i$ for every y^i in X_i. Therefore, for any allocation $x = (x^i)_{i \in N(\tau)} \in X(\tau)$ from the inequalities $x^i \geqslant a^i$ and the identity

$$\sum_{i \in N(\tau)} \tau_i x^i = \sum_{i \in N(\tau)} \tau_i w^i$$

we get

$$x^i = w_i(\tau) - 1/\tau_i \Big[\sum_{j \in N_{-i}(\tau)} \tau_j x^j \Big] \leqslant w_i(\tau) + a_i(\tau),$$

where

$$w_i(\tau) = 1/\tau_i \Big[\sum_{j \in N(\tau)} \tau_j w^j \Big], \quad a_i(\tau) = -1/\tau_i \Big[\sum_{j \in N_{-i}(\tau)} \tau_j a^j \Big],$$

and $N_{-i}(\tau) = N(\tau) \setminus \{i\}$. Setting $b^i(\tau) = w_i(\tau) + a_i(\tau)$, we have $a^i \leqslant x^i \leqslant b^i(\tau)$, $i \in N(\tau)$, for any $x = (x^i)_{i \in N(\tau)} \in X(\tau)$. Consequently, the set $X(\tau)$ is also bounded in the above sense: $\|x^i\|_\infty \leqslant L_\tau$ for all $x = (x^i)_{i \in N(\tau)} \in X(\tau)$, $i \in N(\tau)$, and some $L_\tau > 0$. Finally, by the obvious identity $\|x\|_\infty = \max\limits_{i \in N(\tau)} \|x^i\|_\infty$ for any $x = (x^i)_{i \in N(\tau)} \in X(\tau)$, we obtain the required assertion: $\|y\|_\infty \leqslant L_\tau$ for all $y \in X(\tau)$.

Thus, all sets $X(\tau)$ are bounded. Taking into account that the identity

$$\sum_{i \in N(\tau)} \tau_i x^i = \sum_{i \in N(\tau)} \tau_i w^i$$

defining these sets is preserved under the passage to the limit with respect to x, we conclude that the closedness of the sets X_i for $i \in N(\tau)$ implies the closedness of all sets $X(\tau)$. Therefore, for each coalition $\tau \in \sigma_F$ the set $X(\tau)$ is a nonempty compact set. But the sets $G^F_{\mathscr{E}}(\tau)$ are closed since they are algebraic sums of compact sets $U(\tau)$ and cones $-\mathbb{R}^{N(\tau)}_+$, where $U(\tau) = \{u_i(x^i)_{i \in N(\tau)} \big| (x^i)_{i \in N(\tau)} \in X(\tau)\}$. Indeed, by the definition of $G^F_{\mathscr{E}}(\tau)$, we have

$$G^F_{\mathscr{E}}(\tau) = U(\tau) - \mathbb{R}^{N(\tau)}_+, \quad \tau \in \sigma_F, \tag{6}$$

where $\mathbb{R}^{N(\tau)}_+$ is a closed set and $U(\tau)$ is a compact set because it is the continuous image of the compact set $X(\tau)$. We recall that all functions u_i are continuous under the assumptions of Proposition 3. Since the algebraic sum of a compact set and a closed set is a closed set in a finite-dimensional space, from (6) we obtain the required assertion, i.e., if the sets X_i are closed and bounded from below and the functions u_i are continuous, then all sets $G^F_{\mathscr{E}}(\tau)$ are closed. We note that these sets are comprehensive from below by definition.

To complete the proof of Proposition 3, it remains to note that the nonemptiness of the set $\widehat{G}^F_{\mathscr{E}}(N)$ follows from the inclusion $(w^i)_{i \in N} \in X(N)$ and the relations $u^0_i = \sup \{v_i \in \mathbb{R} \mid v_i \in G^F_{\mathscr{E}}(i)\} = u_i(w^i)$, $i \in N$ (obtained from the assumption $w^i \in X_i$, $i \in N$). The boundedness of $\widehat{G}^F_{\mathscr{E}}(N) = \widehat{G}^F_{\mathscr{E}}(e^N)$ from above follows from the obvious relation $\widehat{G}^F_{\mathscr{E}}(e^N) \leqslant U(e^N)$ and the compactness of $U(\tau)$ (as usual, for $X, Y \subseteq \mathbb{R}^m$ the relation $X \leqslant Y$ is understood in the sense that for any $x \in X$ there is $y = y(x) \in Y$ such that $x \leqslant y$). □

It turns out that, under the same assumptions as in the well-known theorem on the nonemptiness of the usual core of the pure exchange model [3, 5], a much more subtle optimality-stability principle for F-blocking is realizable. Namely, under the standard assumptions guaranteeing the nonemptiness of the usual core, we establish the existence of allocations that are stable not only under standard blocking, but also under blocking by a continual set of fuzzy coalitions. By standard blocking we understand the classical blocking defined only for the set of $(2^n - 1)$ nonempty standard coalitions.

Theorem 2. *If the initial endowments of the participants of the model $\mathscr{E}$ belong to their consumption sets ($w^i \in X_i$, $i \in N$)), X_i are closed and bounded from below for every $i \in N$, and the utility functions u_i are continuous and quasiconcave for all $i \in N$, then the fuzzy core $C_F(\mathscr{E})$ of this model is nonempty.*

Proof. By Propositions 2 and 3, if the assumptions of Theorem 2 hold, then the generalized cooperative game $G^F_{\mathscr{E}}$ satisfies all the assumptions of Theorem 1. Hence the core $C(G^F_{\mathscr{E}})$ is not empty. By Proposition 1, we have the nonemptiness of the core of the cooperative game in the strategic form $\Gamma_{\mathscr{E}} = \langle N, \{X_i, u_i\}_{i \in N}, \{X_{\mathscr{E}}(\tau)\}_{\tau \in \sigma_F}\rangle$, which coincides with the core $C_F(\mathscr{E})$ of the economic exchange model $\mathscr{E}$. □

4 Fuzzy Core of Many-Regional Economic System

The economic model of interaction of regions exchanging m transportable products is expressed by [6]:

$$\mathscr{M} = \langle N, \{A^s, G^s, H^s, b^s, d^s\}_{s \in N}\rangle, \tag{7}$$

where $N = \{1, \dots, n\}$ is the list of regions, A^s is an $n_s \times l_s$-matrix characterizing the production sector of region $s \in N$, G^s and H^s are $n_s \times m$-matrices describing the export and import methods in region $s \in N$, b^s is the n_s-dimensional column-vector characterizing the resource–technological potential of region $s \in N$, d^s is the n_s-dimensional column-vector describing the costs of resources and products caused by the achievement of the goals of development of region $s \in N$.

A detailed discussion of the interpretation and important applications of the model (7), as well as some generalizations can be found in [6] and [11]–[13]. In this paper, we formulate only the definitions necessary to introduce the notions of a core and a fuzzy core of the model $\mathscr{M}$.

The resource–technological potentials Z_s of a region $s \in N$ are defined by

$$Z_s = \{z^s = (x^s, u^s, v^s, \lambda_s) \in \mathbb{R}^{l_s}_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+ \mid A^s x^s + G^s u^s + H^s v^s \geqslant b^s + \lambda_s d^s\},$$

where the nonnegative column-vectors $x^s = (x^s_i)^{l_s}_{i=1}$, $u^s = (u^s_j)^m_{j=1}$, and $v^s = (v^s_j)^m_{j=1}$ are the volumes of production, export, and import respectively, and $\lambda_s \in \mathbb{R}_+$ is the level of achieving the goals of development of region $s \in N$ (as above, $\mathbb{R}$ stands for the set of real numbers and inequalities for vectors are understood in the component sense: $x \geqslant y$ means that $x_k \geqslant y_k$, $k = 1, \dots, l$ for any $x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_l)$ in $\mathbb{R}^l$). Elements of Z_s are called *plans* of region s.

The quality of a plan $z^s \in Z_s$ is estimated by the function t_s sending the vector $z^s = (x^s, u^s, v^s, \lambda_s)$ to its last component λ_s:

$$t_s(z^s) = t_s(x^s, u^s, v^s, \lambda_s) = \lambda_s, \quad (x^s, u^s, v^s, \lambda_s) \in Z_s, \quad s \in N. \tag{8}$$

In other words, the mapping $t_s : Z_s \to \mathbb{R}$ is the objective function of participant $s \in N$ characterizing the level of achieving the goals of regional development.

We set $Z_{\mathscr{M}} = \prod_{s\in N} Z_s$ and denote by $Z_{\mathscr{M}}(N)$ the set of balanced plans of the model $\mathscr{M}$

$$Z_{\mathscr{M}}(N) = \Big\{(x^s, u^s, v^s, \lambda_s)_{s\in N} \in Z_{\mathscr{M}} \;\Big|\; \sum_{s\in N} u^s \geqslant \sum_{s\in N} v^s\Big\}.$$

Together with plans of the grand coalition N, we also consider balanced plans $Z_{\mathscr{M}}(S)$ of other standard coalitions $S \subseteq N$

$$Z_{\mathscr{M}}(S) = \Big\{(x^s, u^s, v^s, \lambda_s)_{s\in S} \in Z_{\mathscr{M}} \;\Big|\; \sum_{s\in S} u^s \geqslant \sum_{s\in S} v^s\Big\}. \tag{9}$$

A particular role in the analysis of the coalition stability of plans in $Z_{\mathscr{M}}(N)$ is played by one-element coalitions $\{s\}$. The corresponding sets of balanced plans

$$Z(s) = Z_{\mathscr{M}}(s) = \Big\{(x^s, u^s, v^s, \lambda_s) \in Z_s \;\Big|\; u^s \geqslant v^s\Big\}, \quad s \in N,$$

will be referred to as the sets of *autarchic plans* of the corresponding regions. Furthermore, to analyze the boundedness conditions for $Z_{\mathscr{M}}(N)$, it is required to consider balanced plans of the *homogeneous model* $\mathscr{M}$ defined by

$$\mathscr{M}_0 = \langle N, \{A^s, G^s, H^s, 0, d^s\}_{s\in N}\rangle. \tag{10}$$

By definition, the model $\mathscr{M}_0$ differs from $\mathscr{M}$ by the fact that its initial resource–technological potential vanishes: $b^s = 0$ for every $s \in N$. Consequently, the set $Z_{\mathscr{M}_0}(N)$ of balanced plans of the model $\mathscr{M}_0$ is the set of solutions to the homogeneous system of linear inequalities

$$A^s x^s + G^s u^s + H^s v^s - \lambda_s d^s \geqslant 0, \quad x^s \geqslant 0, \quad u^s \geqslant 0, \quad v^s \geqslant 0, \quad \lambda_s \geqslant 0, \quad s \in N;$$

$$\sum_{s\in N} u^s - \sum_{s\in N} v^s \geqslant 0.$$

We recall [11] the definition of a standard core of the model $\mathscr{M}$ which will be the basis of the notion of a fuzzy core introduced below.

Definition 10. We say that a plan $z = (z^s)_{s\in N} \in Z_{\mathscr{M}}(N)$ *is blocked* by a (standard) coalition $S \subseteq N$ ($S \neq \varnothing$) if there exist regional plans $\widetilde{z}^s = (\widetilde{x}^s, \widetilde{u}^s, \widetilde{v}^s, \widetilde{\lambda}_s) \in Z_s$, $s \in S$, such that

$$\sum_{s\in S} \widetilde{u}^s \geqslant \sum_{s\in S} \widetilde{v}^s$$

and $t_s(\widetilde{z}^s) > t_s(z^s)$ for all $s \in S$. The set of plans in $Z_{\mathscr{M}}(N)$ that are not blocked by any coalition $S \subseteq N$ is denoted by $C(\mathscr{M})$ and called the (*standard*) *core* of the model $\mathscr{M}$.

We indicate simple conditions (cf. [12]) guaranteeing that the standard core $C(\mathscr{M})$ of the model $\mathscr{M}$ is not empty. These conditions are related to individual properties of regions, as well as to some integral characteristics of the $\mathscr{M}$ in whole:

$(M1)$ $Z_{\mathscr{M}}(s) \neq \varnothing$ for every $s \in N$,

$(M2)$ $Z_{\mathscr{M}_0}(N) = \{0\}$.

Remark 4. Assumption $(M1)$ means a certain autarchy of regions of the model $\mathscr{M}$ in the sense that each region has at least one autarchic plan. Assumption $(M2)$ can be interpreted as the absence of a "horn of plenty" in the system $\mathscr{M}$. We note that the homogeneous component $\mathscr{M}_0$ defined by (10) differs from the model $\mathscr{M}$ only by the fact that the resource–technological potential of every region vanishes. Therefore, the fact that there is no "horn of plenty" in the system means, as in the classical model of equilibrium analysis [14], that if the system has zero economic potential, then the system cannot have any economic activity. We also note that the relation $Z_{\mathscr{M}_0}(N) = \{0\}$ in the formal plan is necessary and sufficient for the boundedness of the set $Z_{\mathscr{M}}(N)$ of balanced plans of the model $\mathscr{M}$ (which is valid because $Z_{\mathscr{M}}(N)$ is polyhedral and in view of results of convex analysis [15]).

We recall [11] the definition of F-blocking on $Z_{\mathscr{M}}(N)$ based on the natural extension on σ_F of the mapping $S \mapsto Z_{\mathscr{M}}(S)$ given by (9) for standard coalitions $S \subseteq N$. This natural extension $\tau \mapsto Z_{\mathscr{M}}(\tau)$ has the form

$$Z_{\mathscr{M}}(\tau) = \Big\{(x^s, u^s, v^s, \lambda_s)_{s\in N(\tau)} \in \prod_{s\in N(\tau)} Z_s \mid \sum_{s\in N(\tau)} \tau_s u^s \geqslant \sum_{s\in N(\tau)} \tau_s v^s\Big\}, \quad \tau \in \sigma_F.$$

Definition 11. We say that a plan $z = (z^s)_{s\in N} \in Z_{\mathscr{M}}(N)$ is blocked by a fuzzy coalition $\tau = (\tau_1, \dots, \tau_n) \in \sigma_F$ if there exist regional plans $\widetilde{z}^s = (\widetilde{x}^s, \widetilde{u}^s, \widetilde{v}^s, \widetilde{\lambda}_s) \in Z_s$, $s \in N(\tau)$, such that

$$\sum_{s\in N(\tau)} \tau_s \widetilde{u}^s \geqslant \sum_{s\in N(\tau)} \tau_s \widetilde{v}^s$$

and $t_s(\widetilde{z}^s) > t_s(z^s)$ for all $s \in N(\tau)$. The set of plans in $Z_{\mathscr{M}}(N)$ that are not blocked by any fuzzy coalition $\tau \in \sigma_F$ is denoted by $C_F(\mathscr{M})$ and called the *fuzzy core* of the model $\mathscr{M}$.

According to the above scheme, with a cooperative game in the strategic form $\Gamma_{\mathscr{M}} = \langle N, \{Z_s, t_s\}_{s\in N}, \{Z_{\mathscr{M}}(\tau)\}_{\tau\in\sigma_F}\rangle$ we associate the generalized cooperative game $G^F_{\mathscr{M}}$ defined by

$$G^F_{\mathscr{M}}(\tau) = \Big\{v \in \mathbb{R}^{N(\tau)} \mid \exists\, (z^s)_{s\in N(\tau)} \in Z_{\mathscr{M}}(\tau) \left[v_s \leqslant t_s(z^s),\ s \in N(\tau)\right]\Big\}, \quad \tau \in \sigma_F;$$

here and below, t_s is defined by (8).

Proposition 4. *For any data of the model $\mathscr{M}$ the game $G^F_{\mathscr{M}}$ is F-balanced.*

Proof. It is clear that for $G^F_{\mathscr{M}}(\tau) = \varnothing$ for all $\tau \in \sigma_F$ the required assertion immediately follows from the definition of balance. We consider the nontrivial case where for some balanced cover $\{\tau^k\}_{k\in K}$ of the coalition N all sets $G_{\mathscr{M}}(\tau^k)$, $k \in K$, are nonempty and for some vector $\omega \in \mathbb{R}^N$ and all $k \in K$ we have $\omega_{N_k} \in G^F_{\mathscr{M}}(N_k)$, where, as above, $N_k = N(\tau^k)$. Let us show that ω belongs to the set $G^F_{\mathscr{M}}(N)$. Indeed, by the definition of the game $G^F_{\mathscr{M}}$, the inclusions $\omega_{N_k} \in G^F_{\mathscr{M}}(N_k)$, $k \in K$, mean that there exist plans $(x^{ks}, u^{ks}, v^{ks}, \lambda_{ks}) \in Z_s$, $s \in N_k$, $k \in K$, such that

$$\sum_{s\in N_k} \tau^k_s (u^{ks} - v^{ks}) \geqslant 0, \quad k \in K. \tag{11}$$

Moreover, for the plans $z^{ks} = (x^{ks}, u^{ks}, v^{ks}, \lambda_{ks})$ and corresponding components of the vector ω we have

$$\omega_s \leqslant \lambda_{ks}, \quad s \in N_k, \quad k \in K. \tag{12}$$

We construct a plan $\overline{z} = (\overline{z}^s)_{s\in N}$ of the grand coalition N by setting

$$\overline{z}^s = \sum_{k\in K_s} \mu_{ks} z^{ks}, \quad s \in N, \tag{13}$$

where $K_s = \{k \in K \mid s \in N_k\}$, $s \in N$, and μ_{ks} are defined by $\mu_{ks} = \delta_k \tau_s^k$, $k \in K_s$, $s \in N$, where δ_k are weights of fuzzy coalitions τ^k under the condition that the cover $\{\tau^k\}_{k\in K}$ is balanced: $\delta_k \geqslant 0$, $k \in K$, and

$$\sum_{k\in K} \delta_k \tau^k = e^N. \tag{14}$$

By (14) and in view of nonnegativity of δ_k and τ_s^k, all μ_{ks} are nonnegative and

$$\sum_{k\in K_s} \mu_{ks} = 1 \quad \forall\ s \in N.$$

From (13) and the convexity of Z_s it follows that $\overline{z}^s \in Z_s$ for every $s \in N$. Let us show that the plan $\overline{z}$ is balanced. To prove that $\overline{z} = (\overline{x}^s, \overline{u}^s, \overline{v}^s, \overline{\lambda}_s)_{s\in N} \in Z_{\mathscr{M}}(N)$, we again use formula (13) and calculate the volumes of export $\overline{u}^s$ and import $\overline{v}^s$ corresponding to the regional components $\overline{z}^s$ of the plan $\overline{z}$. From the definition of $\overline{z}^s$ it immediately follows that the corresponding expressions (in terms of u^{ks}, v^{ks}, μ_{ks}) have the form

$$\overline{u}^s = \sum_{k\in K_s} \mu_{ks} u^{ks}, \quad \overline{v}^s = \sum_{k\in K_s} \mu_{ks} v^{ks} \quad \forall\ s \in N.$$

We show that

$$\sum_{s\in N} \overline{u}^s \geqslant \sum_{s\in N} \overline{v}^s$$

which implies that the plan $\overline{z}$ is balanced. For this purpose we multiply each inequality in (11) by $\delta_k \geqslant 0$ and summarize the obtained relations. Taking into account the identities $\mu_{ks} = \delta_k \tau_s^k$, we find

$$\sum_{k\in K} \sum_{s\in N_k} \mu_{ks} u^{ks} \geqslant \sum_{k\in K} \sum_{s\in N_k} \mu_{ks} v^{ks}.$$

Hence, exchanging the summation order, we find

$$\sum_{s\in N} \sum_{k\in K_s} \mu_{ks} u^{ks} \geqslant \sum_{s\in N} \sum_{k\in K_s} \mu_{ks} v^{ks},$$

which proves

$$\sum_{s\in N} \overline{u}^s \geqslant \sum_{s\in N} \overline{v}^s$$

in view of (13). Thus, $\overline{z} \in Z_{\mathscr{M}}(N)$ is proved.

Further, multiplying each inequality in (12) by the corresponding nonnegative number μ_{ks} and taking the sum of the obtained relations with respect to $k \in K_s$ for every $s \in N$, we get

$$\omega_s = \omega_s \sum_{k\in K_s} \mu_{ks} \leqslant \sum_{k\in K_s} \lambda_{ks} \mu_{ks}, \quad s \in N.$$

Since

$$\overline{\lambda}_s = \sum_{k \in K_s} \lambda_{ks} \mu_{ks}, \quad s \in N,$$

by construction of the plans $\overline{z}^s = (\overline{x}^s, \overline{u}^s, \overline{v}^s, \overline{\lambda}_s)$, we obtain the required relation

$$\omega_s \leqslant t_s(\overline{z}^s) = \overline{\lambda}_s, \quad s \in N,$$

for $\overline{z} = (\overline{z}^s)_{s \in N}$ in $Z_{\mathscr{M}}(N)$, and, consequently, ω belongs to $G^F_{\mathscr{M}}(N)$. □

Proposition 5. *For each coalition $\tau \in \sigma_F$ the set $G^F_{\mathscr{M}}(\tau)$ is comprehensive from below and closed; moreover, Assumption $(M1)$ guarantees that all sets $G^F_{\mathscr{M}}(\tau)$, $\tau \in \sigma_F$, are nonempty. If, in addition to $(M1)$, the model $\mathscr{M}$ satisfies Assumption $(M2)$, then the set of individually rational imputations $\widehat{G}^F_{\mathscr{M}}(N)$ of the game $G^F_{\mathscr{M}}$ is nonempty and bounded from above.*

Proof. The fact that the sets $G^F_{\mathscr{M}}(\tau)$ are comprehensive from below immediately follows from their definition. To verify the closedness of these sets, we fix arbitrarily $\tau \in \sigma_F$ and show that $Z(\tau)$ is polylhedral. Indeed, since the sets Z_s are polyhedral, the Cartesian product $\prod_{s \in N(\tau)} Z_s$ is also polyhedral. Since the set $Z_{\mathscr{M}}(\tau)$ is obtained (by definition) from the set $\prod_{s \in N(\tau)} Z_s$ by imposing additional linear constraints

$$\sum_{s \in N(\tau)} \tau_s (u^s_k - v^s_k) \geqslant 0, \quad k = 1, \ldots, m,$$

we see that $Z_{\mathscr{M}}(\tau)$ is also polyhedral. But, in this case, since the functions t_s are linear, the set

$$U_{\mathscr{M}}(\tau) = \Big\{ t_s(z^s)_{s \in N(\tau)} \;\Big|\; (z^s)_{s \in N(\tau)} \in Z_{\mathscr{M}}(\tau) \Big\}$$

is polylhedral because it is the linear image of the set $Z_{\mathscr{M}}(\tau)$ (cf. [15]). Since the set

$$G^F_{\mathscr{M}}(\tau) = U_{\mathscr{M}}(\tau) - \mathbb{R}^{N(\tau)}_+ \tag{15}$$

is the algebraic sum of polylhedral sets, it is also polylhedral [15], which implies the closedness of $G^F_{\mathscr{M}}(\tau)$.

Let Assumption $(M1)$ be satisfied for the model $\mathscr{M}$. We fix plans $z^{s0} \in Z_{\mathscr{M}}(s)$, $s \in N$. It is clear that $(z^{s0})_{s \in N(\tau)} \in Z_{\mathscr{M}}(\tau)$ for every $\tau \in \sigma_F$. Therefore, under Assumption $(M1)$, all sets $Z_{\mathscr{M}}(\tau)$ are nonempty and, consequently, all sets $G^F_{\mathscr{M}}(\tau)$ are nonempty in view of (15).

Passing to the proof of the last part of Proposition 5, we recall [15] that Assumption $(M2)$ implies the boundedness of the set $Z_{\mathscr{M}}(N)$ (regarded as the set of solutions to a system of linear inequalities such that the corresponding homogeneous system has a unique solution). Consequently, by Assumption $(M1)$ and the closedness of all sets $Z_{\mathscr{M}}(\tau)$, we conclude that $Z_{\mathscr{M}}(N)$ is a nonempty compact set. But, since it is the image of a compact set, $U_{\mathscr{M}}(N)$ is also a nonempty compact set. By (15), the set $G^F_{\mathscr{M}}(N)$ is bounded from above. By the obvious embedding $\widehat{G}^F_{\mathscr{M}}(N) \subseteq G^F_{\mathscr{M}}(N)$, the set $\widehat{G}^F_{\mathscr{M}}(N)$ is bounded from above. To prove the nonemptiness of this set, we first show that all $Z_{\mathscr{M}}(s)$ are nonempty compact sets provided that the assumptions of Proposition 5 hold. Taking into account that these sets are nonempty by Assumption $(M1)$ and closed in view of the polylhedral convexity, it remains to show the boundedness of $Z_{\mathscr{M}}(s)$, $s \in N$. For this purpose we note that from the definition of $Z_{\mathscr{M}}(N)$ we immediately obtain the inclusions $\widehat{z} = (\widehat{z}^1, \dots, \widehat{z}^n) \in Z_{\mathscr{M}}(N)$ for any $\widehat{z}^s \in Z_{\mathscr{M}}(s)$, $s \in N$. Since the set $Z_{\mathscr{M}}(N)$ is bounded in view of Assumption $(M2)$, we conclude that all sets $Z_{\mathscr{M}}(s)$ are bounded.

Thus, every set $Z_{\mathscr{M}}(s)$ is a nonempty compact set. Consequently, the continuous functions t_s attain their maximal values on the corresponding individual plans $z^{s*} \in Z_{\mathscr{M}}(s)$ realizing the maximal guaranteed playoffs u_s^0 of one-element coalitions $\{s\}$:

$$u_s^0 = \max_{z^s \in Z_{\mathscr{M}}(s)} t_s(z^s) = t_s(z^{s*}), \quad s \in N.$$

Since the collective plan $z^* = (z^{s*})_{s \in N}$ belongs to $Z_{\mathscr{M}}(N)$, it follows that $(u_1^0, \dots, u_n^0) = (t_1(z^{1*}), \dots, t_n(z^{n*})) \in U_{\mathscr{M}}(N)$. By (15), $u^0 = (u_1^0, \dots, u_n^0)$ belongs to $G_{\mathscr{M}}^F(N)$. But, in this case, u^0 also belongs to $\widehat{G}_{\mathscr{M}}^F(N)$ (by the definition of these sets) which implies its nonemptiness. □

Using Theorem 1 and Propositions 4 and 5, we conclude that the assumptions guaranteeing the nonemptiness of the standard core $C(\mathscr{M})$ also guarantee realization of a much more subtle optimality principle. Namely, if Assumptions $(M1)$ and $(M2)$ hold, then there exist collective plans of the model $\mathscr{M}$ that are not blocked even by fuzzy coalitions $\tau \in \sigma_F \setminus \sigma_0$.

Theorem 3. *If a model $\mathscr{M}$ satisfies Assumptions $(M1)$ and $(M2)$, then its fuzzy core $C_F(\mathscr{M})$ is not empty.*

Proof. We argue in the same way as in the proof of Theorem 2. By Propositions 4 and 5, if all the assumptions of Theorem 3 hold, then the generalized cooperative game $G_{\mathscr{M}}^F$ satisfies all the assumptions of Theorem 1. Consequently, the core $C(G_{\mathscr{M}}^F)$ is nonempty. By Proposition 1, the core of the cooperative game in the strategic form $\Gamma_{\mathscr{M}} = \langle N, \{Z_s, t_s\}_{s \in N}, \{Z_{\mathscr{M}}(\tau)\}_{\tau \in \sigma_F} \rangle$ coinciding with the core $C_F(\mathscr{M})$ of the system $\mathscr{M}$ is not empty. □

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