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Series expansions for monogenic functions in Clifford algebras and their application

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Abstract. This paper deals with studying some properties of a monogenic function defined on a vector space with values in the Clifford algebra generated by the space. We provide some expansions of a monogenic function and consider its application to study solutions of second-order partial differential equations.

Keywords. Monogenic function, Clifford algebra, partial differential equation.

1. Introduction

The Clifford algebra is a unital associative algebra, which is the minimal extension of a finite-dimensional vector space V with a quadratic form. In this case, the vector space V is called embedded in the Clifford algebra, and its orthogonal basis determines the basis of the algebra. We are interested in properties of a monogenic function defined on the embedded vector space with values in the Clifford algebra, which is a null solution of the Dirac operator. A monogenic function is one of the basic concepts of Clifford analysis [1]. We will obtain the expansion of a left monogenic function into a series of Fueter-type polynomials. Since any right monogenic function can be expanded in a similar manner, only left monogenic functions are considered. We also study an application of the properties of monogenic functions to solving some second-order partial differential equations.

2. Monogenic functions in Clifford algebras

Suppose E^{d+1} is an $(d+1)$-dimensional linear space over $\mathbb{R}$ with basis $\boldsymbol{e}_i$, $i=0,1,\dots,d$. Suppose E^{d+1} is embedded in the Clifford algebra $Cl_{p,q}$, $p+q=d+1$ with an identity $I=\boldsymbol{e}_0$ and $\boldsymbol{e}_k\boldsymbol{e}_l+\boldsymbol{e}_l\boldsymbol{e}_k=\boldsymbol{0}$, $k\neq l,\ 1\le k,l\le d$. In addition, $\boldsymbol{e}_k^2=I$ for $1\le k\le p$ and $\boldsymbol{e}_k^2=-I$ for $p<k\le d$.

Denote, by $D=\sum_{i=0}^{d}\boldsymbol{e}_i\frac{\partial}{\partial x_i}$, the generalized Cauchy–Riemann operator. It is easily verified that $DD^*=\sum_{i=0}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{i=p+1}^{d}\frac{\partial^2}{\partial x_i^2}$.

We are interested in studing the following equation:

$$DD^*f=0. \tag{2.1}$$

Definition 2.1. *A function $f:E^{d+1}\to Cl_{p,q}$ is called left monogenic, if $Df=0$, and f is called right monogenic, if $fD=0$.*

In what follows, we will consider only left monogenic functions. Any left monogenic function is a solution of the equation $DD^*f=0$.

For $E^{d+1}\ni\boldsymbol{x}=\sum_{i=0}^{d}\boldsymbol{e}_ix_i$, we introduce the following polynomials:

$$p_k(\boldsymbol{x})=x_k-\boldsymbol{e}_kx_0,\quad 1\le k\le d.$$

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Polynomials p_k might be called Fueter-type polynomials, since, in the case of the algebra of quaternions $\mathbb{H} = x_0 + \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{k}x_3$, polynomials $x_1 - \boldsymbol{i}x_0$, $x_2 - \boldsymbol{j}x_0$, $x_3 - \boldsymbol{k}x_0$ were introduced by Fueter in [2] and are known as the Fueter polynomials.

An expansion of hyperholomorphic quaternion functions in Fueter polynomials was reported in [3], and the similar expansion of hyperholomorphic co-quaternion functions was considered in [4]. The relations between monogenic and hyperholomorphic functions was considered in [5].

Theorem 2.1. *Let $f \in C^{\infty}\left(E^{d+1}\right)$, $n \in \mathbb{N}$, and let the function f with its derivatives*

$$\frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}} f(\boldsymbol{x})$$

be left monogenic. Then there exists real θ such that

$$f(\boldsymbol{x}) = f(\boldsymbol{0}) + \sum_{i=1}^{d} p_k(\boldsymbol{x}) \frac{\partial}{\partial x_i} f(\boldsymbol{0})$$

$$+ \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} p_i(\boldsymbol{x}) p_j(\boldsymbol{x}) \frac{\partial^2}{\partial x_i \partial x_j} f(\boldsymbol{0}) + \dots + \frac{1}{(n-1)!}$$

$$\times \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \dots \sum_{i_{n-1}=1}^{d} p_{i_1}(\boldsymbol{x}) p_{i_2}(\boldsymbol{x}) \dots p_{i_{n-1}}(\boldsymbol{x}) \frac{\partial^{n-1}}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_{n-1}}} f(\boldsymbol{0})$$

$$+ \frac{1}{n!} \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \dots \sum_{i_n=1}^{d} p_{i_1}(\boldsymbol{x}) p_{i_2}(\boldsymbol{x}) \dots p_{i_n}(\boldsymbol{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}} f(\theta_n \boldsymbol{x}),$$

where $0 \le \theta_n < 1$.

Proof. Consider cfunction $f(t\boldsymbol{x})$ as a function of $t \in \mathbb{R}$. Taking into account that $f(\boldsymbol{x})$ is left monogenic, i.e., $\frac{\partial}{\partial x_0} f(\boldsymbol{x}) = -\sum_{i=1}^{d} \boldsymbol{e}_i \frac{\partial}{\partial x_i} f(\boldsymbol{x})$, we obtain

$$\frac{d}{dt} f(t\boldsymbol{x}) = \sum_{i=0}^{d} x_i \frac{\partial}{\partial x_i} f(t\boldsymbol{x}) = \sum_{i=1}^{d} (x_i - \boldsymbol{e}_i x_0) \frac{\partial}{\partial x_i} f(t\boldsymbol{x}). \tag{2.2}$$

Setting $t = 0$, we have

$$\frac{d}{dt} f(\boldsymbol{0}) = \sum_{i=1}^{d} p_i(\boldsymbol{x}) \frac{\partial}{\partial x_i} f(\boldsymbol{0}). \tag{2.3}$$

Now, considering that the functions

$$\frac{\partial}{\partial x_i} f(\boldsymbol{x}), \quad i = 1, \dots, n$$

are left monogenic and accounting for Eq. (2.2), we get

$$\frac{d^2}{dt^2} f(t\boldsymbol{x}) = \sum_{i=1}^{d} (x_i - \boldsymbol{e}_i x_0) \frac{d}{dt} \left(\frac{\partial}{\partial x_i} f(t\boldsymbol{x}) \right) = \sum_{i=1}^{d} (x_i - \boldsymbol{e}_i x_0) \sum_{j=1}^{d} (x_j - \boldsymbol{e}_j x_0) \frac{\partial^2}{\partial x_j \partial x_i} f(t\boldsymbol{x}).$$

Hence, for $t=0$,

$$\frac{d^2}{dt^2} f(\mathbf{0}) = \sum_{i=1}^{d} \sum_{j=1}^{d} (x_i - \boldsymbol{e}_i x_0)(x_j - \boldsymbol{e}_j x_0) \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{0}). \tag{2.4}$$

In the same way, we can show that

$$\frac{d^n}{dt^n} f(t\boldsymbol{x}) = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_n=1}^{d} p_{i_1}(\boldsymbol{x})\, p_{i_2}(\boldsymbol{x}) \ldots p_{i_n}(\boldsymbol{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}} f(t\boldsymbol{x}). \tag{2.5}$$

Now, consider the Taylor series for $f(t\boldsymbol{x})$ w.r.t. t:

$$\begin{aligned} f(t\boldsymbol{x}) &= f(\mathbf{0}) + \frac{d}{dt} f(\mathbf{0})\, t + \ldots \\ &\quad + \frac{d^{n-1}}{dt^{n-1}} f(\mathbf{0}) \frac{t^{n-1}}{(n-1)!} + \frac{d^n}{dt^n} f(\theta_n \boldsymbol{x}) \frac{t^n}{n!}, \quad 0 \le \theta_n < 1. \end{aligned} \tag{2.6}$$

Substituting Eqs. (2.3)–(2.5) into Eq. (2.6), we obtain Eq. (2.1).

If

$$\frac{1}{n!} \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_n=1}^{d} p_{i_1}(\boldsymbol{x}) p_{i_2}(\boldsymbol{x}) \ldots p_{i_n}(\boldsymbol{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}} f(\theta_n \boldsymbol{x}) \to \mathbf{0}, \quad n \to \infty,$$

then Eq. (2.1) implies that

$$f(\boldsymbol{x}) = f(\mathbf{0}) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_n=1}^{d} p_{i_1}(\boldsymbol{x}) p_{i_2}(\boldsymbol{x}) \ldots p_{i_n}(\boldsymbol{x}) \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}} f(\mathbf{0}) \right). \tag{2.7}$$

In particular case where

$$\frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}} f(\mathbf{0}) = a_n$$

for all $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_n=1}^{d} p_{i_1}(\boldsymbol{x})\, p_{i_2}(\boldsymbol{x}) \ldots p_{i_n}(\boldsymbol{x})\, a_n \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} (p_{i_1}(\boldsymbol{x}) + p_{i_2}(\boldsymbol{x}) + \cdots + p_{i_n}(\boldsymbol{x}))^n a_n \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right)^n a_n. \end{aligned}$$

Thus, in this case,

$$f(\boldsymbol{x}) = a_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right)^n a_n, \tag{2.8}$$

where $a_0 = f(\mathbf{0})$. □

3. Study of hyperbolic and Laplace equations with the use of monogenic functions

By using Eqs. (2.7) and (2.8), we can construct solutions of Eq. (2.1). For example, setting $a_n = n!$ for $1 \le n \le m$, $m \in \mathbb{N}$ and $a_n = 0$ for all $n > m$ in Eq. (2.8), we get the following solution of Eq. (2.1):

$$f(\boldsymbol{x}) = 1 + \sum_{n=1}^{m} \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right)^n = \sum_{n=0}^{m} \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right)^n .$$

Taking into account that $\left(\sum_{k=1}^{d} \boldsymbol{e}_k\right)^{2n} = (p-q)^n$ in $Cl_{p,q}$, we obtain a real solution of Eq. (2.1) as follows:

$$u(x_1, x_2, \ldots, x_d) = \sum_{n=0}^{m} \left(\left(\sum_{k=1}^{d} x_k \right)^n + C_n^2 \left(\sum_{k=1}^{d} x_k \right)^{n-2} (p-q)^2 x_0^2 \right.$$
$$\left. + C_n^4 \left(\sum_{k=1}^{d} x_k \right)^{n-4} (p-q)^4 x_0^4 + \ldots \right),$$

where we assume that $C_n^k = 0$ for $n < k$.

We can also construct solutions of Eq. (2.1) in the cases of infinite non-zero terms of series (2.7). Indeed, by setting $a_n = 1$ for all $n \in \mathbb{N}$ in Eq. (2.7), we get the following solution of Eq. (2.8):

$$f(\boldsymbol{x}) = \exp\left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right) = \exp\left(\sum_{k=1}^{d} x_k \right) \exp\left(-x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right).$$

Considering that $\left(\sum_{k=1}^{d} \boldsymbol{e}_k\right)^{2n} = (p-q)^n$ and the fact that expressions $\left(\sum_{k=1}^{d} \boldsymbol{e}_k\right)^{2n+1}$ are not real for all $n \in \mathbb{N}$, we obtain the following real solution of Eq. (2.1):

$$u(x_1, x_2, \ldots, x_d) = \exp\left(\sum_{k=1}^{d} x_k \right) \cos\left(x_0 \sqrt{p-q} \right).$$

In the case where $p = 0$ (or $q = 0$), i.e., the Clifford algebra is $Cl_{0,d}$ (or $Cl_{d,0}$), it follows from Eq. (2.1) that a monogenic function f satisfies the d-dimensional Laplace equation

$$\Delta_d f = \sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2} f = 0. \tag{3.1}$$

Consider the case $Cl_{0,3}$ with the embedded vector space E^3 and its basis I, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, where I is the identity and $\boldsymbol{e}_k \boldsymbol{e}_l + \boldsymbol{e}_l \boldsymbol{e}_k = -2I\delta_{kl}$, $1 \le k, l \le 2$. In this case, Eq. (3.1) is the Laplace equation

$$\Delta_3 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f = 0.$$

With regard for Eq. (2.8), it is easily seen that

$$f(x, y, z) = \frac{(\boldsymbol{e}_1 + \boldsymbol{e}_2) x - y - z}{1 - (\boldsymbol{e}_1 + \boldsymbol{e}_2) x + y + z}$$

is monogenic in $E^3 \backslash \{(\boldsymbol{e}_1 + \boldsymbol{e}_2) x + y + z = 1\}$.

Let us find the real and imaginary parts of f:

$$f(x,y,z) = \frac{(\boldsymbol{e}_1+\boldsymbol{e}_2)x - y - z}{1-(\boldsymbol{e}_1+\boldsymbol{e}_2)x + y + z}\,\frac{1+(\boldsymbol{e}_1+\boldsymbol{e}_2)x + y + z}{1+(\boldsymbol{e}_1+\boldsymbol{e}_2)x + y + z}$$
$$= \frac{y+z-2x^2-(y+z)^2}{2x^2+(1-y-z)^2} + \frac{x}{2x^2+(1-y-z)^2}(\boldsymbol{e}_1+\boldsymbol{e}_2).$$

Thus, we have the following solutions of Eq. (3.1):

$$u_1(x,y,z) = \frac{y+z-2x^2-(y+z)^2}{2x^2+(1-y-z)^2}$$

and

$$u_2(x,y,z) = \frac{x}{2x^2+(1-y-z)^2}.$$

Now, consider the case $Cl_{1,3}$ where the embedded vector space E^4 has the basis I, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, I is the identity, and $\boldsymbol{e}_k\boldsymbol{e}_l + \boldsymbol{e}_l\boldsymbol{e}_k = -2I\delta_{kl}$, $1 \le k,l \le 3$.

In this case, Eq. (2.1) takes the form

$$\frac{\partial^2}{\partial t^2}f - \frac{\partial^2}{\partial x^2}f - \frac{\partial^2}{\partial y^2}f - \frac{\partial^2}{\partial z^2}f = 0. \tag{3.2}$$

Consider the function

$$f(w) = \frac{w}{1-w},$$

where $w = (\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3)t - x - y - z$.

It follows from Eq. (2.7) that, for $w \neq 1$, the function $f(w)$ is monogenic.

The real and imaginary parts of f can be calculated as follows:

$$f(w) = \frac{(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)t - x - y - z}{1-(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)t + x + y + z}\,\frac{1+(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)t + x + y + z}{1+(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)t + x + y + z}$$
$$= \frac{3t^2-(x+y+z)+x+y+z}{(1-x-y-z)^2-3t^2} + \frac{t(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)}{(1-x-y-z)^2-3t^2}.$$

Hence, we have obtained two solutions of Eq. (3.2):

$$u_1(t,x,y,z) = \frac{3t^2-(x+y+z)+x+y+z}{(1-x-y-z)^2-3t^2}$$

and

$$u_2(t,x,y,z) = \frac{t}{(1-x-y-z)^2-3t^2}.$$

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