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ON THE INTEGRABILITY OF LATTICE EQUATIONS WITH TWO CONTINUUM LIMITS

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Abstract. We study a new example of a lattice equation, which is one of the key equations of a generalized symmetry classification of five-point differential-difference equations. This equation has two different continuum limits, which are the well-known fifth-order partial-differential equations, namely, the Sawada–Kotera and Kaup-Kupershmidt equations. We justify its integrability by constructing an L-A pair and a hierarchy of conservation laws.

***Keywords and phrases*:** differential-difference equation, integrability, Lax pair, conservation law.

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1. Introduction. We consider the differential-difference equation

$$u_{n,t} = (u_n + 1)\left(\frac{u_{n+2}u_n(u_{n+1}+1)^2}{u_{n+1}} - \frac{u_{n-2}u_n(u_{n-1}+1)^2}{u_{n-1}} + (2u_n+1)(u_{n+1}-u_{n-1})\right), \qquad (1)$$

where $n \in \mathbb{Z}$, $u_n(t)$ is an unknown function of one discrete variable n and one continuous variable t, and the index t in the notation $u_{n,t}$ denotes the time derivative. Equation (1) is obtained in the generalized symmetry classification of five-point differential-difference equations of the form

$$u_{n,t} = F(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}), \qquad (2)$$

performed in [8–10]. Equation (1) coincides with Eq. (E16) in [9]; it was obtained earlier in [2]. Equations of the form (2) play an important role in the study of four-point discrete equations on square lattices, which are very relevant today (see e.g., [1, 5, 6, 16]).

At the present time, there is very little information on Eq. (1). It was proved in [9] that Eq. (1) possesses a nine-point generalized symmetry of the form

$$u_{n,\theta} = G(u_{n+4}, u_{n+3}, \dots, u_{n-4}).$$

As for relations to other known integrable equations of the form (2), nothing useful from the viewpoint of constructing solutions is known (see details in the next section).

However, this equation occupies a special place in the classification (see [8–10]). In particular, it possesses a remarkable property discovered in [7]: this equation possesses two different continuum limits, which are the well-known Kaup–Kupershmidt and Sawada–Kotera equations. For this reason, Eq. (1) deserves a more detailed study.

In Sec. 2, we discuss the known properties of Eq. (1). In order to justify the integrability of (1), we construct an L-A pair in Sec. 3 and show that it provides an infinite hierarchy of conservation laws in Sec. 4.

2. Special place of Eq. (1) **in the classification of [8–10].** In two lists of integrable equations of the form (2) presented in [9, 10], the following four equations occupy a special place: they are

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Eq. (1) and

$$u_{n,t} = (u_n^2 - 1)\Big(u_{n+2}\sqrt{u_{n+1}^2 - 1} - u_{n-2}\sqrt{u_{n-1}^2 - 1}\Big), \tag{3}$$

$$u_{n,t} = u_n^2\big(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}\big) - u_n\big(u_{n+1} - u_{n-1}\big), \tag{4}$$

$$u_{n,t} = u_{n+1}u_n^3u_{n-1}\big(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}\big) - u_n^2\big(u_{n+1} - u_{n-1}\big). \tag{5}$$

Equations (3)–(5) correspond to Eqs. (E17) and (E15) in [9] and Eq. (E14) in [10], respectively. Equation (4) has been known for a long time (see [18]).

In the continuum limit, all other equations from [9, 10] are transformed into third-order equations of the form

$$U_\tau = U_{xxx} + F(U_{xx}, U_x, U), \tag{6}$$

where the indices τ and x denote the partial derivatives in τ and x, and mainly into the Korteweg–de Vries equation. These four equations correspond in the continuum limit to fifth-order equations of the form

$$U_\tau = U_{xxxxx} + F(U_{xxxx}, U_{xxx}, U_{xx}, U_x, U). \tag{7}$$

For all equations (1), (3)–(5), we get in the continuum limit one of the two well-known equations. One of them is the Kaup-Kupershmidt equation (see [4, 12])

$$U_\tau = U_{xxxxx} + 5UU_{xxx} + \frac{25}{2}U_xU_{xx} + 5U^2U_x, \tag{8}$$

and the other is the Sawada-Kotera equation (see [17]):

$$U_\tau = U_{xxxxx} + 5UU_{xxx} + 5U_xU_{xx} + 5U^2U_x. \tag{9}$$

Using the substitution

$$u_n(t) = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\varepsilon^2 U\left(\tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t\right), \quad x = \varepsilon n, \tag{10}$$

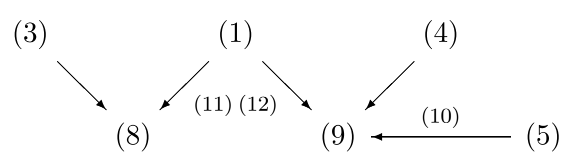
in Eq. (5), we obtain at $\varepsilon \to 0$ the Sawada–Kotera equation (9). All other continuum limits are known (see [1] for (4) and [7] for (1) and (3)). Here we explicitly replicate substitutions for Eq. (1), which has two different continuum limits. The substitution

$$u_n(t) = -\frac{4}{3} - \varepsilon^2 U\left(\tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t\right), \quad x = \varepsilon n \tag{11}$$

in (1) leads to Eq. (8), while the substitution

$$u_n(t) = -\frac{2}{3} + \varepsilon^2 U\left(\tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t\right), \quad x = \varepsilon n \tag{12}$$

leads to Eq. (9). The link between these discrete and continuous equations is shown in the following diagram:



We see that Eq. (8) has two different integrable approximations, while Eq. (9) has three approximations.

As far as we know, there are no relations between Eqs. (1), (3)–(5) and other known equations of the form (2) presented in [9, 10]. More precisely, we mean relations in the form of the transformations

$$\hat{u}_n = \varphi(u_{n+k}, u_{n+k-1}, \ldots, u_{n+m}), \quad k > m, \tag{13}$$

and their compositions (see a detailed discussion of such transformations in [8]). As for relations among (1), (3)–(5), Eq. (5) is transformed into (4) by $\hat{u}_n = u_{n+1}u_n$, i.e., (5) is a simple modification of (4). There is a complicated relation between Eqs. (1) and (4) found in [2]. As was shown in [9], it is a composition of two Miura-type transformations. It is very difficult to use that relation for the construction of solutions because the problem is reduced to solving the discrete Riccati type equations (see [9]).

There is a complete list of integrable equations of the form (7) (see [3, 13, 15]). Equations (8) and (9) play the key role in that list, since all the other are transformed into these two by transformations of the form:

$$\hat{U} = \Phi(U, U_x, U_{xx}, \dots, U_{x\dots x}).$$

3. *L*-*A* pair. As the continuum limit shows, Eq. (1) should be close to Eqs. (3), (4) in its integrability properties, and these equations (3), (4) have L-A pairs defined by (3×3)-matrices (see [1, 7]). Here we construct an L-A pair for Eq. (1) following [7].

We search for an L-A pair of the form

$$L_n\psi_n = 0, \quad \psi_{n,t} = A_n\psi_n, \tag{14}$$

with the operator L_n of the form

$$L_n = T^2 + l_n^{(1)}T + l_n^{(0)} + l_n^{(-1)}T^{-1}; \tag{15}$$

where $l_n^{(k)}$ with $k = -1, 0, 1$ depend on a finite number of functions u_{n+j} and T is the shift operator: $Th_n = h_{n+1}$. In the case (15), the operator A_n can be chosen as follows:

$$A_n = a_n^{(1)}T + a_n^{(0)} + a_n^{(-1)}T^{-1}.$$

The compatibility condition for the system (14) has the form

$$\frac{d(L_n\psi_n)}{dt} = (L_{n,t} + L_nA_n)\psi_n = 0; \tag{16}$$

it can be satisfied by virtue of Eqs. (1) and $L_n\psi_n = 0$.

If we assume that the coefficients $l_n^{(k)}$ depend only on u_n, then we can verify that $a_n^{(k)}$ depend only on u_{n-1} and u_n. However, in this case, the problem for Eq. (1) has no solutions. Therefore, we proceed to the case where the functions $l_n^{(k)}$ depend on u_n and u_{n+1}. Then the coefficients $a_n^{(k)}$ must depend on u_{n-1}, u_n, and u_{n+1}. In this case, we find the operators L_n and A_n with one irremovable arbitrary constant λ playing the role of the spectral parameter:

$$L_n = T^2 - \frac{U_{n+1}}{u_{n+1}}T + \lambda\frac{U_{n+1}}{u_n}\left(1 - \frac{u_n}{U_n}T^{-1}\right), \tag{17}$$

$$A_n = \frac{u_n}{U_n}\big(\lambda T^{-1} - \lambda^{-1}T\big) + \frac{u_n}{U_n^2}\big(u_{n-1} + u_{n+1}T^{-1}\big)(T-1), \tag{18}$$

where

$$U_n = \frac{u_n}{1+u_n}. \tag{19}$$

The L-A pair (14), (17), (18) can be rewritten in the standard matrix form with the (3×3)-matrices $\tilde{L}_n$ and $\tilde{A}_n$:

$$\Psi_{n+1} = \tilde{L}_n\Psi_n, \quad \Psi_{n,t} = \tilde{A}_n\Psi_n,$$

where Ψ_n is a spectral vector-valued function whose standard form is

$$\Psi_n = \begin{pmatrix}\psi_{n+1}\\ \psi_n\\ \psi_{n-1}\end{pmatrix}.$$

Here we slightly change Ψ by a gauge transformation to simplify the matrices $\tilde{L}_n$ and $\tilde{A}_n$:

$$\Psi_n = \begin{pmatrix} U_n\left(\lambda\psi_{n+1} + \dfrac{1}{u_n}\psi_n\right) \\ \psi_n \\ \psi_{n-1} \end{pmatrix},$$

and now $\tilde{L}_n$ and $\tilde{A}_n$ have the form

$$\tilde{L}_n = \begin{pmatrix} 0 & -\dfrac{1}{u_n} & \dfrac{1}{U_n} \\ \lambda U_n & \dfrac{U_n}{u_n} & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{20}$$

$$\tilde{A}_n = \begin{pmatrix} \dfrac{u_{n-1}u_{n-2}}{U_{n-1}^2} - \dfrac{u_{n+1}u_n}{U_n^2} - u_n + u_{n-1} & -\dfrac{u_{n-1}}{\lambda U_{n-1}} - \dfrac{u_{n-1}u_{n-2}}{\lambda U_{n-1}^2} & \dfrac{u_{n+1}u_n + u_{n+1} + u_n}{U_n} \\ \lambda(1+u_n)u_{n-1} - u_n & \dfrac{u_{n+1}u_n + u_{n+1} - u_n u_{n-1}}{U_n} - \dfrac{1}{\lambda} & \dfrac{\lambda u_n}{U_n} - \dfrac{u_{n+1}u_n}{U_n^2} \\ \lambda u_{n-1} - (1+u_{n-1})u_n & \dfrac{u_{n-1}u_{n-2}}{U_{n-1}^2} - \dfrac{u_{n-1}}{\lambda U_{n-1}} & \lambda - \dfrac{u_{n-1}u_{n-2}}{U_{n-1}^2} + \dfrac{u_n u_{n-1}}{U_{n-1}} \end{pmatrix}. \tag{21}$$

In this case, in contrast to (16), the compatibility condition can be represented in a form free of the spectral vector-valued function Ψ_n:

$$\tilde{L}_{n,t} = \tilde{A}_{n+1}\tilde{L}_n - \tilde{L}_n\tilde{A}_n. \tag{22}$$

4. Conservation laws. As far as we know, there exist two methods of constructing conservation laws by using the matrix L-A pair (22) (see [5, 11, 14]). However, it is not clear how to apply these methods in the case of the matrices (20) and (21). Here we will use another scheme of constructing conservation laws from the L-A pair (14) presented in [7], where it was applied to Eq. (3). Here we apply it to Eq. (1).

The structure of the operators (17) and (18) allows us to rewrite the L-A pair (14) in the form of the Lax pair. The operator L_n linearly depends on λ:

$$L_n = P_n - \lambda Q_n, \tag{23}$$

where

$$P_n = T^2 - \frac{U_{n+1}}{u_{n+1}}T, \quad Q_n = -\frac{U_{n+1}}{u_n}\left(1 - \frac{u_n}{U_n}T^{-1}\right),$$

and U_n is defined by (19). Introducing the notation $\hat{L}_n = Q_n^{-1}P_n$, we obtain the equation

$$\hat{L}_n\psi_n = \lambda\psi_n. \tag{24}$$

The functions $\lambda\psi_n$ and $\lambda^{-1}\psi_n$ in the second equation of (14) can be expressed in terms of $\hat{L}_n$ and ψ_n by using (24) and its consequence $\lambda^{-1}\psi_n = \hat{L}_n^{-1}\psi_n$. As a result, we have

$$\psi_{n,t} = \hat{A}_n\psi_n, \tag{25}$$

where

$$\hat{A}_n = \frac{u_n}{U_n}\Big(T^{-1}Q_n^{-1}P_n - TP_n^{-1}Q_n\Big) + \frac{u_n}{U_n^2}\Big(u_{n-1} + u_{n+1}T^{-1}\Big)(T-1).$$

It is important that the new operators $\hat{L}_n$ and $\hat{A}_n$ in the L-A pair (24), (25) are independent of the spectral parameter λ. For this reason, the compatibility condition can be written in the operator form without using the ψ-function:

$$\hat{L}_{n,t} = \hat{A}_n\hat{L}_n - \hat{L}_n\hat{A}_n = [\hat{A}_n, \hat{L}_n]; \tag{26}$$

this is exactly the Lax equation. The difference between this L-A pair and the well-known Lax pairs for the Toda and Volterra equations is that the operators $\hat{L}_n$ and $\hat{A}_n$ are nonlocal. Nevertheless, using the definition of inverse operators:

$$P_nP_n^{-1} = P_n^{-1}P_n = 1, \quad Q_nQ_n^{-1} = Q_n^{-1}Q_n = 1 \tag{27}$$

and the fact that they are linear, we can verify by direct calculations that (26) is valid.

The conservation laws for Eq. (1), which are expressions of the form

$$\rho_{n,t}^{(k)} = (T-1)\sigma_n^{(k)}, \quad k \geq 0,$$

can be derived from the Lax equation (26), despite the nonlocal structure of $\hat{L}_n$ and $\hat{A}_n$ (see [19]). For this, we must first represent the operators $\hat{L}_n$ and $\hat{A}_n$ as formal series in powers of T^{-1}:

$$H_n = \sum_{k\leq N} h_n^{(k)}T^k. \tag{28}$$

Such formal series can be multiplied according the rule $(a_nT^k)(b_nT^j) = a_nb_{n+k}T^{k+j}$. The inverse series of the form (28) can be easily obtained by definition (27), for example,

$$Q_n^{-1} = -\Big(1 + q_nT^{-1} + (q_nT^{-1})^2 + \ldots + (q_nT^{-1})^k + \ldots\Big)\frac{u_n}{U_{n+1}}, \quad q_n = \frac{u_n}{U_n}.$$

The series $\hat{L}_n$ has the second order:

$$\hat{L}_n = \sum_{k\leq 2} l_n^{(k)}T^k = -\left(\frac{u_n}{U_{n+1}}T^2 + u_n\left(\frac{u_{n-1}}{U_n^2} - \frac{1}{u_{n+1}}\right)T + \frac{u_{n-1}}{U_n}\left(\frac{u_nu_{n-2}}{U_{n-1}^2} - 1\right)T^0 + \ldots\right).$$

The conserved densities $\rho_n^{(k)}$ of Eq. (1) can be found as follows:

$$\rho_n^{(0)} = \log l_n^{(2)}, \qquad \rho_n^{(k)} = \operatorname{res}\hat{L}_n^k, \quad k \geq 1, \tag{29}$$

where the residue of the formal series (28) is defined by the rule $\operatorname{res} H_n = h_n^{(0)}$ (see [19]). The corresponding functions $\sigma_n^{(k)}$ can be easily found by direct calculations.

The conserved densities $\hat{\rho}_n^{(k)}$ below have been found in this way and then simplified in accordance with the rule

$$\hat{\rho}_n^{(k)} = c_k\rho_n^{(k)} + (T-1)g_n^{(k)},$$

where c_k is a constant and $g_n^{(k)}$ is a function. The first three densities of Eq. (1) are

$$\hat{\rho}_n^{(0)} = \log(u_n + 1), \quad \hat{\rho}_n^{(1)} = \frac{V_{n+1}u_{n-1}}{U_n},$$

$$\hat{\rho}_n^{(2)} = \frac{u_{n+2}u_{n+1}u_n^2u_{n-1}u_{n-2}}{U_{n+1}^2U_nU_{n-1}^2} + \frac{u_{n+1}u_{n-2}(V_n^2 - u_nu_{n-1})}{U_nU_{n-1}} + \frac{u_{n+1}^2u_n^2u_{n-1}}{2U_{n+1}^2U_n^3} - \frac{u_{n+1}u_nu_{n-1}}{U_{n+1}U_n} + \frac{u_{n+1}u_{n-1}(V_{n+1}-1)V_n}{2U_{n+1}U_n^2} + \frac{u_{n-1}^2}{2U_n^2},$$

where

$$V_n = u_n u_{n-1} + u_n + u_{n-1}.$$

We can easily verify that

$$\frac{{}^2\,\hat{\rho}_n^{(1)}}{u_{n+1}\partial u_{n-1}} \neq 0, \quad \frac{{}^2\,\hat{\rho}_n^{(2)}}{u_{n+2}\partial u_{n-2}} \neq 0.$$

Therefore, in accordance with the theory developed in [19], the conserved densities $\hat{\rho}_n^{(0)}$, $\hat{\rho}_n^{(1)}$, and $\hat{\rho}_n^{(2)}$ have the orders 0, 2, and 4, respectively. This means that we have obtained three conserved densities, which are nontrivial and essentially different.

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