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COHOMOLOGY OF ALGEBRAS OF DIHEDRAL TYPE. V

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The Yoneda algebras for a family of local algebras of dihedral type (from the famous K. Erdmann's list) are described in terms of generators and relations. Bibliography: 21 titles.

1. Introduction

The Yoneda algebras were calculated in [1–17] for several families of algebras of dihedral or semidihedral type, which are contained in Erdmann's classification [18]. The present paper continues this series. We recall that along with the diagrammatic method of D. Benson and J. Carlson [19], used, for example, in [1–3, 5, 10, 12, 15], we often used the approach of paper [4]. The essence of this approach is that based on some empirical observations, we state some hypothesis about the structure of minimal projective resolutions of simple modules. Then after the study of this hypothesis, we read the "cohomology information" from the found resolutions. As a result we get the description of the Yoneda algebras of the algebras under consideration.

In the present paper, we use the approach from [4] to describe the Yoneda algebra for a family of local algebras presented in Erdmann's classification [18].

We note that the technique of paper [4] was also applied in calculating the Hochschild cohomology algebra for a significant number of families of finite-dimensional algebras and for the integral group rings of dihedral and semidihedral groups, see [21] and references therein.

2. Formulation of the main results

Let R be a finite-dimensional algebra over a field K. All modules under consideration are left. Denote by

$$\mathcal{E}(M) = \bigoplus_{m\geq 0} \mathrm{Ext}_R^m(M, M)$$

the Ext-algebra of the R-module M. For a basic K-algebra R with Jacobson radical $J(R)$, the Ext-algebra $\mathcal{E}(R/J(R))$ is called the Yoneda algebra of R and is denoted by $\mathcal{Y}(R)$. In the sequel, we assume that the basic field K is algebraically closed.

We define algebras $R_{m,n} := K[X, Y]/J$, where $m, n \in \mathbb{N}$, $m \geq n \geq 2$, $m + n > 4$, and the ideal J is defined by the elements

$$XY,\ YX,\ X^m - Y^n.$$

The images of the elements X, Y under the canonical map from $K[X, Y]$ to R are denoted by x and y, respectively. Since the algebra $R_{m,n}$ is local, $\mathcal{Y}(R_{m,n})$ is the Ext-algebra of a unique simple $R_{m,n}$-module S.

To describe the Yoneda algebra $\mathcal{Y}(R_{m,n})$, we construct several graded algebras. We introduce a grading on the free K-algebra $K\langle u_1, u_2, v\rangle$, such that

$$\deg u_1 = \deg u_2 = 1, \quad \deg v = 2,$$

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and define an algebra $\mathcal{E} := K\langle u_1, u_2, v\rangle / I$, where the ideal I is generated by the (homogeneous) elements

$$u_1 v - v u_1,\ u_2 v - v u_2,\ u_1^2,\ u_2^2. \tag{2.1}$$

We introduce a grading on $\mathcal{E}$, induced by the grading of $K\langle u_1, u_2, v\rangle$.

Moreover, we define an algebra $\mathcal{E}' := K\langle u_1, u_2\rangle / I'$, where the algebra $K\langle u_1, u_2\rangle$ is considered with a grading such that $\deg u_1 = \deg u_2 = 1$, and the ideal I' is generated by the elements

$$u_1 u_2^2 - u_2^2 u_1, u_1^2;$$

the algebra $\mathcal{E}'$ is considered with the induced grading.

Theorem 2.1. *Let $R = R_{m,n}$, where $m, n \in \mathbb{N}$, $m \geq n \geq 2$, and $m + n > 4$. If $n > 2$, then the Yoneda algebra $\mathcal{Y}(R)$ of the algebra R is isomorphic to the algebra $\mathcal{E}$ as a graded algebra. If $n = 2$, then $\mathcal{Y}(R) \simeq \mathcal{E}'$ as graded algebras.*

3. Resolution

Let $R = R_{m,n}$, where $m, n \in \mathbb{N}$, $m \geq n \geq 2$, and $m+n > 4$. It is clear that $\dim_K R = m+n$ and R admits the set

$$\{1\} \cup \{x^i\}_{i=1}^{m} \cup \{y^j\}_{j=1}^{n-1}$$

as a K-basis. The homomorphism $w^* \colon R \to R$ of the right multiplication by $w \in R$ is denoted, for simplicity, again as w.

We consider the following bicomplex $B_{\bullet\bullet}$ lying in the first quadrant of the plane (i.e., its rows and columns are enumerated by $0, 1, 2, \dots$):

$$\begin{array}{ccccccccccc}
\dots & & \dots & & \dots & & \dots & & \dots & & \\
x\big\downarrow & & y\big\downarrow & & x\big\downarrow & & y\big\downarrow & & x^{m-1}\big\downarrow & & \\
R & \xleftarrow{-y^{n-1}} & R & \xleftarrow{-x^{m-1}} & R & \xleftarrow{-y^{n-1}} & R & \xleftarrow{-x} & R & \xleftarrow{-y} & \dots \\
y\big\downarrow & & x\big\downarrow & & y\big\downarrow & & x^{m-1}\big\downarrow & & y^{n-1}\big\downarrow & & \\
R & \xleftarrow{x^{m-1}} & R & \xleftarrow{y^{n-1}} & R & \xleftarrow{x} & R & \xleftarrow{y} & R & \xleftarrow{x} & \dots \\
x\big\downarrow & & y\big\downarrow & & x^{m-1}\big\downarrow & & y^{n-1}\big\downarrow & & x^{m-1}\big\downarrow & & \\
R & \xleftarrow{-y^{n-1}} & R & \xleftarrow{-x} & R & \xleftarrow{-y} & R & \xleftarrow{-x} & R & \xleftarrow{-y} & \dots \\
y\big\downarrow & & x^{m-1}\big\downarrow & & y^{n-1}\big\downarrow & & x^{m-1}\big\downarrow & & y^{n-1}\big\downarrow & & \\
R & \xleftarrow{x} & R & \xleftarrow{y} & R & \xleftarrow{x} & R & \xleftarrow{y} & R & \xleftarrow{x} & \dots
\end{array} \tag{3.1}$$

Proposition 3.1. *The total complex*

$$\mathrm{Tot}(B_{\bullet\bullet}) =: Q_\bullet = (Q_t, d_t \colon Q_{t+1} \to Q_t)_{t \geq 0}$$

of the bicomplex $B_{\bullet\bullet}$ in (3.1) *is the minimal projective resolution of the simple R-module S.*

Proof. The acyclicity of the complex $\mathrm{Tot}(B_{\bullet\bullet}) \to S \to 0$ is verified by a direct calculation. $\square$

Remark 3.2. To prove Proposition 3.1, one can also consider the spectral sequence of bicomplex (3.1) and observe that the second sheet of this spectral sequence degenerates (cf. [17, proof of Proposition 2.1]).

Corollary 3.3. $\dim_K \mathrm{Ext}_R^t(S, S) = t + 1$.

Remark 3.4. By construction of $Q_\bullet$, we have $Q_t = \bigoplus_{i+j=t} B_{ij}$; in the sequel, we arrange the direct summands in this decomposition in ascending order of the second index j.

4. Generators of the Yoneda algebra

As before, let $R = R_{m,n}$ with $m, n \in \mathbb{N}$, $m \geq n \geq 2$, and $m + n > 4$. In this section, we specify a (finite) set of generators for the Yoneda algebra

$$\mathcal{Y}(R) = \mathcal{E}(R/J(R)) = \bigoplus_{t\geq 0} \operatorname{Ext}^t_R(S, S).$$

We recall the following interpretation of the Yoneda product in the algebra $\mathcal{Y}(R)$ (for the case of the family of local algebras under consideration, cf. [17]). Let $Q_\bullet(S)$ be the minimal projective resolution of the simple R-module S. Using the isomorphism

$$\operatorname{Ext}^t_R(S, S) \simeq \operatorname{Hom}_R(\Omega^t(S), S),$$

we represent any element $f \in \operatorname{Ext}^t_R(S, S)$ by the corresponding homomorphism $\widetilde{f}\colon\ \Omega^m(S) \to S$. Moreover, $\widetilde{f}$ can be extended to the chain map $\{\varphi_l\colon Q_{t+l} \to Q_l\}_{l\geq 0}$. In its turn, $\widehat{f} = \varphi_0$ uniquely defines the homomorphism $\widetilde{f}$. The homomorphism φ_i is called the ith translate of the element f and is denoted by $\mathrm{T}^i(f)$ or $\mathrm{T}^i(\widehat{f})$. Then the Yoneda product of the elements $f \in \operatorname{Ext}^t_R(S, S)$ and $g \in \operatorname{Ext}^s_R(S, S)$ is defined by the map

$$(g \cdot f)\widehat{\ } = \widehat{g} \cdot \mathrm{T}^s(\widehat{f}). \tag{4.1}$$

We introduce the following homogeneous elements of the algebra $\mathcal{Y}(R)$:

$$\widehat{u}_1 = (1\,,\, 0)\colon Q_1 = R^2 \to R, \qquad u_1 \in \operatorname{Ext}^1_R(S, S);$$

$$\widehat{u}_2 = (0\,,\, 1)\colon Q_1 = R^2 \to R, \qquad u_2 \in \operatorname{Ext}^1_R(S, S);$$

$$\widehat{v} = (0,\, 1\,,\, 0)\colon Q_2 = R^3 \to R, \qquad v \in \operatorname{Ext}^2_R(S, S).$$

Proposition 4.1. *As the translates (of suitable orders) of the elements u_1, u_2, v, we can take the following maps:*

$$\mathrm{T}^1(u_1) = \begin{pmatrix} 0 & x^{m-2} & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \mathrm{T}^2(u_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

$$\mathrm{T}^1(u_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -y^{n-2} & 0 \end{pmatrix},$$

$$\mathrm{T}^2(u_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $n > 2$, and

$$\mathrm{T}^2(u_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

for $n = 2$,
and

$$\mathrm{T}^1(v) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathrm{T}^2(v) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. Follows by a direct verification of commutativity of some squares corresponding to chain maps between the resolutions. ∎

Proposition 4.2. (a) *If $n > 2$, then the elements u_1, u_2, v of the algebra $\mathcal{Y}(R)$ satisfy the relations*

$$u_1^2 = u_2^2 = 0, \;\; u_1 v = v u_1, \;\; u_2 v = v u_2.$$

(b) *If $n = 2$, then the following relations are satisfied in $\mathcal{Y}(R)$:*

$$u_1^2 = 0, \;\; u_1 u_2^2 = u_2^2 u_1.$$

Proof. The proof is based on formula (4.1) and elementary calculation using formulas in Proposition 4.1. We leave the corresponding detailed calculation to the reader. □

Proposition 4.3. (a) *Assume that $n > 2$. The* Ext*-groups presented below admit the following K-bases:*

$$\mathrm{Ext}^1_R(S,S) = \langle u_1, u_2\rangle, \mathrm{Ext}^2_R(S,S) = \langle u_2u_1, v, u_1u_2\rangle,$$
$$\mathrm{Ext}^3_R(S,S) = \langle u_1u_2u_1, u_1v, u_2v, u_2u_1u_2\rangle.$$

(b) *Assume that $n = 2$. Then*

$$\mathrm{Ext}^1_R(S,S) = \langle u_1, u_2\rangle, \mathrm{Ext}^2_R(S,S) = \langle u_2u_1, u_2^2, u_1u_2\rangle,$$
$$\mathrm{Ext}^3_R(S,S) = \langle u_1u_2u_1, u_1u_2^2, u_2^3, u_2u_1u_2\rangle.$$

Proof. The desired equalities follow directly from the definition of the elements u_1, u_2, v and from calculation of their products with using suitable translates of these elements (cf. the proof of Proposition 4.2). We leave the details of calculation to the reader. □

Proposition 4.4. (a) *Assume that $n > 2$. The set $\{1, u_1, u_2, v\}$ generates the Yoneda algebra $\mathcal{Y}(R)$ as K-algebra.*

(b) *Assume that $n = 2$. Then the set $\{1, u_1, u_2\}$ generates the Yoneda algebra $\mathcal{Y}(R)$ as K-algebra.*

Proof. Since

$$\mathrm{Ext}^t_R(S,S) \simeq \mathrm{Hom}_R(Q_t, S),$$

any homogeneous element of the Yoneda algebra $\mathcal{Y}(R)$ is given by a (unique) homomorphism $f\colon Q_t = \bigoplus\limits_{i+j=t} B_{ij} \to S$. Using Proposition 4.3 and Corollary 3.3, we may assume that $t > 3$. Now we prove by induction on t that f is represented as a sum of products of elements of smaller degrees.

We may assume that $f(B_{t-j,j}) \neq 0$ for exactly one value of j. According to a "local" configuration of bicomplex (3.1) in which the module $B_{t-j,j}$ is included, we construct commutative diagrams of the form

$$\begin{array}{ccc} Q_{t+1} & \xrightarrow{d_t} & Q_t \\ {\scriptstyle\varphi_1}\big\downarrow & & {\scriptstyle\varphi_0}\big\downarrow \\ Q_{\ell+1} & \xrightarrow{d_\ell} & Q_\ell \end{array}, \qquad (4.2)$$

such that $\ell < t$ and $\mathrm{Ker}\,\varphi_0 \subset \mathrm{Ker}\, f$. For such a diagram, there exists a homomorphism $f'\colon Q_\ell \longrightarrow S$ for which $f'\varphi_0 = f$. Moreover, since R is a QF-algebra, $\varphi_0 = \mathrm{T}^\ell(\tilde{\varphi})$ for some $\tilde{\varphi}\colon Q_{t-\ell} \longrightarrow S$. Thus, we can apply the induction hypothesis to f' and $\tilde{\varphi}$.

The decomposition

$$Q_t = R \oplus Q_{t-2} \oplus R, \qquad (4.3)$$

implies that the matrix of the differential d_{2s}, $s \geq 2$, has block form corresponding to the decomposition

$$d_{2s} = \left(\begin{array}{c|c|c} x & \mathrm{O} & \mathrm{O} \\ \hline \mathrm{O} & \widetilde{d}_{2s-2} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & y \end{array}\right), \tag{4.4}$$

where $\widetilde{d}_{2s-2} = C_1 \cdot d_{2s-2} \cdot C_2$ and

$$\left.\begin{array}{l} C_1 = \operatorname{diag}(-1, 1, \ldots, 1, -1) \in M_{2s-1}(R), \\ C_2 = \operatorname{diag}(1, -1, \ldots, 1, -1) \in M_{2s}(R). \end{array}\right\} \tag{4.5}$$

Similarly, for the differentials d_{2s+1} we have

$$d_{2s+1} = \left(\begin{array}{c|c|c} y & \mathrm{O} & \mathrm{O} \\ \hline \mathrm{O} & \widetilde{d}_{2s-1} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & x \end{array}\right), \tag{4.6}$$

where $\widetilde{d}_{2s-1} = D_1 \cdot d_{2s-1} \cdot D_2$ and

$$\left.\begin{array}{l} D_1 = \operatorname{diag}(-1, 1, \ldots, -1, 1) \in M_{2s}(R), \\ D_2 = \operatorname{diag}(1, -1, \ldots, -1, 1) \in M_{2s+1}(R). \end{array}\right\} \tag{4.7}$$

First assume that $1 \leq j \leq 2s-1$. If $t = 2s$, $s \geq 2$, then we construct the diagram of form (4.2), in which $\ell = 2s-2$, φ_0 is the composition of the projection onto the direct summand $Q_{2s} \to Q_{2s-2}$ (see decomposition (4.3)) and the automorphism of Q_{2s-2} with matrix C_1 (in (4.5)), and φ_1 is the composition of the projection onto the direct summand $Q_{2s+1} \to Q_{2s-1}$ (see decomposition (4.3)) and the automorphism of the module Q_{2s-1} with matrix C_2.

Similarly, for odd $t = 2s+1$, $s \geq 1$, there is diagram (4.2), in which $\ell = 2s-1$, φ_0 is the composition of the projection onto the direct summand $Q_{2s+1} \to Q_{2s-1}$ and the automorphism of Q_{2s-1} with matrix D_1 (in (4.7)), and φ_1 is the composition of the projection onto the direct summand $Q_{2s+2} \to Q_{2s}$ with the automorphism of Q_{2s} with matrix D_2.

Now we consider the case $j = 0$. The configurations in bicomplex (3.1), which include the module $B_{t,0}$, have the following forms:

$$\text{(I)}\quad \begin{array}{ccc} R & & \\ {\scriptstyle y^{n-1}}\big\downarrow & & \\ R & \xleftarrow{\;x\;} & R \end{array} \qquad\qquad \text{(II)}\quad \begin{array}{ccc} R & & \\ {\scriptstyle x^{m-1}}\big\downarrow & & \\ R & \xleftarrow{\;y\;} & R \end{array}$$

(the diagrams contain all arrows ending at the corresponding module $B_{t,0}$).

a) Assume that we have the configuration I. Then $t = 2s$ is even (and $s \geq 2$). If $n > 2$, then we construct a diagram of form (4.2), in which $\ell = 2$, φ_0 is the projection of Q_t onto the first two direct summands $B_{t,0} \oplus B_{t-1,1} \simeq B_{2,0} \oplus B_{1,1}$, and φ_1 is the composition of the projection of Q_{t+1} onto the first three direct summands $B_{m+1,0} \oplus B_{m,1} \oplus B_{m-1,2} \simeq B_{3,0} \oplus B_{2,1} \oplus B_{1,2}$ and the map defined by the matrix $\operatorname{diag}(1, 1, y^{n-2})$.

If $n = 2$, then we construct diagram (4.2) in which $\ell = 2s-2$, φ_0 is the composition of the projection of Q_{2s} onto the first $s+1$ direct summands $\bigoplus_{j=0}^{s} B_{2s-j,j} \simeq X := \bigoplus_{j=0}^{s} B_{2s-2-j,j}$ and the endomorphism of X with matrix $\operatorname{diag}(1, 1, \ldots, 1, x^{m-2})$, and φ_1 is the projection of Q_{2s+1} onto the first $s+1$ direct summands $\bigoplus_{j=0}^{s} B_{2s+1-j,j}$.

b) Let us consider the configuration II. In this case, $t = 2s+1$ is odd ($s \geq 2$). Again, first we assume that $n > 2$. Then we construct square (4.2), in which $\ell = 1$, φ_0 is the composition of the projection of Q_{2s+1} onto the first two direct summands $B_{2s+1,0} \oplus B_{2s,1} \simeq X := B_{1,0} \oplus B_{0,1}$ and the endomorphism of X, defined by the matrix $\operatorname{diag}(1, y^{n-2})$, and φ_1 is the projection of Q_{2s+2} onto the first two direct summands $B_{2s+2,0} \oplus B_{2s+1,1} \simeq B_{2,0} \oplus B_{1,1}$.

In the case $n = 2$, we construct square (4.2), in which $\ell = 2s - 1$, φ_0 is the projection of Q_{2s+1} onto the first $s+1$ direct summands, and φ_1 is the composition of the projection of Q_{2s+2} onto the first $s+1$ direct summands $\bigoplus_{j=0}^{s} B_{2s+2-j,j} \simeq Y := \bigoplus_{j=0}^{s} B_{2s-j,j}$ and the endomorphism of Y with matrix $\operatorname{diag}(1, 1, \dots, 1, x^{m-2})$.

Finally, if $j = t$ (i.e., $f(B_{0,t}) \neq 0$), then the argument is similar to the case $j = 0$.

□

5. End of the proof of Theorem 2.1

Put $R = R_{m,n}$. We consider the case $n > 2$ in detail; minor modifications are needed in the corresponding arguments for $n = 2$ only. Let $\mathcal{E} = K[u_1, u_2, v]/I$ be a graded K-algebra described in Sec. 2. From Propositions 4.2 and 4.4, it follows that there exists a surjective homomorphism $\mathcal{E} \longrightarrow \mathcal{Y}(R)$ of graded K-algebras, that takes the canonical generators of $\mathcal{E}$ to the corresponding generators of $\mathcal{Y}(R)$, introduced in the beginning of Sec. 4. Let $\mathcal{E} = \bigoplus_{t \geq 0} \mathcal{E}^t$ be the direct decomposition of the algebra $\mathcal{E}$ into homogeneous direct summands. Now, the following statement completes the proof of Theorem 2.1.

Proposition 5.1. *For any $t \geq 0$,* $\quad \dim_K \mathcal{E}^t = t + 1$.

Proof. We assume that $t > 0$. By the defining relations of the algebra $\mathcal{E}$ (see (2.1)), each monomial $f \in \mathcal{E}$ is presented in the form

$$f = u_1^{\varepsilon_1} u_2 u_1 u_2 \dots u_1 u_2^{\varepsilon_2} v^k, \tag{5.1}$$

where $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. It easily follows that $\mathcal{E}^{2s}$, $s \geq 1$, is generated by the set

$$\{(u_1u_2)^{s-j} v^j\}_{j=0}^{s} \cup \{(u_2u_1)^{s-j} v^j\}_{j=0}^{s-1},$$

and $\mathcal{E}^{2s+1}$, $s \geq 0$, is generated by the set

$$\{(u_1u_2)^{s-j} u_1 v^j\}_{j=0}^{s} \cup \{(u_2u_1)^{s-j} u_2 v^j\}_{j=0}^{s-1}.$$

Thus, for any t, we have $\dim_K \mathcal{E}^t \leq t+1$. Since there is a surjective linear map $\mathcal{E}^t \longrightarrow \mathcal{Y}(R)^t = \operatorname{Ext}_R^t(S, S)$, we obtain the desired equality.

If $n = 2$ (i.e., $R = R_{m,2}$), then there is a surjective homomorphism $\mathcal{E}' \longrightarrow \mathcal{Y}(R)$ of graded K-algebras, where the algebra $\mathcal{E}'$ is defined in Sec. 2. Slightly modifying the above argument (namely, using u_2^2 instead of v), we can prove that in this case $\dim_K (\mathcal{E}')^t = t + 1$ again. □

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