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ON A CERTAIN CLASS OF HYPERBOLIC EQUATIONS WITH SECOND-ORDER INTEGRALS

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Abstract. In this paper, we examine a special class of nonlinear hyperbolic equations possessing a second-order y-integral. We clarify the structure of x-integrals and prove that they are x-integrals of a hyperbolic equation with a first-order y-integral. We also prove that this class contains the well-known Laine equation.

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1. Introduction. We consider equations of the following form:

$$u_{xy} = \frac{p - \varphi_u}{\varphi_{u_y}} u_x + \frac{q}{\varphi_{u_y}} \sqrt{u_x}, \tag{1}$$

where p and q are functions of the variables x, y, and u and φ is a function of the variables x, y, u, and u_y. In [5] it was shown that if Eq. (1) has a second-order y-integral

$$\bar{W} = \bar{W}\big(x, y, u, u_y, u_{yy}\big), \quad D\bar{W} = 0, \tag{2}$$

then the function φ is independent of the variable x. Here D is the operator of complete differentiation with respect to the variable x. Note that Eqs. (1) have not been examined in [4].

We assume that

$$v = \varphi(y, u, u_y). \tag{3}$$

Then from Eq. (3) it follows that

$$u_y = \Phi(y, u, v), \tag{4}$$

and Eq. (1) can be written as follows:

$$Dv = p \cdot u_x + q\sqrt{u_x}. \tag{5}$$

In [5], necessary and sufficient conditions under which Eq. (1) has a second-order y-integral were obtained. Taking into account (4), we can write the integral (2) as follows:

$$D\bar{W} = 0, \quad \bar{W} = \bar{W}(x, y, u, v, v_y). \tag{6}$$

Then Eq. (6) is equivalent to the system of equations

$$L_1\bar{W} = 0, \quad L_2\bar{W} = 0, \quad L_3\bar{W} = 0, \tag{7}$$

where the operators

$$\begin{aligned} L_1 &= \frac{\partial}{\partial x} + \frac{1}{2} q^2 \Phi_v \cdot \frac{\partial}{\partial \bar{v}_1}, \\ L_2 &= \frac{\partial}{\partial u} + p\frac{\partial}{\partial v} + \big[p_y + p_u\Phi + p\Phi_u + p^2\Phi_v\big] \frac{\partial}{\partial \bar{v}_1}, \\ L_3 &= \frac{\partial}{\partial v} + \left[\frac{3}{2} p\Phi_v + \frac{q_y}{q} + \frac{q_u}{q}\Phi + \frac{1}{2}\Phi_u\right] \frac{\partial}{\partial \bar{v}_1}, \quad \bar{v}_1 = v_y, \end{aligned} \tag{8}$$

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and the conditions of the existence of a y-integral (6) for Eq. (1) have the following form:

$$\Phi_{vv} = 3\frac{p_x}{q^2}\Phi_v + \frac{2}{q^2}(\ln q)''_{xu} \cdot \Phi + \frac{2}{q^2}(\ln q)''_{xy}, \tag{9}$$

$$\left(\frac{1}{2}p_u + p\frac{q_u}{q}\right)\Phi_v + p\Phi_{vu} + (\ln q)''_{yu} + \frac{q_u}{q}\Phi_u + (\ln q)''_{uu} \cdot \Phi + \frac{1}{2}\Phi_{uu} + \frac{1}{2}p^2\Phi_{vv} = 0, \tag{10}$$

$$p_{xy} - \frac{p_x q_y}{q} + \left(p_{ux} - \frac{p_x q_u}{q}\right)\Phi + \frac{1}{2}p_x\Phi_u + \left(\frac{1}{2}pp_x - qq_u\right)\Phi_v - \frac{1}{2}q^2\Phi_{uv} - \frac{1}{2}pq^2\Phi_{vv} = 0. \tag{11}$$

In [5], a partial analysis of the conditions (9)–(10) was performed in the case where the solution of Eq. (9) is determined by the formula

$$\Phi = -\frac{C}{B} + D(y,u)e^{S_1 v} + R(y,u)e^{S_2 v}, \tag{12}$$

where

$$S_{1,2} = \frac{A \pm \sqrt{\Delta}}{2}, \quad \Delta = A^2 + 4B \neq 0, \quad B \neq 0, \quad R \cdot D \neq 0 \tag{13}$$

and

$$A = 3\frac{p_x}{q^2}, \quad B = \frac{2}{q^2}(\ln q)''_{xu}, \quad C = \frac{2}{q^2}(\ln q)''_{xy}, \tag{14}$$

under the assumption

$$A_x = 0. \tag{15}$$

In this case (see [5, Theorem 4.1]), the following relations are possible:

$$S_1 = 2, \quad S_2 = 1, \quad A = 3, \quad B = -2, \quad D = 1, \quad R = -S(x,y)\frac{q_x}{q^2} + \frac{H(x,y)}{q}, \quad p = -\frac{q_u}{q}. \tag{16}$$

In this paper, we describe a class of Eqs. (1) with second-order y-integrals satisfying the conditions (12)–(16).

It is shown that one of examples of such equations is the following equation constructed by Laine in 1926 (see [1–3]):

$$u_{xy} = 2\left[(u+Y)^2 + u_y + (u+Y)\sqrt{(u+Y)^2 + u_y}\right] \times \left[\frac{\sqrt{u_x} + u_x}{u - x} - \frac{u_x}{\sqrt{(u+Y)^2 + u_y}}\right], \tag{17}$$

where $Y = Y(y)$.

2. Equations with second-order y-integrals. It follows from (9) that if $A_x = 0$, then $B_x = 0$ and $C_x = 0$. We rewrite the conditions (12)–(16) using the substitution $2v + \ln D \to 2v$ (see [5]):

$$\Phi = \frac{C}{2} + e^{2v} + Re^v, \tag{18}$$

$$R(y,u) = -S(x,y)\frac{q_x}{q^2} + \frac{H(x,y)}{q}, \tag{19}$$

$$p = -\frac{q_u}{q}, \quad \frac{p_x}{q^2} = 1, \tag{20}$$

$$C = \frac{2}{q^2}(\ln q)''_{xy}, \quad C_x = 0. \tag{21}$$

It is immediately verified that if the relations (18)–(21) are valid, then Eqs. (9)–(11) are satisfied. Thus, the original equation (1) possesses a second-order y-integral, which has the following form (see [5]):

$$\bar{W} = \bar{v}_1 + \beta(x,y,u,v). \tag{22}$$

In order to calculate the y-integral (22), we consider the system of equations (7) with the operators (8). Taking into account (18), we can write this system as follows:

$$\beta_x + \frac{1}{2}q^2(2e^{2v} + Re^v) = 0, \tag{23}$$

$$\beta_u + p\beta_v + p_y + p_u\left[\frac{C}{2} + e^{2v} + Re^v\right] + p\left[\frac{C_u}{2} + R_u e^v\right] + p^2\left[2e^{2v} + Re^v\right] = 0, \tag{24}$$

$$\beta_v + \frac{3}{2}p\left[2e^{2v} + Re^v\right] + \frac{q_y}{q} + \frac{q_u}{q}\left[\frac{C}{2} + e^{2v} + Re^v\right] + \frac{1}{2}\left[R_u e^v + \frac{C_u}{2}\right] = 0. \tag{25}$$

From Eq. (25) we obtain that

$$\beta = -\left(\frac{q_y}{q} + \frac{1}{2}\frac{q_u}{q}C + \frac{1}{4}C_u\right)v - \left(\frac{3}{2}p + \frac{1}{2}\frac{q_u}{q}\right)e^{2v} - \left(\frac{3}{2}pR + \frac{q_u}{q}R + \frac{1}{2}R_u\right)e^v + E(x, y, u).$$

Further, taking into account the relations (20) and (21), it is easy to show that

$$\beta = -pe^{2v} - \left(\frac{1}{2}pR + \frac{1}{2}R_u\right)e^v + E. \tag{26}$$

The substitution of the function β defined by the formula (26) in Eq. (23) reduces it to the relation $E_x = 0$. Finally, according to (26), Eq. (24) is written as follows:

$$E_u + p_y + \frac{1}{2}(p\cdot C)'_u + \left[-\left(\frac{3}{2}p + \frac{1}{2}\frac{q_u}{u}\right)'_u - 2\left(\frac{3}{2}p^2 + \frac{1}{2}p\frac{q_u}{q}\right) + p_u + 2p^2\right]e^{2v}$$
$$+\left[-\left(\frac{3}{2}pR + \frac{q_u}{q}R + \frac{1}{2}R_u\right)'_u - \left(\frac{3}{2}p^2 + p\frac{q_u}{q}\right)R + \frac{1}{2}pR_u + p_uR + p^2R\right]e^v. \tag{27}$$

Since $p = -q_u/q$ (see (20)), the coefficient e^{2v} in the formula (27) is zero. To determine the coefficient of e^v, we calculate the derivative R_{uu}. Differentiating Eq. (19) and taking into account (20), we obtain that

$$\frac{(qR)'_u}{q^2} = S(x, y).$$

Now differentiating the last equality by the variable u, we arrive at the relation

$$(qR)''_{uu} = 2\frac{q_u}{q}(Rq)'_u,$$

which can be written in the form

$$R_{uu} = p_uR + p^2R \tag{28}$$

(here we take into account (20)). Further, using (28), we see that the coefficient of e^v in the formula (27) is zero. Hence, we obtain

$$E_u + p_y + \frac{1}{2}(pC)'_u = 0$$

or, according to (20),

$$E_u = (\ln q)''_{uy} - \frac{1}{2}(pC)'_u.$$

Thus,

$$E = \frac{q_y}{q} - \frac{1}{2}pC + f(y).$$

Since

$$\frac{q_y}{q} + \frac{1}{2}\frac{q_u}{q}C + \frac{1}{4}C_u = 0, \tag{29}$$

the function E can be represented in the form

$$E = -\frac{1}{4}C_u + f(y)$$

and, consequently, the y-integral (22), according to (26), is determined by the following formula:

$$\bar{W} = \bar{v}_1 - pe^{2v} - \frac{1}{2}\left(pR + R_u\right)e^v - \frac{1}{4}C_u, \tag{30}$$

where the function C is defined by Eq. (21).

We write the integral (30) in another way. For this, we assume that

$$v = r - \ln q. \tag{31}$$

Then

$$\bar{W} = \bar{r}_1 + \left(-\frac{q_y}{q} + \frac{1}{2}pC - \frac{1}{4}C_u\right) + \frac{1}{2}\left(pR - R_u\right)\frac{1}{q}\cdot e^r; \tag{32}$$

moreover, the integral (32) is independent of the variable u and, due to (29), (32), and (20), is specified as follows:

$$\bar{W} = \bar{r}_1 + a(x,y)e^r, \quad a(x,y) = \frac{1}{2q}\left(pR - R_u\right). \tag{33}$$

Next, we define the right-hand side of the original equation (1), for which the conditions (18)–(21) are satisfied. As was shown above, these relations possess the second-order y-integral (33).

We consider Eq. (19). Note that $S(x,y) \neq 0$ and $q_x \neq 0$; otherwise from (20) we obtain $q = 0$. We rewrite Eq. (19) in the following form:

$$\frac{\partial}{\partial x}\left(\frac{1}{q}\right) = \frac{R(y,u)}{S(x,y)} - \frac{H(x,y)}{S(x,y)}\frac{1}{q}.$$

A solution of this equation is determined by the formula

$$\frac{1}{q} = R(y,u)\lambda(x,y) + \Phi(y,u)\mu(x,y); \tag{34}$$

moreover,

$$H = -\frac{\mu_x}{\mu^2}\cdot\frac{1}{\left(\frac{\lambda}{\mu}\right)'_x}, \quad S = \frac{1}{\mu\cdot\left(\frac{\lambda}{\mu}\right)'_x}. \tag{35}$$

Further, the conditions (20) imply that the function q satisfies the Liouville equation

$$-q^2 = (\ln q)'_{xu}. \tag{36}$$

Now, using the representation of (34) from (36), we easily obtain the following formula for the function q:

$$\frac{1}{q} = R\cdot\lambda\left[1 + \int\frac{dx}{\lambda^2}\cdot\int\frac{du}{R^2}\right], \tag{37}$$

where the functions $H(y,u)$ and $S(x,y)$ (see (35)) are defined as follows:

$$S = -\lambda\int\frac{dx}{\lambda^2}, \quad H = \frac{1}{\lambda} + \lambda x\int\frac{dx}{\lambda^2}. \tag{38}$$

Thus, the relations (19) and (20) are reduced to (37) and (38).

It remains to consider the condition (21). For this, assuming that

$$\int\frac{dx}{\lambda^2} = \alpha(x,y), \quad \int\frac{du}{R^2} = \gamma(y,u), \tag{39}$$

we rewrite the formula (37) as follows:

$$\frac{1}{q} = \frac{1+\alpha\gamma}{\sqrt{\alpha_x}\cdot\sqrt{\gamma_u}}.$$

Thus,

$$q^2 = \frac{\alpha_x\cdot\gamma_u}{(1+\alpha\gamma)^2}. \tag{40}$$

Further, the condition (21) can be written in the following form:

$$(\ln q^2)''_{xy} = h(y,u)q^2. \tag{41}$$

Taking into account (40), we rewrite the relation (41) in the following form:

$$h\cdot\gamma_u = \frac{(\ln\alpha_x)''_{xy}}{\alpha_x} - 2\gamma_y + 2\left(\frac{\alpha}{\alpha_x}(\ln\alpha_x)''_{xy} - \frac{\alpha_{xy}}{\alpha_x}\right)\gamma + \left[\frac{\alpha^2}{\alpha_x}(\ln\alpha_x)''_{xy} - 2\left(\frac{\alpha\alpha_{xy}}{\alpha_x} - \alpha_y\right)\right]\gamma^2.$$

We differentiate the last equality with respect to variable x and, taking into account the facts that $h = h(y,u)$ and $\gamma = \gamma(y,u)$, we obtain

$$\left[\frac{(\ln\alpha_x)''_{xy}}{\alpha_x}\right]'_x + 2\left(\frac{\alpha}{\alpha_x}(\ln\alpha_x)''_{xy} - \frac{\alpha_{xy}}{\alpha_x}\right)'_x\cdot\gamma + \left[\frac{\alpha^2}{\alpha_x}(\ln\alpha_x)''_{xy} - 2\left(\frac{\alpha\alpha_{xy}}{\alpha_x} - \alpha_y\right)\right]'_x\cdot\gamma^2 = 0.$$

Since $\alpha = \alpha(x,y)$ and $\gamma_u \neq 0$ (see (39)), we rewrite the last equation as follows:

$$\left[\frac{(\ln\alpha_x)''_{xy}}{\alpha_x}\right]'_x = 0, \tag{42}$$

$$\left[\frac{\alpha}{\alpha_x}(\ln\alpha_x)''_{xy} - \frac{\alpha_{xy}}{\alpha_x}\right]'_x = 0, \tag{43}$$

$$\left[\frac{\alpha^2}{\alpha_x}(\ln\alpha_x)''_{xy} - 2\left(\frac{\alpha\alpha_{xy}}{\alpha_x} - \alpha_y\right)\right]'_x = 0. \tag{44}$$

It is not difficult to show that the relations (43) and (44) are satisfied by (42). Due to (39), Eq. (42) is reduced to the following form:

$$(\ln\lambda)''_{xy} = \kappa(y)\frac{1}{\lambda^2}. \tag{45}$$

Thus, if a function $\lambda(x,y)$ is a solution of Eq. (45), then the condition (21) is satisfied, i.e., $C_x = 0$. Note that for $\kappa(y) \neq 0$, Eq. (45) is reduced to the Liouville equation $\psi_{xy} = e^{\psi}$ by the substitution $-2\kappa/\lambda^2 = e^{\psi}$.

Thus, if the function q is calculated by the formula (37), where $\lambda(x,y)$ is a solution of Eq. (45), $p = -q_u/q$, and the function $\varphi(y,u,u_y)$ is determined from the equation

$$u_y = \frac{1}{q^2}(\ln q)''_{xy} + e^{2\varphi} + Re^{\varphi}, \tag{46}$$

then the original equation (1) has a second-order y-integral, which, as can be easily shown by using the formula (33), has the following form:

$$\bar{W} = \bar{r}_1 + \frac{1}{2}\lambda\left(\int\frac{dx}{\lambda^2}\right)e^r, \quad a = \frac{1}{2}\lambda\left(\int\frac{dx}{\lambda^2}\right)e^r, \tag{47}$$

where $r = \varphi + \ln q$ (see (31)).

Concluding this section, we describe the structure of x-integral of Eq. (1) in the case considered. The x-integral W has the following form:

$$W = W\big(x,y,u,u_1,u_2,\ldots,u_n\big), \quad \bar{D}W = 0, \quad u_k = D^k u_k, \quad k = 1,2,\ldots,n. \tag{48}$$

Further, taking into account the formula (31), we can write Eq. (5) as follows:

$$Dr = \frac{q_x}{q} + q\sqrt{u_x}. \tag{49}$$

According to (49), from the variables x, y, u, u_1, u_2, …, one can obtain the new variables, namely, x, y, u, r_1, r_2, …. Consequently, the integral (48) can be represented in the following form:

$$W = W\big(x,y,u,r_1,r_2,\ldots,r_n\big).$$

Further, taking into account (46) and (47), we have

$$\bar{D}W = W_y + W_u\left[\frac{1}{q^2}(\ln q)''_{xy} + \frac{1}{q^2}e^{2r} + \frac{1}{q}Re^{\varphi}\right] - W_{r_1}D\left(ae^r\right) - W_{r_2}D^2(ae^r) - \ldots - W_{r_n}D^n(ae^r).$$

Therefore, since x-integral is independent of the variable r, it follows that $W_y = 0$, $W_u = 0$, and the last equality takes the form

$$W_{r_1}D(ae^r) + W_{r_2}D^2(ae^r) + \ldots + W_{r_n}D^n(ae^r) = 0. \tag{50}$$

Note that by virtue of (50), the x-integral W can be considered as an x-integral of the hyperbolic equation

$$r_{xy} + (a_x + ar_x)\, e^r = 0,$$

which possesses a first-order y-integral $\bar{W} = \bar{r}_1 + ae^r$.

3. Equation (17)**.** In this section, we show that the Laine equation (17) is contained in the above-considered class of Eqs. (1), and we present its x- and y-integrals. Equation (17) coincides with (1) if

$$p = q = \frac{1}{u-x}, \quad \varphi = \ln\left[(u+Y) + \sqrt{u_y + (u+Y)^2}\right].$$

In this case, $C = 0$, $R = -2(u+Y)$, and the second-order y-integral (47) has the following form:

$$\bar{W} = \bar{r}_1 - (x - Y(y))\, e^r, \quad \left(\lambda = -\frac{1}{2}\right), \tag{51}$$

where

$$r = \varphi + \ln\frac{1}{u-x}, \quad a = -\left(x - Y(y)\right).$$

Next, we calculate a third-order x-integral. According to (51), we can rewrite Eq. (50) in the form

$$\left\{\left(1 + (x-Y)r_1\right)\frac{\partial}{\partial r_1} + \left[2r_1 + (x-Y)(r_1^2 + r_2)\right]\frac{\partial}{\partial r_2}\right.$$
$$\left. + \left[3(r_2 + r_1^2) + (x-Y)\left(r_3 + 3r_1r_2 + r_1^3\right)\right]\frac{\partial}{\partial r_3}\right\}W = 0. \tag{52}$$

For $Y'(y) \neq 0$, Eq. (52) is reduced to the following two equations:

$$L_1W = \left[-r_1\frac{\partial}{\partial r_1} - (r_1^2 + r_2)\frac{\partial}{\partial r_2} - (r_3 + 3r_1r_2 + r_1^3)\frac{\partial}{\partial r_3}\right]W = 0, \tag{53}$$

$$L_2W = \left[(1 + xr_1)\frac{\partial}{\partial r_1} + \left(2r_1 + x(r_1^2 + r_2)\right)\frac{\partial}{\partial r_2} + \left(3(r_2 + r_1^2) + x(r_3 + 3r_1r_2 + r_1^3)\right)\frac{\partial}{\partial r_3}\right]W = 0. \tag{54}$$

We assume that $L_3 = L_2 + xL_1$; then

$$L_3W = \left[\frac{\partial}{\partial r_1} + 2r_1\frac{\partial}{\partial r_2} + 3\left(r_2 + r_1^2\right)\frac{\partial}{\partial r_3}\right]W = 0. \tag{55}$$

We introduce the operator $L_4 = -L_1 - r_1L_3$. So we have the following:

$$L_4W = \left[\left(r_2 - r_1^2\right)\frac{\partial}{\partial r_2} + \left(r_3 - 2r_1^3\right)\frac{\partial}{\partial r_3}\right]W = 0. \tag{56}$$

Note that the commutator of the operators L_3 and L_4 is zero, $[L_3, L_4] = 0$. Thus, the system (53), (54) is equivalent to (55), (56).

It is known that if there exists an integral of the order n, $n \geq 2$, then we can assume that it is linear with respect to the leading variable. We assume that

$$W = \alpha\left(x, r_1, r_2\right) r_3 + \beta\left(x, r_1, r_2\right). \tag{57}$$

The system (55), (56) for the integral (57) is reduced to the system

$$\alpha_{r_1} + 2r_1\alpha_{r_2} = 0, \qquad \beta_{r_1} + 2r_1\beta_{r_2} + 3(r_2 + r_1^2)\alpha = 0,$$
$$(r_2 - r_1^2)\alpha_{r_2} + \alpha = 0, \qquad (r_2 - r_1^2)\beta_{r_2} - 2r_1^3\alpha = 0,$$

the following functions are the solutions of this system:

$$\alpha = \frac{1}{r_2 - r_1^2}, \quad \beta = -\frac{2r_1^3}{r_2 - r_1^2} - 3r_1. \tag{58}$$

Thus, according to (57) and (58), the x-integral of Eq. (17) for $Y'(y) \neq 0$ has the following form:

$$W = \frac{r_3}{r_2 - r_1^2} - \frac{2r_1^3}{r_2 - r_1^2} - 3r_1,$$

where, according to (49),

$$r_1 = \frac{1}{u - x} + \frac{u_x}{u - x}.$$

For $Y'(y) = 0$, Eq. (17) possesses a second-order x-integral, which, according to (52), is a solution of the equation

$$\left\{(1 + (x - Y)r_1)\frac{\partial}{\partial r_1} + \left[2r_1 + (x - Y)(r_1^2 + r_2)\right]\frac{\partial}{\partial r_2}\right\} W = 0.$$

Assuming that

$$W = \alpha(x, r_1)r_2 + \beta(x, r_1),$$

we can easily obtain that it is given by the formula

$$W = \frac{r_2}{1 + (x - Y)r_1} - \frac{r_1^2}{1 + (x - Y)r_1}.$$

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