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COLLECTIVE DYNAMICS AND BIFURCATIONS IN SYMMETRIC NETWORKS OF PHASE OSCILLATORS. II

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The present paper is the second part of a brief survey of development of the Kuramoto model of coupled phase oscillators. We consider several systems obtained as generalizations of the classical Kuramoto model and given on symmetric oscillatory networks for different functions of interaction between elements. We describe the collective dynamics and bifurcations of transitions between different regimes of interacting elements, namely, full and partial synchronizations, global antiphase regime, and slow switching. We reveal the relationship between the symmetries of the network and the existence of invariant manifolds of the system, cluster states, and more complicated collective regimes. We also describe dynamics of the model with central element and the systems with circulant and modular networks. The coexistence of conservative and dissipative dynamics, as well as the existence of chimera states and the competition for synchronization regimes are demonstrated.

1. Introduction

The present paper is a continuation of the work [1] in which we considered different collective regimes in the models of coupled oscillators given on symmetric networks. In the first part of the survey, we have briefly outlined the history of investigation of the collective dynamics (various types of synchronous regimes) in complex systems of coupled elements and analyzed the motivation of appearance of mathematical models of these systems. One of the simplest and most popular models of coupled elements (the *Kuramoto model of coupled phase oscillators* [2, 3]) was considered in detail. Various generalizations of the standard Kuramoto model of *identical globally coupled* phase oscillators were also presented. Systems of this type have the maximum number of symmetries responsible for the existence of invariant regions, invariant manifolds, various cluster regimes and heteroclinic cycles.

In the second part of the survey, we consider models of phase oscillators given on symmetric networks but with already *nonglobal* interaction between the elements. We describe some important properties of the collective dynamics of systems with central element, systems with circulant coupling, and systems of indistinguishable elements. The loss of global interaction in these networks naturally leads to the loss of certain symmetries and the corresponding cluster regimes, and also to the destruction of canonical invariant regions. The phase closure of the trajectories in globally coupled systems is also lost as a result of breaking of couplings between the oscillators. In each of the indicated three cases, the systems may have their own specific features, topological structures, and the conditions of collective interaction.

To a significant extent, the aim of the present paper is to demonstrate some interesting collective phenomena observed in oscillatory networks with different architecture of couplings. Note that each phenomenon described in the present paper for simple systems of phase oscillators is fairly universal and, hence, can be also encountered in quite complicated physical and neural networks with similar structures of interaction.

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The generalized Kuramoto model of N coupled phase oscillators has the form

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\Gamma_{ij}(\theta_i - \theta_j), \quad i = 1, \ldots, N, \tag{1}$$

where $\theta_i \in [0, 2\pi) = \mathbb{T}^1$ are phase variables, ω_i are the eigenfrequencies of oscillators, K_{ij} are the parameters (strengths) of couplings between oscillators, and $\Gamma_{ij}(x)$ are smooth 2π-periodic coupling functions. Each variable θ_i runs over a one-dimensional circle. Hence, the phase space of the system is a torus $\mathbb{T}^N$. The right-hand sides of the general system (1) depend on the phase differences $\theta_i - \theta_j$, which reveals the presence of symmetry of the phase shift along the circle $\mathbb{T}^1$ given by the action $\theta_i \mapsto \theta_i + \varepsilon$. Thus, parallel with the original system (1), it is convenient to investigate the system in the phase differences

$$\varphi_i = \theta_1 - \theta_{i+1}, \quad i = 1, \ldots, N-1, \tag{2}$$

which has the form

$$\frac{d\varphi_i}{dt} = \Delta_i + \frac{1}{N}\sum_{j=1}^{N}\left(K_{1,j}\Gamma_{1,j}(\varphi_{j-1}) - K_{i+1,j}\Gamma_{i+1,j}(\varphi_{j-1} - \varphi_{i-1})\right), \quad i = 1, \ldots, N-1,$$

where $\Delta_i = \omega_1 - \omega_{i+1}$, $i = 1, \ldots, N-1$, and $\varphi_0 = 0$. For the sake of convenience of investigations, in order to reduce system (1), parallel with the phase differences (2), it is also possible to use any collection of $N-1$ linearly independent phase differences $\varphi_{ij} = \varphi_i - \varphi_j$ because the dynamics of different reduced systems are topologically equivalent.

As indicated in Part 1 of this survey, the mathematical definition of the networks of dynamically interacting elements (oscillators, neurons, etc.) always includes the following *three main components*:

(1) description of the dynamics of an *individual node* (not coupled with the other nodes) with the help of equations or systems of equations;

(2) description of the *architecture of interaction between separate elements of the network*, which most frequently can be done by using the graph of couplings;

(3) description of the *influence of one element upon another in each link of the coupling* realized with the help of additional terms on the right-hand sides of the system of an individual element or with the use of additional equations or systems of equations.

In the present paper, we mainly consider systems of *equivalent* elements for which the above-mentioned components 1–3 are symmetric. Namely, we consider *identical* oscillators with identical eigenfrequencies

$$\omega_i = \omega, \quad i = 1, \ldots, N, \tag{3}$$

(all nodes of the system are identical) given on *symmetric* networks (the matrix of couplings $K = (K_{ij})_{i,j=1}^{N}$ has certain symmetries) and with the *same* coupling function $\Gamma_{ij}(x) = g(x)$ (the same external influence upon each element). As an exception from this situation, we can mention a system with central element in which the parameters of the central oscillator differ from the parameters of the other identical oscillators. It is clear that, in the system in phase differences obtained for the model of identical oscillators, we have $\Delta_i = 0$, $i = 1, \ldots, N$.

In the previous part of the present paper, we formulated the definitions of various types of collective dynamics for systems of coupled phase oscillators. We now recall some of these definitions. We say that:

- two oscillators θ_i and θ_j are *phase-synchronized* if $|\theta_i(t) - \theta_j(t)| \to 0$ as $t \to \infty$;
- θ_i and θ_j are in the *antiphase* if $|\theta_i(t) - \theta_j(t)| \to \pi$ as $t \to \infty$;
- θ_i and θ_j are *phase-locked* if

$$|\theta_i(t) - \theta_j(t)| \le \text{const} < 2\pi$$

  for any t;
- θ_i and θ_j are *phase-unlocked* if the previous condition is not satisfied;
- θ_i and θ_j are *desynchronized* if they are not phase-locked (the difference can be greater than 2π for large values of t);
- the system has the *regime of full synchronization* if all N oscillators are synchronized with each other;
- the system has an m*-cluster* or *partial synchronization* if m oscillators are synchronized, $1 < m < N$;
- the system is desynchronized if it has at least two desynchronized oscillators;
- system (1) has the *regime of slow switching* if the corresponding system in phase differences has a heteroclinic cycle.

The other necessary definitions are introduced directly prior to their application.

In the present paper, we consider oscillatory models on networks with quite simple symmetries, which can be easily described. In the next section, we describe an oscillatory model on a starlike network with central executive element and give a biological motivation of its appearance. In Sec. 3, we consider an oscillatory model on a circulant network, which has a time-reversal symmetry under the condition of skew-symmetry of the coupling matrix; thus, in turn, leads to the coexistence of conservative and dissipative dynamics in the system. In Sec. 4, we mainly consider a phenomenon but not the type of a model. Indeed, we discuss the concept of chimera states in systems of interacting elements.

All models and phenomena considered in the present paper are, in fact, known and investigated in numerous research works. The proposed survey is far from being complete, it only gives some preliminary information about the collective dynamics in various types of oscillatory models. We also present a list of (selected) references in which the analyzed models are studied in detail.

2. Model with Central Element

In this section, we consider a Kuramoto-type system given on an oscillatory network with central block (element) radially coupled with peripheral elements. Networks with central element are encountered as parts of more complicated networks in various fields, such as communication systems, social networks, and the neural structures of the brains of mammals. In the last case, they are widespread due to the convergent organization of compounds in the hierarchy of brain structures [4, 5]. These networks can play an important role in modelling multisensory integration [6, 7] and attention [8, 9].

2.1. Neural Model of Memory and Attention. Attention is understood as the ability of mammals to extract a part of information (usually most interesting or important) from the large amount of simultaneously incoming data with an aim of its subsequent more detailed processing. The attention system is hierarchic, i.e., it has a subsystem called *central executive component* that controls the formation of the focus of attention [8, 10–12]. The experimental data presented in [13] show that the interaction of the central element of attention system with neural assemblies representing visual objects can be realized by synchronizing their activities at the gamma-band frequencies. A model of attention based on the frequency adjustment in a system of phase oscillators was proposed and analyzed in [14–17]. According to this model, attention is focused on an object encoded in the cortex by an oscillatory activity synchronized with activity of the central element. This model is used for the realization of the subsequent selection of objects in the focus of attention [9] and for the simultaneous tracing of motion of several target objects [18, 19]. The investigation of systems with central element may be useful for understanding the role of synchronization in cognitive functions. Models of phase oscillators give a convenient and mathematically suitable instrument for these investigations.

In networks with central block, global interactions of elements are realized via the *central oscillator*, which has both feedforward and feedback couplings with all other elements called *peripheral oscillators.* In addition to couplings with the central oscillator, peripheral oscillators may also have local couplings with their neighbors, which are, as a rule, much weaker than their couplings with the central oscillator [9, 19–21].

Assume that the central oscillator has index 0 and the peripheral oscillators are marked by the indices $i = 1, \ldots, N$. Consider a model whose dynamics is described by the following system of ordinary differential equations:

$$\frac{d\theta_0}{dt} = \omega_0 + \sum_{i=1}^{N} f(\theta_i - \theta_0), \tag{4}$$

$$\frac{d\theta_i}{dt} = \omega + g(\theta_0 - \theta_i), \quad i = 1, \ldots, N, \tag{5}$$

where $(\theta_0, \theta_1, \ldots, \theta_N) \in \mathbb{T}^{N+1}$ are phase variables on an $(N+1)$-dimensional torus, $\theta_i \in [0, 2\pi)$, ω is the eigenfrequency of (identical) peripheral oscillators, $f(x)$ and $g(x)$ are, respectively, the functions of influence of the peripheral oscillators on the central oscillator, and vice versa. Assume that the functions $f(x)$ and $g(x)$ are odd, 2π-periodic, and smooth. Under these assumptions, in particular, we get

$$f(0) = f(\pi) = g(0) = g(\pi) = 0. \tag{6}$$

System (4), (5) is a special case of the more general system (1) with the coupling matrix $K = \left(K_{ij}\right)_{i,j=0}^{N}$, where $K_{i0} = K_{0j} = 1$, $i, j = 1, \ldots, N$, and $K_{ij} = 0$, otherwise, and the coupling functions $\Gamma_{0j}(x) = f(x)/N$, $j \neq 0$, and $\Gamma_{i0}(x) = g(x)/N$, $i \neq 0$. Subtracting Eq. (4) from Eq. (5), we obtain the following system in phase differences:

$$\frac{d\varphi_i}{dt} = -\sum_{j=1}^{N} f(\varphi_j) - g(\varphi_i), \quad i = 1, \ldots, N, \tag{7}$$

where $\varphi_i = \theta_i - \theta_0$, $i = 1, \ldots, N$. Denote

$$f'(0) = a_1, \qquad f'(\pi) = a_2, \qquad g'(0) = b_1, \qquad f'(\pi) = b_2. \tag{8}$$

System (7) possesses the permutation symmetry $\mathbf{S}_N$, which is the common property with the original system of globally coupled identical oscillators (in the variables θ_i). The main difference is that the reduced system does not have the phase-shift symmetry. Despite the fact that system (7) possesses the invariant cluster regimes $\varphi_i = \varphi_j$, $i \neq j$ [corresponding to the clusters $\theta_i = \theta_j$ of the original system (4), (5)], these clusters no longer select a closed invariant region $\mathcal{C}$ in $\mathbb{T}^N$ (see the description of the invariant regions in Part 1 of the present paper [1]) because $\varphi_i = 0$ are not invariant sets of the system in the analyzed case. Unlike the system with global coupling, the system with starlike coupling possesses *phase-unlocked regimes* corresponding to the trajectories nonhomotopic to zero.

2.2. Antiphase States in the Model with Central Element. The points $\Phi = (\varphi_1, \ldots, \varphi_N)$ with coordinates $\varphi_i \in \{0, \pi\}$, $i = 1, \ldots, N$, are equilibria of system (7), (8) (however, the system may also have some other equilibria). By Φ_k we denote the points that have k coordinates equal to 0 and $(N-k)$ coordinates equal to π. A point of this kind corresponds to a regime in which k peripheral oscillators are winners in the competition for synchronization with the central oscillator, whereas the remaining $N-k$ peripheral oscillators are losers in this competition (they are in the antiphase with the central oscillator). From the viewpoint of attention modeling, the stable regimes Φ_k can be well interpreted because they describe situations in which k objects (properties) are in the focus of attention and the remaining objects are out of the focus. In terms of this notation, the point $\Phi_N = \Phi_{\text{sync}}$ is the regime of full synchronization, the point Φ_0 is the regime in which all peripheral oscillators are in the antiphase with the central oscillator, and Φ_1 is the *"winner-takes-all"* regime (only one peripheral oscillator wins in the competition for synchronization with the central oscillator).

In view of the existing symmetry, all points Φ_k have identical properties. The stability and bifurcation properties of these points can be described by using notation (8). According to [20], the point Φ_k is asymptotically stable if, for the corresponding values of k, the following conditions are satisfied:

$$\begin{array}{ll} k = 0: & b_2 > 0, \quad a_2 > -b_2/N, \\ k = 1: & b_2 > 0, \quad L_1 > 0, \quad L_2 > 0, \\ 2 \le k \le N-2: & b_1 > 0, \quad b_2 > 0, \quad L_1 > 0, \quad L_2 > 0, \\ k = N-1: & b_1 > 0, \quad L_1 > 0, \quad L_2 > 0, \\ k = N: & b_1 > 0, \quad a_1 < -b_1/N, \end{array} \tag{9}$$

where

$$L_1 = k b_2 a_1 + (N-k) b_1 a_2 + b_1 b_2, \qquad L_2 = k a_1 + (N-k) a_2 + (b_1 + b_2).$$

In order that a stable point Φ_k be a node, in addition to inequalities (9), the following relation must be true:

$$\left(\frac{k a_1}{2}\right)^2 + \left(\frac{(N-k) a_2}{2}\right)^2 \ge 0.$$

Otherwise, these points are stable foci.

The surfaces $L_1 = 0$ and $L_2 = 0$ are bifurcation surfaces in the space of parameters (a_1, a_2, b_1, b_2). Each of these surfaces is associated either with the Andronov–Hopf bifurcation or with the pitchfork bifurcation of the point Φ_k depending on the relationship between the parameters and the dimension of the phase space. Note that, in numerous cases, the local bifurcations mentioned above are parts of global heteroclinic bifurcations and lead to complex heteroclinic cycles. In the spaces with dimensions $N \geq 3$, depending on the form of the functions $f(x)$ and $g(x)$, these bifurcations also lead to analogs of Shilnikov's saddle-focus bifurcations and Shilnikov–Hopf-type bifurcations. For specific functions $f(x)$ and $g(x)$, these bifurcations were considered in more detail in [20, 22].

2.3. Model with Two-Harmonic Coupling. We now present a more detailed analysis of the case where coupling functions are two-harmonic, namely,

$$f(x) = a(\sin x + r\sin(2x)), \qquad g(x) = b(\sin x + p\sin(2x)),$$

where a, b, r, and p are parameters. In the analyzed case, the system in phase variables (7) takes the form

$$\frac{d\varphi_i}{dt} = -\sum_{j=1}^{N} a(\sin\varphi_j + r\sin(2\varphi_j)) - b(\sin\varphi_i + p\sin(2\varphi_i)), \quad i = 1,\dots,N. \tag{10}$$

In view of the symmetries of the network and the oddness of the functions f and g, we can find the invariant manifolds of system (10) and describe its bifurcation properties in more detail. This system has two types of invariant cluster manifolds:

- m-dimensional manifolds

$$\mathcal{M}_m = \left\{(\varphi_1,\dots,\varphi_N)\colon \varphi_{k_1} = \varphi_{k_2} = \dots = \varphi_{n-m+1}\right\}, \quad m = 1,\dots,N,$$

  corresponding to $(n-m+1)$-clusters of peripheral oscillators with regard for their permutation symmetry;

- m-dimensional manifolds

$$\mathcal{Q}_m = \left\{(\varphi_1,\dots,\varphi_N)\colon \varphi_{k_i} + \varphi_{k_j} = 0, \varphi_{k_l} \in \{0,\pi\}\right\},$$

$$i = 1,\dots,m, \quad j \neq i, \quad l = 2m+1, \quad m = 1,\dots,[N/2],$$

  (where $[x]$ is the integer part of x) appearing due to the oddness symmetry of the right-hand sides of the system.

System (10) has the following bifurcations of the points Φ_k [among other (mainly global) bifurcations]:

- Andronov–Hopf bifurcations at the point Φ_k given in the parametric space with the surface

$$AH(\Phi_k) = \{(a,b,r,p)\colon 4bp + 2Nar + (2k-N)a = 0\}, \quad k = 1,\dots N-1;$$

- pitchfork bifurcations given, depending on the parameter k, by the expressions

$$PF(\Phi_k) = \big\{(a,b,r,p)\colon a(N(4pr + 2r - 2p - 1) + 4k(p-r)) + b(4p^2 - 1) = 0\big\}, \quad k = 1,\dots N-1,$$

$$PF(\Phi_N) = \{(a, b, r, p): b(2p+1) + Na(2r+1) = 0\},$$

$$PF(\Phi_0) = \{(a, b, r, p): b(2p-1) + Na(2r-1) = 0\}$$

and leading to the appearance (disappearance) of two new points inside the invariant manifold $\mathcal{M}_m$;

– pitchfork bifurcations observed on the bifurcation surfaces

$$PF_*(\Phi_k) = \{(a, b, r, p): p = -1/2\}, \quad k = 1, \ldots, N,$$

$$PF^*(\Phi_k) = \{(a, b, r, p): p = 1/2\}, \quad k = 0, \ldots, N-1,$$

in the directions transcritical for the invariant manifolds and leading to the appearance of $k-1$ for $PF_*(\Phi_k)$ and $N-k-1$ for $PF^*(\Phi_k)$ pairs of new equilibria from each fixed point Φ_k;

– degenerate "switch" bifurcation in the hyperplane

$$SW(\Phi_k) = \{(a, b, r, p): b = 0\}, \quad k = 0, \ldots, N,$$

which changes the direction of influence of the central oscillator on peripheral oscillators.

Numerous collective properties described above are also preserved for a more general system with starlike structure of couplings but with different functions of influence $f_1, \ldots, f_N$ of peripheral oscillators upon the central oscillator. The corresponding system in phase differences takes the form

$$\frac{d\varphi_i}{dt} = -\sum_{j=1}^{N} f_j(\varphi_j) - g(\varphi_i), \quad i = 1, \ldots, N. \tag{11}$$

This system loses some of its symmetry properties depending on the relations between the functions f_i. Under conditions (6), this system also has the equilibria Φ_k (parallel with some other equilibria). Moreover, system (11) has invariant manifolds $\mathcal{M}_m$ and $\mathcal{Q}_m$ for any functions $f_1, \ldots, f_N$, g. The *hierarchical structure* of system (11) relative to its invariant manifolds $\mathcal{M}_m$ is an important specific feature, which is absent in the major part of oscillatory systems given on networks with different architecture. This means that any m-dimensional system (11) has the same dynamics as the dynamics on a certain invariant manifold $\mathcal{M}_m$ of the general N-dimensional system (11) ($m < N$) with new odd functions $\widetilde{f}_i$ that are linear combinations of the old functions f_i. This feature enables one to automatically describe the dynamics of the system on its invariant manifolds by using the results already obtained for the same system of lower dimensions. The results of detailed investigations of low-dimensional systems of this kind, including a great variety of phase portraits and bifurcation and schematic diagrams can be found in [20, 22].

In [20, 23], the collective dynamics of systems with central element was investigated in the presence of interaction between the peripheral elements, i.e., system (4), (5) takes a more general form

$$\frac{d\theta_0}{dt} = \omega_0 + \sum_{i=1}^{N} f(\theta_i - \theta_0),$$

$$\frac{d\theta_i}{dt} = \omega + g(\theta_0 - \theta_i) + \sum_{k \in \mathcal{N}_i} h(\theta_k - \theta_i), \qquad i = 1, \ldots, N,$$

where $h(x)$ is the function of local interaction between the peripheral oscillators and $\mathcal{N}_i$ is the set of indices of interactions between the peripheral oscillators. It is clear that systems with weak peripheral couplings give more adequate models of neural processes. In the cited works, much attention was given to the problem of influence of the architecture of peripheral couplings and the strength of interaction between different peripheral oscillators on the total collective dynamics of the system. In particular, it was shown that the stable regime of full synchronization (Φ_N for the system in phase differences) remains stable even for fairly weak couplings between peripheral elements independently of configuration of the couplings. The greater the number of peripheral oscillators in the network, the more difficult is problem of breaking the stability of this regime. At the same time, the stability of antiphase regimes can be easily violated as a result of the appearance of weak couplings between peripheral elements.

2.4. Conservative Chaos. An oscillatory starlike system with three or more peripheral oscillators may exhibit a quite complicated chaotic behavior. We can indicate the relationship between the vector field (4), (5) and the well-known ABC-flow (Arnold–Beltrami–Childress flow). For the first time, this flow was investigated by Arnold in [24]. Consider a system

$$\begin{aligned} \frac{d\varphi_1}{dt} &= A\sin\varphi_3 + C\cos(\varphi_2 - \delta), \\ \frac{d\varphi_2}{dt} &= B\sin\varphi_1 + A\cos(\varphi_3 - \delta), \\ \frac{d\varphi_3}{dt} &= C\sin\varphi_2 + B\cos(\varphi_1 - \delta). \end{aligned} \tag{12}$$

System (12) is an ABC flow with parameters A, B, and C for $\delta = 0$ and coincides with the system in phase differences (7) for the model with three peripheral oscillators and the coupling functions

$$f(x) = g(x) = -\sin x$$

for $A = B = C = 1$ and $\delta = \pi/2$. The chaotic behavior of the ABC-flow was studied and described in the literature from different points of view [25–28]. The system has so-called conservative chaos when a chaotic trajectory fills almost the entire phase torus $\mathbb{T}^3$ with the exception of one- and two-dimensional invariant manifolds of this system. A similar behavior is observed for the phase flows in system (4), (5) of three peripheral oscillators for a certain distribution of the parameters. Thus, the system has three invariant planes $\mathcal{M}_2$ of the form $\varphi_i = \varphi_j$ decomposing the phase space $\mathbb{T}^3$ into two invariant regions filled with chaotic phase-unlocked trajectories [22]. The phase space is also pierced by one-dimensional invariant sets $\mathcal{M}_1\colon \varphi_1 = \varphi_2 = \varphi_3$ and $\mathcal{Q}_1$ of the form

$$\varphi_i + \varphi_j = 0, \quad \varphi_k \in \{0, \pi\}, \quad i, j, k = 1, 2, 3.$$

By analyzing the conditions of stability of equilibria (9), one can find an open region in the space of parameters such that all points $\Phi_k \in \mathbb{T}^3$ of the system are saddle points. In [22], one can find a description of the situation in which the system has equilibria only in the form of saddle points Φ_0 and Φ_3 and the saddle-node points Φ_1 and Φ_2 (in view of the symmetry, we have three points in each case) and all these points are connected into a heteroclinic cycle with the help of one-dimensional invariant manifolds $W^u(\Phi_k)$ and $W^s(\Phi_k)$ some of which coincide with the above-mentioned manifolds $\mathcal{M}_1$ and $\mathcal{Q}_1$ and the other (focal) ones belong to manifolds of the form $\mathcal{M}_2$. In this case, the entire phase space is pierced by a network of heteroclinic cycles. These cycles serve as a basis for the existence of chaotic trajectories by analogy with the situation of chaos in the vicinity of a saddle-focus loop described by Shilnikov [29]. Bifurcations of codimension two described, in particular, in [30, 31] lead to the appearance of heteroclinic cycles whose nodes are saddle points and saddle cycles. These bifurcations are responsible for the deformation and compression of chaos, which may, in this case, occupy not the entire toroidal phase space. Note that the motion of a phase point along the heteroclinic trajectory with different nodes Φ_k corresponds to the model of attention switching from one object to another in the neural system.

According to the above-mentioned hierarchy of the structure relative to the proper invariant manifolds, the system in phase differences of dimension $N \geq 4$ has a chaotic structure inside the invariant manifolds $\mathcal{M}_m$, $m = 3, \ldots, N-1$, if it has a structure of this kind for $N = 3$. Hence, for a system of phase oscillators with central element, it is possible to find a coupling function, which has either a conservative chaos similar to the ABC-flow or a chaotic attractor for any dimensions $N \geq 3$.

2.5. Other Models with Central Element. Based on the biological motivation for a more exact description of the neural processes of memory and attention, an oscillatory system with *central element and adaptation* [obtained as a natural generalization of system (4), (5)] was proposed and studied in [32, 33]. In the cited works, the authors, in fact, studied a system of coupled oscillators with central element and coupling strengths $K_{ij} = K_{ij}(t)$. In the analyzed case, the indicated strengths are unknown functions described by additional differential equations rather than constants. In this model, the frequency of the central oscillator $\omega_0 = \omega_0(t)$ is also determined from an additional differential equation, which is used to adapt this frequency to the frequency of the peripheral oscillator synchronized with the central oscillator (it turns out that, in the considered case, this oscillator is unique). This model gives an adequate description of the "winner-takes-all" computational principle, which is frequently used in the investigations of artificial neural networks and neural processes running in the brain [34–38].

For the physical and biological reasons, the researchers also studied various systems with central element and several hubs. Thus, in particular, collective regimes were investigated in discrete starlike networks [39], in starlike Stuart–Landau systems [40], in recurrent neural networks [41, 42], and in various oscillatory and neural networks with several hubs [7, 43–47].

3. Conservative-Dissipative Dynamics in Oscillatory Models

In the present section, we consider oscillatory networks with cyclic symmetry. The investigations of networks of this kind are aimed at the description of various natural phenomena and demonstration of interesting mathematical effects. We study the case where the interactions between each pair of oscillators have opposite directions but the same strength. This network has reversal symmetry, which, in turn, results in coexistence of two opposite dynamics (conservative and dissipative) in the same phase space.

3.1. Oscillatory Model with Circulant Coupling. Consider a translationally invariant ring of coupled phase oscillators with periodic boundary conditions

$$\frac{d\theta}{dt} = \omega_i + \sum_{j=1}^{N} K_{j-i}\, g(\theta_i - \theta_j), \quad i = 1, \ldots, N, \tag{13}$$

where $\theta_i \in [0, 2\pi)$ are phase variables, ω_i are eigenfrequencies, $g(x)$ is a smooth 2π-periodic coupling function, K_j, $j = 1, \ldots, N$, are the parameters of strength of the interaction between oscillators, and all subscripts are determined modulo N. The coefficient $K_N = K_0$ specifies the self-coupling of the oscillator. System (13) is a special case of system (1) with the following *circulant matrix* of couplings:

$$K = \mathrm{circ}(K_0, K_1, \ldots, K_{N-1}) = \begin{pmatrix} K_0 & K_1 & \ldots & K_{N-2} & K_{N-1} \\ K_{N-1} & K_0 & K_1 & \ddots & K_{N-2} \\ \vdots & K_{N-1} & K_0 & \ddots & \vdots \\ K_2 & \ddots & \ddots & \ddots & K_1 \\ K_1 & K_2 & \ldots & K_{N-1} & K_0 \end{pmatrix}, \tag{14}$$

(in other words, $K_{ij} = K_{i-j}$) and a single coupling function between the elements $\Gamma_{ij}(x) = g(x)/N$.

The system in phase differences (2) corresponding to (13) takes the form

$$\frac{d\varphi_i}{dt} = \Delta_i + \sum_{j=1}^{N-1} K_j (g(\varphi_j) - g(\varphi_{i+j} - \varphi_i)), \quad i = 1, \ldots N-1, \tag{15}$$

where the subscripts are considered modulo N. We formulate our main results for the reduced system (13) in the phase space $\mathbb{T}^{N-1}$.

We describe some properties of system (13) in more detail for the case of identical oscillators, i.e., for the case where equality (3) is true. The circulant structure of the network and the identity of oscillators induce the equivariance of the analyzed system under the cyclic group $\mathbb{Z}_N$ specified by the action

$$\gamma\colon (\theta_1, \theta_2, \ldots, \theta_N) \longmapsto (\theta_N, \theta_1, \ldots, \theta_{N-1}).$$

The cyclic symmetry of system (13), (3) leads to the existence of invariant *splay states* (or *rotating waves*)

$$\Theta^{(k)}_{\mathrm{splay}}(t) = \left(\theta(t,k), \theta(t,k) - \frac{2\pi k}{N}, \ldots, \theta(t,k) - \frac{(N-1)2\pi k}{N}\right), \quad k = 0, \ldots, N-1,$$

where

$$\theta(t,k) = \left(\omega + \sum_{j=1}^{N} K_{j-i}\, g\left(\frac{2jk\pi}{N}\right)\right) t,$$

for any coupling functions $g(x)$. For the system in phase differences (15), (3) corresponding to (13), (3), the splay states $\Theta^{(k)}_{\mathrm{splay}}(t)$ correspond to the equilibria

$$\Phi^{(k)}_{\mathrm{splay}} = \left(\frac{2k\pi}{N}, \frac{4k\pi}{N}, \ldots, \frac{2(N-1)k\pi}{N}\right).$$

As a special case of splay state with the wave number $k = 0$, we can mention the state of *full synchronization* of all oscillators in the system

$$\Theta_{\mathrm{sync}}(t) = \Theta^{(0)}_{\mathrm{splay}}(t) = (\theta(t), \ldots, \theta(t)).$$

For the system in phase variables, this state is the origin $\Phi_{\mathrm{sync}} = \Phi^{(0)}_{\mathrm{splay}} = (0, \ldots, 0)$. Note that, in addition to $\Phi^{(k)}_{\mathrm{splay}}$, system (15), (3) may also have other equilibria depending on the coupling function $g(x)$.

The analysis of the equilibria $\Phi^{(k)}_{\text{splay}}$, $k = 0, \dots, N-1$, demonstrates that the Andronov–Hopf (degenerate or regular) bifurcation is a typical bifurcation for these points. Moreover, in the case of skew-symmetry of the coupling matrix, the synchronous state Φ_{sync} is neutral, whereas the other states $\Phi^{(k)}_{\text{splay}}$ are either attractors or repellers [48]. This means that the conservative dynamics may coexist in the vicinity of Φ_{sync} with the dissipative dynamics in the system under certain additional restrictions imposed on the coupling function.

3.2. Reversibility of the Oscillatory System and the Coexistence of Conservative and Dissipative Dynamics. An oscillatory system given on a circulant network serves as a source of numerous nontrivial collective dynamics. All synchronous regimes described in [1] are possible for this system because a *network of globally coupled identical elements* is a partial case of *circulant network*. In what follows, we also show that a *ring network of identical elements* is a source of *chimera states* in the system under certain additional conditions. In this section, we describe only some properties of this system in the case where the couplings between any two elements of the system in the same or different directions have identical absolute values but the opposite signs. We present some results for the systems in phase differences (15) and note that the original system (13) has the same properties.

The system in phase differences (15) for *identical* phase oscillators with a *circulant* and *skew-symmetric* coupling matrix K (with $K_j = -K_{-j}$) has the *time-reversible symmetry* $\mathcal{R}$: $\mathbb{T}^{N-1} \longrightarrow \mathbb{T}^{N-1}$, where

$$\mathcal{R}(\varphi_1, \dots, \varphi_{N-1}) = (\varphi_{N-1}, \dots, \varphi_1), \quad t \longmapsto -t.$$

Recall that the involution $\mathcal{R}$ is a time-reversible symmetry of the system $d\Phi/dt = G(\Phi)$ if

$$G(\mathcal{R}\Phi) = -\mathcal{R}(G\Phi)$$

and $\mathcal{R}^2 = id$, where id stands for the identical transformation [49], i.e., $\mathcal{R}$ maps any solution $\Phi(t)$ into another solution $\mathcal{R}\Phi(-t)$ of this system with the opposite direction of the phase flow. The systems with time-reversible symmetry form a fairly broad class of dynamical systems with geometry of the phase space typical only of these systems. The properties of reversible systems have been quite extensively studied for several last decades [50–57]. Numerous classical results deal with the existence of families of periodic solutions, elliptic fixed points, and invariant tori. For the detailed bibliography of works in this field, see the survey [58].

As one of the most interesting specific features of time-reversible systems, we can mention the coexistence of conservative (Hamilton-like) and dissipative dynamics in the same phase space. The phenomenon of coexistence of conservative and dissipative dynamics was described for various specific physical systems, in particular, for a three-dimensional laser system [59], for coupled superconducting Josephson junction arrays [60, 61], and for a system of anisotropically coupled spheres (Stokeslet model) [62]. In [63, 64], the existence of conservative-dissipative dynamics was demonstrated for chains and rings of oscillatory systems and a possible physical interpretation of this phenomenon was given.

A system with time-reversible symmetry can be completely conservative without attractors and repellers, and its phase space is filled with families of neutral periodic, quasiperiodic, or even chaotic trajectories and heteroclinic cycles. The dynamics of this system resembles the dynamics of a Hamiltonian system despite the fact that a conservative time-reversible system is not necessarily even-dimensional. On the other hand, a time-reversible system can be also dissipative with attracting and repelling elements of the phase space. Thus, the indicated two types of dynamics may take place for the same reversible system but for different values of the parameters. However, the most interesting case is represented by the third type of reversible systems whose phase space is split into two regions the first of which is conservative and filled with neutral trajectories, whereas the second region is dissipative and contains repellers and attractors and also the trajectories that connect repellers and attractors. The boundary between conservative and dissipative regions is, as a rule, a heteroclinic cycle or a multidimensional invariant set of heteroclinic cycles (if the dimension of the system is greater than two). As a rule, a time-reversible system with parameter can be conservative-dissipative for the major part of values of this parameter and only conservative (or only dissipative) for the limit values of the parameter. The indicated properties for the systems of coupled oscillators (13) and (15) were described in detail in [48].

A fixed space of involution $\mathcal{R}$ for system (15) is defined as follows:

$$\operatorname{Fix}\mathcal{R} = \{\Phi \in \mathbb{T}^{N-1}: \mathcal{R}\Phi = \Phi\} = \left\{\Phi \in \mathbb{T}^{N-1}: \varphi_i = \varphi_{N-i},\ 1 \le i \le \left[\frac{N-1}{2}\right]\right\}.$$

Its dimension depends on the evenness of the phase space of the system: $\dim(\operatorname{Fix}\mathcal{R}) = [N/2]$.

We now present several properties of solutions of the system depending on their intersections with a fixed subspace of $\mathcal{R}$.

If an orbit intersects $\operatorname{Fix}\mathcal{R}$ at two points, then it is *periodic* and consists of two parts mapped into each other by the involution $\mathcal{R}$.

Any *nonperiodic* trajectory can intersect $\operatorname{Fix}\mathcal{R}$ only once.

The *attractors* and *repellers* of a reversible system do not belong to $\operatorname{Fix}\mathcal{R}$.

If a reversible system has an *attractor* $\mathcal{A}$, then it also has a *repeller* $\mathcal{R}\mathcal{A}$.

If a trajectory starts from a *source* and intersects $\operatorname{Fix}\mathcal{R}$, then it approaches a symmetrically located *sink* and, moreover, this trajectory is *heteroclinic*.

The superposition of the symmetries $\mathbb{Z}_N$ and $\mathcal{R}$ implies the existence of $N-1$ more reversible symmetries $\mathcal{R}_i$, $i = 2, \ldots, N$. Thus, there exist $N-1$ hyperplanes $\operatorname{Fix}\mathcal{R}_i = \gamma^i \operatorname{Fix}\mathcal{R}_1$, $\mathcal{R}_1 := \mathcal{R}$, invariant under the transformations of $\mathcal{R}_i$. All $\mathcal{R}_i$ intersect at the point Φ_{sync} if N is odd. At the same time, if N is even, then they intersect along the one-dimensional line $V_0 = (\varphi, 0, \varphi, 0, \ldots, \varphi, 0, \varphi)$, $\varphi \in \mathbb{T}^1$. The intersections described above are the centers of conservative regions for system (15).

In [48], by using the theoretical results from [62, 65], it was shown that system (15) with a *circulant skew-symmetric* coupling matrix K and a coupling function $g(x)$ such that $g'(0) \neq 0$ has the following dynamical regimes:

(A) *Families of periodic solutions* in the vicinity of the point Φ_{sync}. There exists a one-parameter family of periodic solutions $\Phi_\sigma(t)$ in the vicinity of Φ_{sync} if N is odd and a two-parameter family of periodic solutions $\Phi_{(\sigma_1,\sigma_2)}(t)$ with period close to $2\pi/\Omega_m$ if N is even; here,

$$\Omega_m = 2g'(0) \sum_{j=1}^{[(N-1)/2]} K_j \sin\left(\frac{2mj\pi}{N}\right).$$

(B) A *dense set of invariant tori in the vicinity of* Φ_{sync}. In any vicinity of Φ_{sync}, there exist analytic $[(N-1)/2]$-dimensional quasiperiodic tori with incommensurable frequencies close to $\Omega_1, \ldots, \Omega_{[(N-1)/2]}$.

(C) *Dissipative dynamics*. The equilibrium $\Phi_{\text{splay}}^{(k)}$, $k \neq 0$, is a sink if the following condition is satisfied:

$$\sum_{j=1}^{[(N-1)/2]} K_j \left(g'\left(\frac{2\pi k}{N} j\right) - g'\left(-\frac{2\pi k}{N} j\right)\right)\left(1 - \cos\left(\frac{2mj\pi}{N}\right)\right) < 0, \quad m = 1, \ldots, N-1.$$

In this case, $\Phi_{\text{splay}}^{(-k)}$ is a source.

Items (A) and (B) describe the conservative dynamics in the vicinity of the state of full synchronization, whereas item (C) describes the possibility of dissipative dynamics in the vicinity of splay states. System (15) is characterized by the coexistence of regimes (A) and (B) with regime (C) for almost all coupling functions $g(x)$ because one can almost always find a circulant skew-symmetric coupling matrix for which all necessary additional inequalities are satisfied.

By $\mathcal{D}_0$ we denote a conservative region in the vicinity of the synchronous fixed point Φ_0. The structure of the boundary $\partial\mathcal{D}_0$ of the indicated region is quite complicated because, in the general case, it depends on the parameters of the matrix K and the function $g(x)$. Nevertheless, we can say that $\partial\mathcal{D}_0$ is an $(N-2)$-dimensional hypersurface in the phase space $\mathbb{T}^{N-1}$ of system (15) completely composed of heteroclinic cycles.

In the two-dimensional case, the region $\mathcal{D}_0$ is bounded by a $\mathbb{Z}_3$-symmetric heteroclinic cycle and filled with neutral periodic trajectories, which concentrically surround the point Φ_{sync}. In the three-dimensional case, the region $\mathcal{D}_0$ is bounded by a tetrahedron-like surface formed by heteroclinic cycles and completely filled with a two-parameter family of periodic orbits surrounding a one-parameter family of equilibria with coordinates $(\varphi, 0, \varphi)$, $\varphi \in \mathbb{T}$. As indicated in item (A), the families of periodic orbits are two-dimensional for odd N and three-dimensional for even N. Hence, for the system in phase differences whose phase space is $(N-1)$-dimensional, the conservative region is completely filled with periodic orbits solely in two- and three-dimensional cases of $\mathbb{T}^{N-1}$. For higher dimensions $N \geq 5$, the region $\mathcal{D}_0$ is already filled with quasiperiodic or chaotic solutions with an " interwoven" two- or three-dimensional surface of periodic orbits.

System (15) may have several conservative regions. In particular, for the even dimensions, the fixed point $\Phi_{\text{splay}}^{N/2}$ is always the center of a conservative region. Thus, for $N = 4$, the phase space has two conservative regions around the points Φ_{sync} and $\Phi_{\text{splay}}^{(2)}$ and a dissipative region that contains heteroclinic trajectories connecting the attracting (repelling) fixed point $\Phi_{\text{splay}}^{(1)}$ with the repelling (attracting) fixed point $\Phi_{\text{splay}}^{(3)}$ (as shown in [48]). In the general case, the number of conservative regions increases with the number of harmonics of the coupling function $g(x)$. As a rule, the dissipative region remains simply connected independently of the number of conservative "islands" in the system.

System (15) with a *circulant skew-symmetric* coupling matrix K and an *odd* coupling function $g(x)$ is *conservative* (divergence free). Moreover, this system has the first integral

$$E_1(\varphi_1, \ldots, \varphi_{N-1}) = \sum_{i=0}^{N-1} h(\varphi_i - \varphi_{i+1}),$$

where $h'(\varphi) = g(\varphi)$ and $\varphi_N = \varphi_0 = 0$. In the case of even N, odd $g(x)$, and a sparse skew-symmetric matrix (14) with $K_j = 0$ for even j, system (15) also has another first integral

$$E_2(\varphi_1, \ldots, \varphi_{N-1}) = \sum_{i=1}^{N-1} (-1)^{i-1} \varphi_i.$$

System (15) with a *circulant skew-symmetric* matrix K and an *even* function $g(x)$ is *gradient* and, hence, *dissipative* in the entire space $\mathbb{T}^{N-1}$. The indicated two cases are limit cases for the function $g(x)$. For the conservative-dissipative dynamics, it is necessary that the function $g(x)$ be neither even, nor odd. Note that system (15) with a *symmetric matrix* K (in particular, in the case of global coupling described in [1]) is *gradient* for *odd* $g(x)$ and *divergence free* for *even* $g(x)$. Hence, for symmetric and skew-symmetric matrices, we observe the opposite situations. It is worth noting that the system with symmetric coupling does not have conservative-dissipative dynamics for any coupling functions.

3.3. Ring Oscillatory Networks with the Opposite Feedback Coupling. A special case of system (13) is a system given on a ring network, where each oscillator acts with identical strengths upon its l neighbors located clockwise along the circle in the direction from the oscillator and with the opposite strengths upon the neighbors located anticlockwise. Choosing the strengths $K_j = -K_j = 1$, $j = 1, \ldots, l$, $l < N/2$, we can represent the system of identical oscillators in the form

$$\frac{d\theta_i}{dt} = \omega + \sum_{j=1}^{l} g(\theta_i - \theta_{i+j}) - \sum_{j=N-l}^{N-1} g(\theta_i - \theta_{i+j}), \quad j = 1, \ldots N. \tag{16}$$

Then the corresponding reduced system takes the form

$$\frac{d\varphi_i}{dt} = \sum_{j=1}^{l} \big(g(\varphi_j) - g(\varphi_{i+j} - \varphi_i)\big) - \sum_{j=N-1}^{N-l} \big(g(\varphi_j) - g(\varphi_{i+j} - \varphi_i)\big). \tag{17}$$

A *system of three coupled oscillators* of the form (16) with one-harmonic coupling function (see Kuramoto–Sakaguchi [66]):

$$g(x) = -\sin(x - \alpha), \tag{18}$$

where α is a phase shift, serves as the simplest example of a system with coexistence of conservative and dissipative dynamics. In this case, the system in phase differences is two-dimensional

$$\begin{aligned} \frac{d\varphi_1}{dt} &= -\sin(\varphi_1 - \alpha) - b\sin(\varphi_2 - \alpha) - b\sin(\varphi_1 + \alpha) - \sin(\varphi_1 - \varphi_2 + \alpha), \\ \frac{d\varphi_2}{dt} &= -\sin(\varphi_1 - \alpha) - b\sin(\varphi_2 - \alpha) - \sin(\varphi_2 + \alpha) - b\sin(\varphi_2 - \varphi_1 + \alpha). \end{aligned} \tag{19}$$

This system has the cyclic symmetry $\mathbb{Z}_3$ generated by the action $\gamma_{\mathbb{Z}_3}\colon (\varphi_1, \varphi_2) \mapsto (-\varphi_2, \varphi_1 - \varphi_2)$, the time-reversal symmetry $\mathcal{R}$ given by the action $\gamma_{\mathcal{R}}\colon (\varphi_1, \varphi_2, t) \mapsto (\varphi_2, \varphi_1, -t)$, and the reversible symmetry of parametric shift given by the action $\gamma_\alpha\colon (\varphi_1, \varphi_2, \alpha, t) \mapsto (\varphi_1, \varphi_2, \alpha + \pi, -t)$. The superposition of $\mathbb{Z}_3$ and $\mathcal{R}$ yields the existence of three involutions $\mathcal{R}_i$, $i = 1, 2, 3$, with the corresponding fixed spaces $\mathrm{Fix}\,\mathcal{R}_1\colon \varphi_1 = \varphi_2$, $\mathrm{Fix}\,\mathcal{R}_2\colon \varphi_2 = 0$, and $\mathrm{Fix}\,\mathcal{R}_3\colon \varphi_1 = 0$.

System (19) is conservative for $\alpha = 0$ and $\alpha = \pi$ and has the first integral

$$E(\varphi_1, \varphi_2) = \cos\varphi_1 + \cos\varphi_2 + \cos(\varphi_1 - \varphi_2).$$

For $\alpha = \pm\pi/2$, the system is gradient (and, hence, dissipative). If the parameter α is unequal to the values indicated above, then we observe the coexistence of conservative dynamics in a certain region of the phase space of the system with the dissipative dynamics in the remaining part of this space. The conservative region $\mathcal{D}_0$ is formed around the synchronous equilibrium $\Phi_{\text{sync}} = (0, 0)$. This region is bounded by a $\mathbb{Z}_3$-symmetric heteroclinic cycle (curvilinear triangle) formed by three saddles: $S_1\,(\widetilde{\varphi}, \widetilde{\varphi}) \in \mathrm{Fix}\,\mathcal{R}_1$, $S_2\,(-\widetilde{\varphi}, 0) \in \mathrm{Fix}\,\mathcal{R}_2$, and $S_3\,(0, -\widetilde{\varphi}) \in \mathrm{Fix}\,\mathcal{R}_3$, where $\widetilde{\varphi} = \pi - 2\alpha$, and their one-dimensional invariant manifolds $W^u(S_i) = W^s(S_{i+1})$, $i = 1, 2, 3$. The region $\mathcal{D}_0$ is filled with a one-parameter family of concentric orbits with center at the point Φ_{sync}. The dissipative region $\mathbb{T}^2 \setminus \mathcal{D}_0$ of the dynamical system is formed around two equilibria $\Phi^{(1)}_{\text{splay}} = (2\pi/3, 4\pi/3)$ and $\Phi^{(2)}_{\text{splay}} =$

$(4\pi/3, 2\pi/3)$ one of which is attracting and the other is repelling for $\alpha \neq \pm\pi/2$. The entire dissipative region is filled with heteroclinic trajectories connecting the points $\Phi^{(1)}_{\rm splay}$ and $\Phi^{(2)}_{\rm splay}$.

The conservative region $\mathcal{D}_0$ occupies the entire phase space $\mathbb{T}^2$ for $\alpha = 0$, begins to decrease as $|\alpha|$ increases, and contracts into the point $\Phi_{\rm sync}$ (i.e., disappears) for $|\alpha| = \pi/2$. At the same time, the dissipative region, which was absent for $\alpha = 0$, appears around $\Phi^{(1)}_{\rm splay}$ and $\Phi^{(2)}_{\rm splay}$ for $|\alpha| \neq 0$, increases with $|\alpha|$, and occupies the entire phase space as $|\alpha|$ approaches $\pi/2$. The subsequent increase in $|\alpha|$ from $\pi/2$ to π is accompanied by symmetric processes running in the opposite direction. We can easily trace the process of contraction of the conservative region by watching the motion of three saddle points S_i of the heteroclinic cycle along fixed lines of the reversible symmetry $\operatorname{Fix}\mathcal{R}_i$ observed as the parameter α varies. We also note that $\alpha = 0$ and $\alpha = \pi$ are the bifurcation values for the fixed points $\Phi^{(1)}_{\rm splay}$ and $\Phi^{(2)}_{\rm splay}$ corresponding to the degenerate Andronov–Hopf bifurcation. At the time of bifurcation, the points $\Phi^{(1)}_{\rm splay}$ and $\Phi^{(2)}_{\rm splay}$ are surrounded with neutral families of concentric periodic orbits. The values $\alpha = \pm\pi/2$ are bifurcation values for the point $\Phi_{\rm sync}$ corresponding to a triple degenerate transcritical bifurcation realized along each line $\operatorname{Fix}\mathcal{R}_i$, $i = 1, 2, 3$. At the time of bifurcation, the point $\Phi_{\rm sync}$ is transformed from the center (for $\alpha \neq \pm\pi/2$) into a degenerate saddle with six saddle cells.

A *system of four coupled oscillators* of the form (16) with coupling function (18) also exhibits conservative-dissipative dynamics, which has both common and absolutely different specific features as compared with lower-dimensional systems. The corresponding three-dimensional system in phase variables (17) with $N = 4$ and $l = 1$ has two conservative regions, namely, $\mathcal{D}_0$ around the synchronous state $\Phi_{\rm sync} = (0, 0, 0)$ and $\mathcal{D}_2$ around the double antiphase state $\Phi^{(2)}_{\rm splay}(\pi, 0, \pi)$ (two oscillators are in the antiphase relative to the other two oscillators). The other two fixed points $\Phi^{(1)}_{\rm splay} = (\pi/2, \pi, 3\pi/2)$ and $\Phi^{(3)}_{\rm splay} = (3\pi/2, \pi, \pi/2)$ are, respectively, an attractor and a repeller for $\alpha \neq 0$ and $\alpha \neq \pi$. The dissipative region $\mathbb{T}^3 \backslash (\mathcal{D}_0 \cup \mathcal{D}_2)$ is formed by the heteroclinic trajectories connecting $\Phi^{(1)}_{\rm splay}$ and $\Phi^{(3)}_{\rm splay}$.

The conservative regions $\mathcal{D}_0$ and $\mathcal{D}_2$ have identical structures due to the symmetry $\mathbb{Z}_2$ and differ only by the opposite directions of the vector field inside the regions. Each of these regions is bounded by a tetrahedron-like surface ($\partial\mathcal{D}_0$ or $\partial\mathcal{D}_2$, respectively). Four edges of this surface form a $\mathbb{Z}_4$-symmetric heteroclinic cycle and the other two edges are completely formed by saddle points that are degenerate saddles (neutrally stable along the edges). The faces of each tetrahedron-like surface $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_2$ are filled with a one-parameter family of heteroclinic trajectories (the arcs connecting two saddle points of one of the above-mentioned two edges). Any pair of heteroclinic trajectories forms a heteroclinic cycle. Thus, the described boundary of the region $\partial\mathcal{D}_0$ (or $\partial\mathcal{D}_2$) consists of two two-dimensional sets of two-saddle heteroclinic cycles (two pairs of glued faces) connected by a four-saddle heteroclinic cycle into a tetrahedron-like surface. The interior of $\partial\mathcal{D}_0$ (or $\partial\mathcal{D}_2$) is filled with a two-parameter family of neutral limit cycles concentrically located around neutral fixed points of the straight line

$$V_0 = \bigcap_{i=1}^{4} \operatorname{Fix}\mathcal{R}_i = (\varphi_1, 0, \varphi_1)$$

(note that $\operatorname{Fix}\mathcal{R}_1 = \operatorname{Fix}\mathcal{R}_3$ and $\operatorname{Fix}\mathcal{R}_2 = \operatorname{Fix}\mathcal{R}_4$). Every periodic trajectory crosses $\operatorname{Fix}\mathcal{R}_1$ at two points

$$(\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_1) \quad \text{and} \quad (\overline{\varphi}_1 - \overline{\varphi}_2, -\overline{\varphi}_1, \overline{\varphi}_1 - \overline{\varphi}_2)$$

and the plane $\operatorname{Fix}\mathcal{R}_2$ at the points

$$(-\overline{\varphi}_1, 0, \overline{\varphi}_2 - \overline{\varphi}_1) \quad \text{and} \quad (\overline{\varphi}_2 - \overline{\varphi}_1, 0, -\overline{\varphi}_1),$$

where $\overline{\varphi}_1$ and $\overline{\varphi}_2$ are the initial values of the trajectories on $\operatorname{Fix}\mathcal{R}_1$.

As in the case of lower dimension, for $\alpha = 0$ (and, symmetrically, for $\alpha = \pi$), system (17) is completely conservative and, hence, $\mathcal{D}_0 \bigcup \mathcal{D}_2 = \mathbb{T}^3$. As the parameter $|\alpha|$ increases from zero, each conservative region is contracted and, hence, the space occupied by the dissipative region increases. In the case where $|\alpha|$ reaches the value $\pi/2$, both conservative regions shrink into the fixed points $\Phi^{(1)}_{\text{splay}}$ and $\Phi^{(3)}_{\text{splay}}$ (in this case, they are degenerate saddles) and create the possibility for the dissipative region to undergo maximum possible expansion and fill the entire phase space.

Five oscillators can be coupled in a ring oscillatory network of the form (16) in two ways: locally (each oscillator is coupled only with two nearest neighbors; $l = 1$) and globally (each oscillator interacts with all other oscillators with two types of bonds; $l = 2$). In both cases $l = 1$ and $l = 2$, the system in phase differences (17) exhibits conservative-dissipative dynamics. The conservative region $\mathcal{D}_0 \in \mathbb{T}^4$ is concentrated around the neutral synchronous point Φ_{sync}. It is bounded by a $\mathbb{Z}_5$-symmetric hypersurface (polyhedron) $\partial\mathcal{D}_0$ completely formed by heteroclinic cycles. The boundary $\partial\mathcal{D}_0$ can be described with the help of a complete directed graph based on five vertices (fixed saddle points) S_i and has ten edges. The three-dimensional faces of $\partial\mathcal{D}_0$ are completely filled with heteroclinic trajectories connecting two different saddle vertices S_i. As a result, we observe the appearance of two- and three-dimensional sets of heteroclinic cycles formed by heteroclinic trajectories from one or several faces. The region $\mathcal{D}_0$ is filled with a dense two-parameter set of concentric (around Φ_{sync}) quasiperiodic tori with five interwoven one-parameter (identical with respect to the symmetry $\mathbb{Z}_5$) sets of periodic orbits. The presence of quasiperiodic trajectories makes the cases with $N \geq 5$ strongly different from the cases $N = 3$ and $N = 4$. The dissipative region $\mathbb{T}^4 \setminus \mathcal{D}_0$ is formed around the equilibria $\Phi^{(i)}_{\text{splay}}$, $i = 1, \ldots, 4$, two of which are attractors and the other two are repellers. It is completely filled with heteroclinic trajectories connecting the pairs of points $\Phi^{(i)}_{\text{splay}}$.

Systems of arbitrarily many (N) *oscillators* of the form (16) and (17) also reveal the coexistence of conservative and dissipative dynamics for

$$g(x) = -\sin(x - \alpha)$$

independently of the number of couples between the neighbors l. There is a principal difference between the structures of the conservative region $\mathcal{D}_0 \in \mathbb{T}^{N-1}$ for (17) and its boundary $\partial\mathcal{D}_0$ observed for even and odd N. The boundary $\partial\mathcal{D}_0$ is a $\mathbb{Z}_N$-symmetric and $(N-2)$-dimensional hypersurface in $\mathbb{T}^{N-1}$ or a hyperpolyhedron that can be described by a complete graph based on the saddle points S_i, $i = 1, \ldots, N$, playing the role of vertices. All $N(N-1)/2$ one-dimensional edges of $\partial\mathcal{D}_0$ are invariant manifolds of the saddles S_i and form heteroclinic cycles in the case of odd N. Note that, in the case of an even number of oscillators, only $N(N-3)/2$ edges of $\partial\mathcal{D}_0$ are heteroclinic trajectories, whereas the other N edges are completely formed by one-parameter sets of degenerate saddles. In the case of odd N, all periodic and quasiperiodic orbits interior for $\mathcal{D}_0$ are concentrated around the point Φ_{sync}. At the same time, for even N, these orbits are concentrated around the one-parameter set of central fixed points

$$V_0 = (\varphi, 0, \varphi, 0, \ldots, \varphi, 0, \varphi),$$

where φ is the parameter of location of a fixed point on the straight line. The family of periodic orbits inside $\mathcal{D}_0$ is one-parameter for odd N and two-parameter for even N. In the case of even N, parallel with the conservative region $\mathcal{D}_0$, we always encounter another symmetric region $\mathcal{D}_{N/2}$ around the equilibrium $\Phi^{(N/2)}_{\text{splay}}$.

As the parameter $|\alpha|$ changes from 0 to $\pi/2$, we observe the transformation of the system from completely conservative into conservative-dissipative and, finally, into completely dissipative (as in the low-dimension cases described above). A similar situation with nonhomotopic regions is observed, in particular, for the other type of reversible systems [60, 61, 63]. The results of additional numerical analysis reveal the presence of chaotic trajectories in the conservative region (at least for $N \geq 6$) mixed with periodic and quasiperiodic orbits. Note that, in this case, chaotic trajectories are trajectories of the neutral type, i.e., they are similar to classical ABC-flows [26, 67].

3.4. Systems of Nonidentical Elements. It turns out that system (13) may also have conservative-dissipative dynamics in the case of partial symmetry breaking. This specific type of dynamics is preserved for certain relations between the frequencies of oscillators ω_i and symmetries of the function $g(x)$.

System (13) [as well as (15)] with s *circulant skew-symmetric* coupling matrix K and an *odd* coupling function $g(x)$ is *divergence free* for *any eigenfrequencies* ω_i and any dimensions N. Thus, the entire phase space $\mathbb{T}^{N-1}$ of system (15) is filled with neutral (relative to each other) trajectories (that can be also chaotic). Hence, unlike the case of identical frequencies $\omega_i = \omega$, the periodic and quasiperiodic orbits or heteroclinic cycles can be both homologous to zero and nonhomologous to zero.

A system can be conservative-dissipative if a perturbation of eigenfrequencies breaks the cyclic symmetry $\mathbb{Z}_N$ but preserves at least one time-reversal symmetry $\mathcal{R}$ (for N available $\mathcal{R}_i$ in the case of identical oscillators). This symmetry $\mathcal{R}$ is preserved if the eigenfrequencies ω_i are pairwise equidistant from a certain mean value. By $\Phi_\Delta^{(k)}$ we denote a fixed point formed from the state of distributed phases $\Phi_{\text{splay}}^{(k)}$, $k = 0, \dots, N-1$, as a result of perturbation of the frequency vector $\Delta = (\Delta_1, \dots, \Delta_{N-1})$, $\Delta_i = \omega_1 - \omega_i$, $i = 1, \dots, N-1$. In this notation, $\Phi_\Delta^{(0)}$ are formed by the perturbations

$$\Phi_{\text{sync}} = \Phi_{\text{splay}}^{(0)}.$$

The point $\Phi_\Delta^{(k)}$ remains an attractor (repeller) for small perturbations of Δ from the position $(0, \dots, 0)$ in the case where the point $\Phi_{\text{splay}}^{(k)}$ is an attractor (repeller). Thus, system (15) with circulant skew-symmetric K may exhibit conservative-dissipative dynamics if the frequency differences satisfy the relations

$$\Delta_{N-i} = -\Delta_i, \quad i = 1, \dots, [N/2], \tag{20}$$

and at least one point $\Phi_\Delta^{(k)}$, $k \neq 0$, is either a sink or a source. Note that, under condition (20), in the presence of a perturbation Δ, the point $\Phi_\Delta^{(0)}$ moves along Fix $\mathcal{R}$ together with the entire conservative part. The bifurcations of the parameters of Δ may lead to the disappearance of both conservative and dissipative regions.

System (15) may have numerous conservative "islands" if the coupling function $g(x)$ has higher harmonics in its expansion in the Fourier series. In the case of equidistant eigenfrequencies, conservative regions can be topologically nontrivial without internal central fixed points. Moreover, the boundaries of conservative regions may have quite complicated structures, and some of these regions with different topologies can be located inside each other [48].

At the end of the section, we note that numerous theoretical results were also obtained for the circulant networks whose nodes can be specified by complicated multidimensional systems or systems with delay. The inversion symmetry of these networks is a source of rich dynamical behaviors, including rotating waves [68–74], heteroclinic cycles [75–80], symmetric chaos [68, 81–83], chimera states [84, 85], and compactons [64]. As applications to neuroscience, the bifurcation mechanisms in rings of coupled Hodgkin–Huxley neurons with inhibitory and excitatory synapses were studied in [86–89]. Note that the complicated dynamical scenarios and multistability were also discovered in the cited works. The relationship between the dynamics of infinite-dimensional oscillatory systems given on circulant networks and the solutions of the complex Ginzburg–Landau equation and nonlinear Schrödinger equation was established in [48, 90, 91].

4. Chimeras in Models of Coupled Oscillators

4.1. Concept of Chimera States. In the present section, we mainly describe a relatively new collective phenomena of significant interest, rather than analyze a specific oscillatory model. In [84], Kuramoto and Battogtokh showed, for the first time, that the set of identical oscillators given on a ring network and subjected to the action of a dynamical system may decompose into two groups of oscillators such that all oscillators from one group are coherent and phase-locked, whereas the oscillators from the other group are incoherent and desynchronized. This type of collective behavior seems to be impossible because all elements of the network are equivalent. Later, Abrams and Strogatz called this collective behavior a *chimera state* (or simply *chimera*) [85], which emphasizes the presence of a combination of incongruous elements. The existence of chimera states turned into a subject of extensive theoretical and experimental investigations. Note that about 700 works devoted to this research area were published for last two decades. The results of numerical investigations show that chimera states may appear in systems on the networks with absolutely different architectures and for diverse objects each of which is characterized by a complicated individual dynamics. It was shown that chimera states appear in mechanics [92, 93], chemistry [94], biology [95], neuroscience [96, 97], electronics [98, 99], optics [100], electrochemistry [101], sociology [102], and in analyzing the behaviors of financial markets [103]. In our opinion, the theory proposed by Ott and Antonsen (the Ott–Antonsen ansatz) [104] proves to be very important for the theoretical investigation of chimera states. It is worth noting that the first experimental confirmations of existence of the chimera states [94, 105] appeared ten years later than the first theoretical work [84]. In recent years, chimera states have been extensively investigated, new concepts of dynamical regimes related to chimeras have been introduced, chimeras have been studied for networks with various architectures and different individual nodes, and chimera states have been described for broader classes of dynamical systems [102, 106–112].

4.2. Definition of Weak Chimeras. The first results on coherent-incoherent states were introduced in [84] for a complex-valued oscillatory system given by the Ginzburg–Landau-type equation with periodic boundary conditions. Note that the same chimera states are obtained for the phase part of the Ginzburg–Landau equation, which has the following form:

$$\frac{\partial \theta(y,t)}{\partial t} = \omega + \int_{-\pi}^{\pi} \mathcal{G}(y-\overline{y}) \sin(\theta(y,t) - \theta(\overline{y},t) + \alpha) d\overline{y}, \tag{21}$$

where $y \in \mathbb{T}^1$ is the space variable, $\theta(y,t)$ is the phase of the oscillator at a point y, ω is frequency, α is the phase shift, and $\mathcal{G}(y-\overline{y})$ is an even function (kernel) guaranteeing the nonlocal coupling between oscillators in a ring depending on the distance between them $|y-\overline{y}|$. In the first works [84, 85], the kernels had the form $\mathcal{G}(y) = \kappa \exp(-\kappa|y|)/2$ and $\mathcal{G}(y) = (1+\kappa\cos y)/(2\pi)$, where κ is a parameter, respectively. The theoretical results obtained for chimera states were confirmed by the data of numerical simulations carried out for a large number of coupled oscillators and the parameter of phase shift α close to $\pi/2$.

Equation (21) can be reduced to the following finite-dimensional Kuramoto–Sakaguchi system with the coupling function $g(x) = -\sin(x-\alpha)$:

$$\frac{d\theta_i}{dt} = \omega + \sum_{j=1}^{N} K_{ij}\, g(\theta_i - \theta_j), \quad i = 1, \dots, N, \tag{22}$$

where $K = (K_{ij})_{i,j=1}^N$ is a circulant symmetric matrix whose elements are distributed by analogy with $\mathcal{G}(y)$ and $|y|$ is a discrete analog of the distance between an element of the matrix and a diagonal element. The necessary condition for the collective regime to be regarded as a chimera state is the equivalence of all elements of the network. It is clear that the ring structure of the network is not the most general structure in the analyzed case and, moreover, the results of investigations show that chimeras appear not only for ring networks. In the most general case, it is reasonable to consider *indistinguishable oscillators*, i.e., oscillators that are identical and coupled in a sense that they have the same number and strength of couplings [113].

In the first theoretical works dealing with chimeras, the researchers, as a rule, considered the arrays of infinitely many (or of a large number of) coupled oscillators. For the complete understanding of the structure of these dynamical regimes, it is important to determine the minimal number of elements guaranteeing the existence of chimeras and the degree of complexity of each element of the network required for the existence of this state. However, in the major part of works devoted to chimeras, the authors did not make attempts to give a rigorous analytic definition of the chimera state, which can be easily applied to the case of low-dimensional systems. As implicit definitions of chimera states, we can mention the examples of regimes given in the first classical works in this field [84, 85, 114]. The first strict definition of a (somewhat simplified) concept of chimeras was given in [115].

We say that the oscillators i and j on the trajectory of system (22) are *frequency synchronized* if

$$\Omega_{ij} = \lim_{t\to\infty} \frac{1}{t}\left(\theta_i(t) - \theta_j(t)\right) = 0,$$

where we choose continuous representations for $\theta_i(t)$ and $\theta_j(t)$. The set $\mathcal{A}$ is a *weak chimera state* [115] for a system of indistinguishable phase oscillators if it is a connected chain-recurrent [116] flow-invariant set such that, in each trajectory from the set $\mathcal{A}$, one can find i, j, and k such that $\Omega_{ij} \neq 0$ and $\Omega_{ik} = 0$.

This definition does not impose any restrictions on the dynamical behavior and stability of the solutions corresponding to weak chimeras. It also does not pretend to give a description of chimera states in a broad sense with regard for the transitivity and chaotic behavior of the corresponding solutions, and also as $N \to \infty$. However, it can be regarded as an important step in understanding the behavior of coherent-incoherent regimes in low-dimensional systems and as an auxiliary tool for the investigation of these regimes in the general case.

4.3. Minimal Chimeras in Oscillatory Systems. It follows from the definition that the trajectory of a weak chimera state must be phase-unlocked. This means that the minimal necessary dimension of system (22) is equal to *four* because the three-dimensional system of this kind is globally connected and has only phase-locked trajectories. For the same reason, an additional condition should be imposed on the system of four phase oscillators: it is necessary to have at least *two types of different couplings* K_{ij}. A more nontrivial condition for the existence of chimeras follows from the results of additional investigations and can be formulated as follows: the coupling function must have at least two harmonics in its expansion in the Fourier series

$$g(x) = -\sin(x - \alpha) + r\sin(2x) \tag{23}$$

with parameters α and r.

An oscillatory network of four phase oscillators $\theta_1, \ldots, \theta_4$ that satisfies all conditions imposed above was constructed in [115]. This system has strong couplings between two pairs of oscillators

$$K_{i,i+2} = K_{i+2,i} = 1$$

and all other couplings are weak, i.e.,

$$K_{i,i+1} = K_{i+1,i} = \varepsilon,$$

and the index is taken modulo 4. It is clear that, in this case, the oscillatory network is formed by indistinguishable elements and system (22) takes the form

$$
\begin{aligned}
\frac{d\theta_1}{dt} &= \omega + (g(\theta_1-\theta_3)+g(0)) + \varepsilon\,(g(\theta_1-\theta_2)+g(\theta_1-\theta_4)\,,\\
\frac{d\theta_2}{dt} &= \omega + (g(\theta_2-\theta_4)+g(0)) + \varepsilon\,(g(\theta_2-\theta_3)+g(\theta_2-\theta_1)\,,\\
\frac{d\theta_3}{dt} &= \omega + (g(\theta_3-\theta_1)+g(0)) + \varepsilon\,(g(\theta_3-\theta_4)+g(\theta_3-\theta_2)\,,\\
\frac{d\theta_4}{dt} &= \omega + (g(\theta_4-\theta_2)+g(0)) + \varepsilon\,(g(\theta_4-\theta_1)+g(\theta_4-\theta_2)\,.
\end{aligned}
\tag{24}
$$

It was proved that there exists an open set of parameters (r,α) such that the four-oscillator system (22), (23) has a *stable weak chimera state* for sufficiently small $|\varepsilon|$. The idea of construction of the minimal system with chimera dynamics can be described as follows: For $\varepsilon = 0$, network (24) consists of two separate oscillatory blocks (modules) (θ_1,θ_3) and (θ_2,θ_4) given by identical two-dimensional systems. The two-dimensional system is characterized by the bistability of the inphase regime with the antiphase regime for $r < -|\cos\alpha|/2$. Hence, for the corresponding initial data, system (24) has two clusters (coherent regimes) $\theta_1 = \theta_3$ and $\theta_2 = \theta_4 + \pi$ that can freely rotate along the phase circle relative to each other because the modules are not coupled with each other for $\varepsilon = 0$, i.e., they are incoherent. Since the system is structurally stable for $\varepsilon = 0$, the coherent-incoherent regime is also preserved for perturbations ε such that $|\varepsilon| < \varepsilon_0(r,\alpha)$. Note that, for the existence of weak chimeras, the parameter ε must be smaller than 1 because, otherwise, the system is global with identical interactions between the elements.

The idea of construction of modular systems with chimeras is quite fruitful and enables one to give an analytic proof of the existence of complex stable chimeras for systems of higher dimensions. Consider a network with $N = mk$ oscillators formed by m modules each of which contains k oscillators. This network is described by the following system of equations:

$$
\frac{d\theta_{ij}}{dt} = \omega + \sum_{q=1}^{k}\left[K_{ij,iq}\,g(\theta_{ij}-\theta_{iq}) + \varepsilon K_{ij,pq}\sum_{p=1,p\neq i}^{m} K_{ij,iq}\,g(\theta_{ij}-\theta_{pq})\right],
$$

where $i = 1,\ldots,m$, $j = 1,\ldots,k$, and the coupling strengths $K_{ij,pq}$ are chosen to guarantee that the oscillators in the network are indistinguishable. By using the results on the multistability of a separate uncoupled k-dimensional module in the system, we can prove, e.g., the existence either of heteroclinic chimeras [115, 117] or of chaotic chimeras [110].

The existence, stability, and bifurcations of weak chimeras in a system of six coupled oscillators (22) were investigated in detail for three possible (nonmodular) configurations of the network of indistinguishable elements of the system in [115, 118]. As a convenient method for the investigation of low-dimensional systems of this kind, we can mention the analysis of possible symmetries of the system and the corresponding invariant cluster manifolds. Thus, for the system of six indistinguishable oscillators, the subspaces

$$
(\theta_1,\ldots,\theta_6) = (\theta_1,\theta_2,\theta_1,\theta_1+\pi,\theta_2,\theta_1+\pi)
$$

and

$$
(\theta_1,\ldots,\theta_6) = (\theta_1,\theta_1+\pi,\theta_2,\theta_1,\theta_1+\pi,\theta_2+\pi)
$$

are invariant (however, this is true not only for these subspaces).

In low-dimensional systems, weak chimeras may correspond to several (at least two) oscillatory clusters that freely move relative to each other along the phase circle. Therefore, the investigation of chimera states is reduced to the analysis of the solutions of systems on manifolds of lower dimensions. The application of phase differences decreases the dimension of the original system by one and can also reduce the dimension of the required manifold. The existence of a minimal chimera corresponds to a trajectory nonhomologous to zero on a two-dimensional toroidal manifold of the system in phase differences [115]. The existence of an invariant cluster manifold corresponds to the *coherence* of oscillators inside the cluster, while the nonhomology of a trajectory to zero on the invariant torus guarantees, in this case, the phase unlockness of oscillators from different clusters and, hence, the incoherence of the collective regime. The procedure of reduction to the two-dimensional system makes it possible to better understand the structure of trajectories and the bifurcation mechanisms of appearance of minimal chimeras in the study of these problems, first inside an invariant manifold and then in the transverse directions. A weak chimera can be heteroclinic if the corresponding trajectory on the invariant manifold is heteroclinic. In [118], it was shown that the system of six indistinguishable oscillators can be integrable and, moreover, can have a continuous family of neutral chimera solutions. Note that the Kuramoto–Sakaguchi system of five locally coupled oscillators on a ring may also have a weak chimera state of the form

$$(\theta_1, \ldots, \theta_5) = (\theta_1, \theta_2, \theta_3, 2\theta_3 - \theta_2, 2\theta_3 - \theta_1)$$

for the values of the parameter of phase shift close to $\pi/2$.

As already indicated, a system of *globally* coupled identical phase oscillators each of which (together with the coupling) is given by a single equation of the form (1) cannot have chimera states. In the case where the oscillators are given by more complicated equations (one of the imposed conditions is violated), chimeras may also exist on the global networks of identical couplings. Thus, in particular, the existence of chimeras in the Kuramoto–Sakaguchi model with global coupling and *delay* was shown in [119]. Chimeras in globally coupled complex-valued Stuart–Landau oscillators were described in [120]. The existence of chimeras in the network of four globally coupled lasers was shown in [121].

The existence of weak chimeras in the extended Kuramoto–Sakaguchi model with inertia

$$m\frac{d^2\theta_i}{dt^2} + \varepsilon\frac{d\theta_i}{dt} = \omega - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j - \alpha), \quad i = 1, \ldots, N,$$

where m and ε are additional parameters, was proved for $N = 3$ in [122]. In this case, two requirements imposed above are violated, namely, the system is globally coupled and the minimal number of oscillators is equal to three. At the same time, each oscillator is described by a second-order equation (or by a two-dimensional system). The results obtained in [122] show that the minimal number of elements (of any kind) for the existence of chimeras in a dynamical system is equal to three.

The investigations of chimera regimes are carried out in different directions: construction of more complicated networks of identical elements in which chimeras exist; verification of the existence of chimeras in known networks but with different nodal elements; comparison of the corresponding results for finite and infinite-dimensional systems; experimental investigations and their comparison with theoretical results and the data of computer simulations; description of chimeras with certain additional properties (chaotic, fractal, heteroclinic, transitive, etc.), and investigation of the stability and bifurcations of chimera states. In particular, a classification of the chimera states according to various criteria can be found in [123]. Numerous recent works appearing in this field demonstrate that chimeras are now one of the most popular objects in the investigation of the collective regimes in complicated systems and reveal broad prospects for their subsequent development.

5. Discussion

In the present paper, we study systems of coupled identical phase oscillators of the Kuramoto type (1) given on networks with symmetries of various types. It was shown how various types of architecture of the networks and interaction between the elements affect the existence of different types of collective behavior in the systems. In particular, it was proved that the presence of symmetries in the interactions between the elements always promotes the appearance of certain cluster regimes. In turn, cluster regimes correspond to the invariant manifolds of the system in the case where the clusters are not destroyed under the action of a vector field of the system (this happens not for all networks). Much attention is given to the description of bifurcations of transitions between different regimes of collective dynamics.

In the first part of the present paper (see [1]), we described several globally coupled oscillatory systems. It was shown that the permutation symmetry $\mathbf{S}_N$ of all oscillators plays a decisive role in the formation of dynamics of the system. The presence of permutation symmetry results in the existence of all cluster invariant manifolds with any number of oscillators, which, in turn, leads to the appearance of canonical invariant regions, which bound the phase-locked trajectories. The subgroups of the symmetry group $\mathbf{S}_N$ also result in the existence of collective regimes of various kinds, such as, e.g., the splay state, the global antiphase regime, and various regimes of slow switching (the existence of these regimes also depends on the parameters of the coupling function). Most of these regimes are unstable under violations of the identity of eigenfrequencies of oscillators ω_i. However, symmetric networks can be regraded as the most convenient starting point for the theoretical investigations and subsequent analyses of nonsymmetric networks under perturbations of the parameters and in the case of application of the bifurcation theory.

In the second part of the paper, we have considered the systems with nonglobal interaction between the elements but in the presence of certain symmetries. It is shown that the absence of some couplings between the elements strongly affects their collective behavior. In particular, the absence of global interaction in each part of the analyzed system leads to the breaking of permutation symmetry, a smaller number of possible cluster regimes, the absence of closed invariant regions, and the appearance of phase-unlocked orbits. The absence of global couplings in the network leads to the disappearance of certain collective regimes accompanied by the formation of an even greater number of new regimes with very nontrivial specific features. Thus, it was shown that the oscillatory system with central element has the property of multistability of the antiphase "competition for synchronization" regimes and that the system of indistinguishable elements may have either chimera states or the conservative-dissipative dynamics. It was also shown that parity or oddness of the coupling function may turn an oscillatory system into a gradient or conservative system. It is also worth noting that the systems with global and circulant skew-symmetric coupling have the opposite properties in this respect.

Note that various models of Kuramoto-type phase oscillators are the simplest dynamical systems with one-dimensional nodes and without additional equations specifying the interaction between the elements. Nevertheless, these systems have a very broad spectrum of collective dynamics corresponding to the most important regimes observed in more complicated systems. In addition, the Kuramoto-type systems prove to be among the systems most convenient for the analytic investigation of the collective behaviors. In many cases, it turns out that the obtained and well-studied regimes typical of simple oscillatory models also appear in a certain (possibly modified) form in much more complicated physical, chemical, or neural systems. In the most evident way, this observation is illustrated by the phenomenon of chimera states whose existence was established for various and often fairly complicated natural systems. The investigation of simpler models of interacting elements often opens new directions for their analysis and gives tools for the study of more complicated and general systems with similar specific features of the architecture of networks. The aim of the present survey was not only to describe some theoretical results obtained for relatively complicated symmetric dynamical systems but also to demonstrate the potentialities and broad prospects of the investigation and description of the natural phenomena of collective interaction on the basis of various oscillatory Kuramoto-type models.

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