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TENSOR PRODUCTS OF QUANTUM MAPPINGS

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Abstract. In this paper, we examine properties of the tensor powers of quantum mappings Φ. In particular, we review positivity properties of unitary and nonunitary qubit mappings $\Phi^{\otimes 2}$. For arbitrary finite-dimensional systems, we present the relationship between the positive and completely positive divisibility of dynamical mappings $\Phi_t^{\otimes 2}$ and Φ_t. A criterion of annihilation of entanglement by an arbitrary qubit mapping $\Phi^{\otimes 2}$ is found.

***Keywords and phrases*:** quantum channel, complete positivity, positive mapping, divisibility, tensor product.

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1. Introduction. Linear mappings naturally arise in the problems of quantum evolution of density operators (see [6, 27, 39]) since the fundamental quantum equations of motion are linear differential equations of the first order in time. In this paper, we consider finite-dimensional quantum systems whose states are described by linear density operators $\varrho(t) \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional Hilbert (unitary) space, $\dim \mathcal{H} = d < \infty$, and $\mathcal{B}(\mathcal{H})$ is the set of operators acting on $\mathcal{H}$. The density operator $\varrho(t)$ is Hermitian ($\varrho^\dagger(t) = \varrho(t)$) and nonnegative definite ($\langle\psi|\varrho^\dagger(t)|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$; here and below we use the standard Dirac notation), which has a unit trace ($\mathrm{tr}[\varrho(t)] = 1$).

In the case of the absence of initial correlations between a quantum system and its environment, the evolution is described by a dynamic mapping $\varrho(t) = \Phi_t[\varrho(0)]$, where Φ_t is a completely positive, trace-preserving mapping called a *quantum channel* (see, e.g., [27]). The physical requirement of complete positivity instead of simple positivity of the mapping is explained by the fact that the initial state of the system considered can be entangled with another (auxiliary) system (see [31, 48, 49]). The auxiliary system can have an arbitrary dimension k; its trivial evolution is governed by the identity mapping Id_k. The density matrix of the initial entangled state must turn into a certain density matrix; therefore, the mapping $\Phi_t \otimes \mathrm{Id}_k$ must be positive for any k: this is the definition of complete positivity of the mapping Φ_t. It is remarkable that for completely positive mappings, there exists a convenient criterion based on the Choi–Jamiołkowski isomorphism (see [7, 9, 32]), while for positive mappings such a criterion has not been found in the general case.

The existence of entangled states implies the structure of the tensor product of the space $\mathcal{H}$. An *entangled state* of a two-part system is a density operator $\varrho_{AB} \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$, which cannot be represented as the closure of the convex sum of tensor products of local density operators (see [53]), i.e.,

$$\varrho_{AB} \neq \sum_k p_k \varrho_A^{(k)} \otimes \varrho_B^{(k)}. \tag{1}$$

States described by the right-hand side of Eq. (1) are said to be *separable* (or disentangled); they can be prepared by using local operations and classical communication in remote laboratories A and B.

Assume that two physical carriers of information A and B evolve independently. In particular, this occurs when propagating a quantum signal from a source to two remote signal receivers A and B via individual communication lines. In this case, the dynamic mapping is given by a local quantum channel of the form $\Phi_1^A \otimes \Phi_2^B$. It is easy to show that $\Phi_1^A \otimes \Phi_2^B$ is completely positive if and only if both Φ_1^A and Φ_2^B are completely positive.

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Another important case of tensor mapping occurs when information encoded in several physical carriers of information is transmitted via the same communication channel Φ. In case of the absence of memory effects, the consecutive transfers of n physical carriers via the channel lead to the mapping $\Phi^{\otimes n}$. Such tensor constructions are used in the definition of the capacity of quantum channels (see [28]). Obviously, $\Phi^{\otimes n}$ is completely positive if and only if Φ is completely positive. Channels that break entanglement have the same property of invariance with respect to the tensor product (see [25, 26, 30, 33, 44, 50]). However, the positivity property is not invariant with respect to the tensor product, i.e., $\Phi^{\otimes n}$ may be nonpositive even if the linear mapping Φ is positive (see [13, 15, 20, 38]). Similarly, the mappings $\Phi^{\otimes n}$ and $\Phi^{\otimes m}$, $m > n \geq 2$, can have different effects on entangled states: $\Phi^{\otimes n}$ can lead to annihilation of the entanglement (see [11, 12, 17, 19–21, 37]) or to the absolute separability of the output state (see [16]), while $\Phi^{\otimes m}$ does not possess these properties.

The properties of positivity and complete positivity of intermediate mappings (see [54]) $\Theta_{t,t+s} = \Phi_{t+s} \circ \Phi_t^{-1}$, $s > 0$, are often used for characterizing the Markov property of the dynamic process Φ_t (see [5, 10, 43]). In particular, if $\Theta_{t,t+s}$ is completely positive for any $t, s \geq 0$, then the process Φ_t is said to be *completely positively divisible* (or *Markov* process), and the corresponding kinetic equation has the form of the Gorini–Kossakovsky–Sudarshan—Lindblad equation (see [23, 34]) with time-dependent coefficients (in this case, the decoherence rates remain nonnegative, see [24]). If $\Theta_{t,t+s}$ is positive for all $t, s \geq 0$, but is not completely positive for some $t, s \geq 0$, then the process Φ_t is called a *weakly non-Markov* process (see [8, 42, 55]). If $\Theta_{t,t+s}$ is not positive for some $t, s \geq 0$, then the process Φ_t is called a *substantially non-Markov process* (see [8]). The physical difference between these processes is clearly demonstrated in collision models, where the quantum system sequentially interacts with the microscopic particles of the reservoir (see [41, 47, 56, 57]). Markov processes can be easily implemented by the factorized state of the reservoir (see [18]) in continuous or stroboscopic limit (see [22, 35, 36]); at the same time, for non-Markov (weakly or substantially) processes, a correlated environment is needed (see [18, 46]) (the correlations can be quantum or classical).

This paper is devoted to the analysis of one-parameter families of mappings Φ_t, $t \geq 0$, $\Phi_0 = \mathrm{Id}$, and their tensor products $\Phi_t^{\otimes 2}$.

Let Φ_t be a semigroup with some linear generator $\mathcal{L}$, i.e., $\Phi_t = \exp(\mathcal{L}t)$. It was proved in [2] that the positivity of the mapping $\exp(\mathcal{L}t)^{\otimes 2}$ is equivalent to the complete positivity of the mapping $\exp(\mathcal{L}t)$ (in this case, the generator $\mathcal{L}$ is given by the Gorini–Kossakovsky—Sudarshan—Lindblad equation with constant coefficients). In this paper, we show that if Φ_t is not a semigroup, then this property does not hold. The first aim of this paper is to establish a relationship between the positivity properties of the mappings Φ_t and $\Phi_t^{\otimes 2}$.

The examples mentioned above show that the processes Φ_t and $\Phi_t^{\otimes n}$ can possess different divisibility. The second aim of this paper is to establish a correspondence between the divisibility properties of the mappings Φ_t and $\Phi_t^{\otimes 2}$.

The third aim of this paper is to find a condition of annihilation of an entanglement by an arbitrary two-qubit quantum channel $\Phi_t^{\otimes 2}$.

2. Positivity of the tensor product of mappings. Let $\Phi : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2)$ be a linear qubit mapping. It was proved in [38] that $\Phi^{\otimes n}$ is positive for any $n \in \mathbb{N}$ if and only if Φ is completely positive or completely copositive (i.e., is obtained by successive application of a completely positive mapping and transposition in a certain basis). For the case $n = 1$, it is also known that any positive mapping $\Phi : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2)$ is a conical combination of a completely positive mapping and a completely copositive mapping (see [51]). The case of arbitrary n was examined in the recent work [15]. In particular, if Υ is a unitary mapping of the form

$$\Upsilon[\varrho] = \frac{1}{2}\left(\mathrm{tr}[\varrho]I + \sum_{j=1}^{3} \widetilde{\lambda}_j \,\mathrm{tr}\left[\sigma_j \varrho\right] \sigma_j\right), \tag{2}$$

where σ_1, σ_2, and σ_3 are the standard Pauli operators and I is the identity operator on $\mathcal{H}_2$. Then $\Upsilon^{\otimes 2}$ is positive if and only if Υ^2 is completely positive (see [15]). Since the criterion of the complete positivity of qubit mappings is known (see [45]), we find that $\Upsilon^{\otimes 2}$ is positive if and only if

$$1 + \widetilde{\lambda}_1^2 - \widetilde{\lambda}_2^2 - \widetilde{\lambda}_3^2 \geq 0, \quad 1 - \widetilde{\lambda}_1^2 + \widetilde{\lambda}_2^2 - \widetilde{\lambda}_3^2 \geq 0, \quad 1 - \widetilde{\lambda}_1^2 - \widetilde{\lambda}_2^2 + \widetilde{\lambda}_3^2 \geq 0.$$

Example 1. Consider a time-dependent dissipator

$$\mathcal{L}_t[\varrho] = \frac{\alpha}{2} \sum_{k=1}^{3} \gamma_k(t) \big(\sigma_k \varrho \sigma_k - \varrho\big), \tag{3}$$

where $\alpha > 0$, $\gamma_1(t) = \gamma_2(t) = 1$, and $\gamma_3(t) = -\tanh(t)$. Since the mappings $\mathcal{L}_{t_1}$ and $\mathcal{L}_{t_2}$ for different instants of time commute, the resulting mapping is

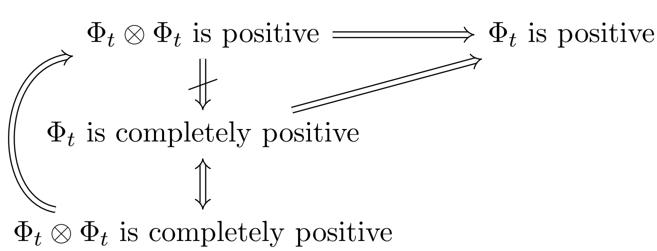
$$\Upsilon_t = \exp\left(\int_0^t \mathcal{L}(t')dt' \right).$$

It is easy to obtain that

$$\widetilde{\lambda}_1(t) = \widetilde{\lambda}_2(t) = \cosh^{\alpha}(t) e^{-\alpha t}, \quad \widetilde{\lambda}_3(t) = e^{-2\alpha t}.$$

The resulting mapping Υ_t is completely positive if and only if $\alpha \geq 1$ and is simply positive for $\alpha > 0$ (see [3]). Using the positivity criterion for the mapping $\Upsilon_t^{\otimes 2}$ (see [15]), we conclude that $\Upsilon_t^{\otimes 2}$ is positive if and only if $\alpha \geq 1/2$. Thus, if $1/2 \leq \alpha < 1$, then all the mappings $\{\Upsilon_t^{\otimes 2}\}_{t>0}$ are positive, while the mappings $\{\Upsilon_t\}_{t>0}$ are not completely positive. This shows that the positivity of the mapping $\Upsilon_t^{\otimes 2}$ in the general case does not imply the complete positivity of Υ_t, in contrast to the case of the semigroup considered in [2].

The obtained results can be presented in the form of the following diagram:



It was proved in [1] that for any positive qubit mapping Φ that does not belong to the boundary of positive mappings, there exist positive definite operators A and B such that

$$\Phi_A \circ \Phi \circ \Phi_B = \Upsilon, \tag{4}$$

where $\Phi_A[X] = AXA^\dagger$, $\Phi_B[X] = BXB^\dagger$, and Υ has the form (2). Due to the nondegeneracy of the operators A and B, the mappings $\Phi^{\otimes n}$ and $\Upsilon^{\otimes n}$ have the same positivity properties. In [14], the explicit forms of the operators A and B and the parameters $\widetilde{\lambda}_j$ were found. In particular, for the nonunital mapping

$$\Phi[X] = \frac{1}{2} \left(\operatorname{tr}[X]\big(I + t_3 \sigma_3\big) + \sum_{j=1}^{3} \lambda_j \operatorname{tr}\left[\sigma_j \varrho\right] \sigma_j \right), \tag{5}$$

the parameters $\widetilde{\lambda}_j$ are defined by the expressions

$$\widetilde{\lambda}_1 = \frac{2\lambda_1}{\sqrt{(1+\lambda_3)^2 - t_3^2} + \sqrt{(1-\lambda_3)^2 - t_3^2}}, \tag{6}$$

$$\widetilde{\lambda}_2 = \frac{2\lambda_2}{\sqrt{(1+\lambda_3)^2 - t_3^2} + \sqrt{(1-\lambda_3)^2 - t_3^2}}, \tag{7}$$

$$\widetilde{\lambda}_3 = \frac{4\lambda_3}{\left(\sqrt{(1+\lambda_3)^2 - t_3^2} + \sqrt{(1-\lambda_3)^2 - t_3^2}\right)^2}. \tag{8}$$

As a result, we obtain the following criterion of the positivity of the mapping $\Phi^{\otimes 2}$.

Proposition 1. *The mapping $\Phi^{\otimes 2}$, where Φ is given by Eq.* (5) *and $|t_3| + |\lambda_3| < 1$, is positive if and only if the parameters* (6)–(8) *satisfy the conditions*

$$1 + \widetilde{\lambda}_1^2 - \widetilde{\lambda}_2^2 - \widetilde{\lambda}_3^2 \geq 0, \quad 1 - \widetilde{\lambda}_1^2 + \widetilde{\lambda}_2^2 - \widetilde{\lambda}_3^2 \geq 0, \quad 1 - \widetilde{\lambda}_1^2 - \widetilde{\lambda}_2^2 + \widetilde{\lambda}_3^2 \geq 0.$$

Similarly, combining the results of [14, 15], we can obtain a criterion of the positivity of the mapping $\Phi^{\otimes 3}$. We do not present this criterion here because of its cumbersome form. Necessary and (separately) sufficient conditions for the positivity of an arbitrary qubit mapping $\Phi^{\otimes n}$ can be obtained in the same way.

3. Divisibility of the tensor product of mappings. Properties considered in the previous section are applicable to the intermediate mapping $\Theta_{t,t+s} = \Phi_{t+s} \circ \Phi_t^{-1}$, which transforms the state at the time t into the state at the time $t+s$. Obviously, if the process Φ_t is completely positively divisible, then the process $\Phi_t \otimes \mathrm{Id}$ is also completely positively divisible. However, if the process Φ_t is simply positively divisible, then this does not imply the positive divisibility of the process $\Phi_t \otimes \mathrm{Id}$ since the mapping $\Theta_{t,t+s} \otimes \mathrm{Id}$ is not always positive for positive mappings $\Theta_{t,t+s}$. This example shows that adding a trivial auxiliary system (governed by the identity evolution transformation Id) to this system can affect the property of positive divisibility of the mapping.

If we pass to the tensor-product level of a self-mapping, then the following result is valid for the dynamics of an arbitrary finite-dimensional system (see [3]).

Proposition 2. *A one-parameter family $\{\Phi_t\}_{t\geq 0}$ of dynamical mappings of a finite-dimensional quantum system is completely positively divisible if and only if the tensor product $\{\Phi_t \otimes \Phi_t\}_{t\geq 0}$ is positively divisible.*

On the other hand, $\Theta_{t,t+s} \otimes \Theta_{t,t+s}$ is completely positive if and only if $\Theta_{t,t+s}$ is completely positive. Thus, we obtain the following statement.

Proposition 3. *The positive divisibility and the completely positive divisibility of a finite-dimensional dynamical mapping $\Phi_t \otimes \Phi_t$ are equivalent.*

These results can be presented in the diagram form:

$\Phi_t \otimes \Phi_t$ is positive divisible

$\Updownarrow$

Φ_t is completely positively divisible

$\Updownarrow$

$\Phi_t \otimes \Phi_t$ is completely positively divisible.

Since the positivity of the mapping $\Theta_{t,t+s}^{\otimes n+1}$ implies the positivity of the mapping $\Theta_{t,t+s}^{\otimes n}$, $n \in \mathbb{N}$, we conclude that Φ_t is completely positively divisible (Markov) mapping if $\Phi_t^{\otimes n}$ is positively divisible, where $n \geq 2$.

4. Annihilation of entanglement by the tensor product of qubit mappings. A mapping

$$\Phi \otimes \Phi' : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}') \mapsto \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}')$$

is called a *positive mapping annihilating entanglement* if $\Phi \otimes \Phi'[\varrho]$ is a separable density operator with respect to the partition $\mathcal{H}|\mathcal{H}'$ for any density operators $\varrho \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}')$ (see [20]). The properties of such mappings were partially investigated in [11, 12, 19, 21]. In this paper, we completely characterize two-qubit mappings

$$\Phi \otimes \Phi' : \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{B}(\mathcal{H}_2') \mapsto \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{B}(\mathcal{H}_2'),$$

possessing the property of annihilation of entanglement. Since almost all nonunital positive qubit mappings Φ are reduced to unital mappings of the form (2) by the formula (4) and the mappings Φ_A and Φ_B do not change the entanglement property, our problem is reduced to the analysis of the tensor product $\Upsilon \otimes \Upsilon'$. To the mappings Υ and Υ' we assign the vectors $\boldsymbol{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3)^\top$ and $\widetilde{\boldsymbol{\lambda}} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3)^\top$.

Proposition 4. *Let Υ and Υ' be positive qubit mappings. The mapping $\Upsilon \otimes \Upsilon'$ is a positive mapping annihilating entanglement if and only if $|\widetilde{\boldsymbol{\lambda}}^\top P R \widetilde{\boldsymbol{\lambda}}'| \leq 1$ for all permutation (3×3)-matrices and the matrices*

$$R \in \Big\{ I,\ \mathrm{diag}(1,-1,-1),\ \mathrm{diag}(-1,1,-1),\ \mathrm{diag}(-1,-1,1) \Big\}.$$

Proof. Due to the convex structure of separable states, the mapping $\Upsilon \otimes \Upsilon'$ annihilates entanglement if and only if $\Upsilon \otimes \Upsilon'[|\psi\rangle\langle\psi|]$ is separable for all pure states $|\psi\rangle$. The Schmidt decomposition of an arbitrary two-qubit pure state $|\psi\rangle$ has the form

$$|\psi\rangle = \sqrt{p}|\varphi\rangle \otimes |\chi\rangle + \sqrt{1-p}|\varphi_\perp\rangle \otimes |\chi_\perp\rangle = U \otimes V |\psi_p\rangle,$$

where $0 \leq p \leq 1$,

$$|\psi_p\rangle = \sqrt{p}|0\rangle \otimes |0\rangle + \sqrt{1-p}|1\rangle \otimes |1\rangle$$

and the operators U and V are unitary (the transition matrices from the basis $\{|0\rangle, |1\rangle\}$ to the basis $\{|\varphi\rangle, |\varphi_\perp\rangle\}$ and from the basis $\{|0\rangle, |1\rangle\}$ to the basis $\{|\chi\rangle, |\chi_\perp\rangle\}$).

We assume that $\Phi_U[X] = UXU^\dagger$; then

$$\Upsilon \otimes \Upsilon' \Big[|\psi\rangle\langle\psi| \Big] = \big(\Upsilon \circ \Phi_U\big) \otimes \big(\Upsilon' \circ \Phi_V\big) \Big[|\psi_p\rangle\langle\psi_p| \Big].$$

Since local unitary operations preserve separability, we apply additional unitary operations $\Phi_{U^\dagger} \otimes \Phi_{V^\dagger}$. In this case, $\Upsilon \otimes \Upsilon' \big[|\psi\rangle\langle\psi| \big]$ is separable if and only if

$$\Big(\Phi_{U^\dagger} \circ \Upsilon \circ \Phi_U\Big) \otimes \Big(\Phi_{V^\dagger} \circ \Upsilon' \circ \Phi_V\Big) \Big[|\psi_p\rangle\langle\psi_p| \Big]$$

is separable.

The matrix representation

$$M_{ij}(\Upsilon) = \frac{1}{2} \operatorname{tr} \Big[\sigma_i \Upsilon[\sigma_j] \Big], \quad i, j = 0, \ldots, 3,$$

of the mapping $\Phi_{U^\dagger} \circ \Upsilon \circ \Phi_U$ is not diagonal:

$$M(\Phi_{U^\dagger} \circ \Upsilon \circ \Phi_U) = \left(\begin{array}{c|c} 1 & \mathbf{0}^\top \\ \hline \mathbf{0} & Q_U^\top \end{array} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \widetilde{\lambda}_1 & 0 & 0 \\ 0 & 0 & \widetilde{\lambda}_2 & 0 \\ 0 & 0 & 0 & \widetilde{\lambda}_3 \end{pmatrix} \left(\begin{array}{c|c} 1 & \mathbf{0}^\top \\ \hline \mathbf{0} & Q_U \end{array} \right), \qquad (9)$$

where Q_U is an orthogonal (3×3)-matrix corresponding to the channel Φ_U and $\mathbf{0}$ is a three-component zero column. In the action on the state $|\psi_p\rangle\langle\psi_p|$, only diagonal elements of the matrices $M(\Phi_{U^\dagger} \circ \Upsilon \circ \Phi_U)$ and $M(\Phi_{V^\dagger} \circ \Upsilon' \circ \Phi_V)$ are significant; therefore,

$$\big(\Phi_{U^\dagger} \circ \Upsilon \circ \Phi_U\big) \otimes \big(\Phi_{V^\dagger} \circ \Upsilon' \circ \Phi_V\big) \big[|\psi_p\rangle\langle\psi_p| \big] = \Upsilon_U \otimes \Upsilon'_V \big[|\psi_p\rangle\langle\psi_p| \big],$$

where

$$M(\Upsilon_U) = \mathrm{diag}\left(1, (\mathcal{Q}_U \widetilde{\boldsymbol{\lambda}})^\top\right), \quad (\mathcal{Q}_U)_{ij} = \left|(Q_U^\top)_{ij}\right|^2, \tag{10}$$

$$M(\Upsilon'_V) = \mathrm{diag}\left(1, (\mathcal{Q}_V \widetilde{\boldsymbol{\lambda}}')^\top\right), \quad (\mathcal{Q}_V)_{ij} = \left|(Q_V^\top)_{ij}\right|^2. \tag{11}$$

Note that the matrices $\mathcal{Q}_U$ and $\mathcal{Q}_V$ are bistochastic. For brevity, we introduce the notation $\boldsymbol{\mu} = \mathcal{Q}_U \widetilde{\boldsymbol{\lambda}}$ and $\boldsymbol{\mu}' = \mathcal{Q}_V \widetilde{\boldsymbol{\lambda}}'$. In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the density operator $\Upsilon_U \otimes \Upsilon'_V[|\psi_p\rangle\langle\psi_p|]$ has the so-called X-form; therefore, it is nonnegative definite and separable by the Perez–Horodecki criterion (see [29, 40]) if and only if

$$\left(1 + \mu_3\mu'_3\right)^2 \geq \left(\mu_1\mu'_1 \pm \mu_2\mu'_2\right)^2 + (2p-1)^2\left[\left(\mu_3 + \mu'_3\right)^2 - \left(\mu_1\mu'_1 \pm \mu_2\mu'_2\right)^2\right], \tag{12}$$

$$\left(1 - \mu_3\mu'_3\right)^2 \geq \left(\mu_1\mu'_1 \pm \mu_2\mu'_2\right)^2 + (2p-1)^2\left[\left(\mu_3 - \mu'_3\right)^2 - \left(\mu_1\mu'_1 \pm \mu_2\mu'_2\right)^2\right]. \tag{13}$$

The conditions (12) and (13) are automatically fulfilled for $p = 0$ or $p = 1$ since the mappings Υ and Υ' are positive and the state $|\psi_p\rangle$ in this case is factorized. Due to the monotonicity with respect to the parameter $(2p-1)^2$, the conditions (12) and (13) are satisfied for all $0 \leq p \leq 1$ if and only if they are satisfied for $p = 1/2$:

$$\left|\boldsymbol{\mu}^\top \widetilde{P}^{-1} \widetilde{R} \widetilde{P} \boldsymbol{\mu}'\right| \leq 1, \tag{14}$$

where $\widetilde{R} = I$ or $\widetilde{R} = \mathrm{diag}(1, -1, -1)$, and $\widetilde{P}$ is a permutation (3×3)-matrix. Recalling that $\boldsymbol{\mu} = \mathcal{Q}_U \widetilde{\boldsymbol{\lambda}}$ and $\boldsymbol{\mu}' = \mathcal{Q}_V \widetilde{\boldsymbol{\lambda}}'$, we reduce the inequality (14) to the form

$$\left|\widetilde{\boldsymbol{\lambda}}^\top \mathcal{Q}_U^\top \widetilde{P}^{-1} \widetilde{R} \widetilde{P} \mathcal{Q}_V \widetilde{\boldsymbol{\lambda}}'\right| \leq 1. \tag{15}$$

By the Birkhoff–von Neumann theorem (see [4, 52]), the bistochastic matrices $\mathcal{Q}_U^\top$ and $\mathcal{Q}_V$ are the convex sums of permutation matrices, and the inequality (15) holds for all $\mathcal{Q}_U^\top$ and $\mathcal{Q}_V$ if and only if the condition

$$\left|\widetilde{\boldsymbol{\lambda}}^\top P_1 \widetilde{R} P_2 \widetilde{\boldsymbol{\lambda}}'\right| \leq 1 \tag{16}$$

is satisfied for all permutation matrices P_1 and P_2. On the other hand, $\widetilde{R} P_2 = P_2 R$, where R is one of the matrices I, $\mathrm{diag}(1, -1, -1)$, $\mathrm{diag}(-1, 1, -1)$, or $\mathrm{diag}(-1, -1, 1)$. Introducing the notation $P_1 P_2 = P$, we obtain that

$$\left(\Phi_{U^\dagger} \circ \Upsilon \circ \Phi_U\right) \otimes \left(\Phi_{V^\dagger} \circ \Upsilon' \circ \Phi_V\right)\left[|\psi_p\rangle\langle\psi_p|\right]$$

is separable, and hence $\Upsilon_1 \otimes \Upsilon_2$ annihilates entanglement if and only if $\left|\widetilde{\boldsymbol{\lambda}}^\top P R \widetilde{\boldsymbol{\lambda}}'\right| \leq 1$ for all permutation (3×3)-matrices and the matrices

$$R \in \left\{I,\ \mathrm{diag}(1, -1, -1),\ \mathrm{diag}(-1, 1, -1),\ \mathrm{diag}(-1, -1, 1)\right\}.$$

Proposition 4 is proved. □

Corollary 1. *Let*

$$1 \geq \widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \widetilde{\lambda}_3 \geq 0, \quad 1 \geq \widetilde{\lambda}'_1 \geq \widetilde{\lambda}'_2 \geq \widetilde{\lambda}'_3 \geq 0.$$

A local two-bit unitary mapping $\Upsilon \otimes \Upsilon'$ annihilates entanglement if and only if

$$\widetilde{\boldsymbol{\lambda}}^\top \widetilde{\boldsymbol{\lambda}}' = \widetilde{\lambda}_1\widetilde{\lambda}'_1 + \widetilde{\lambda}_2\widetilde{\lambda}'_2 + \widetilde{\lambda}_3\widetilde{\lambda}'_3 \leq 1.$$

Corollary 2. *A local two-bit unital mapping $\Upsilon^{\otimes 2}$ annihilates entanglement if and only if*

$$\widetilde{\lambda}_1^2 + \widetilde{\lambda}_2^2 + \widetilde{\lambda}_3^2 \leq 1.$$

Corollary 3. *The mapping* $\Phi^{\otimes 2}$*, where* Φ *is given by the formula* (5)*, annihilates entanglement if and only if*

$$\frac{4\big(\lambda_1^2+\lambda_2^2\big)}{\Big(\sqrt{(1+\lambda_3)^2-t_3^2}+\sqrt{(1-\lambda_3)^2-t_3^2}\Big)^2}+\frac{16\lambda_3^2}{\Big(\sqrt{(1+\lambda_3)^2-t_3^2}+\sqrt{(1-\lambda_3)^2-t_3^2}\Big)^4}\leq 1. \qquad (17)$$

5. Conclusion. In this paper, we analyzed the properties of the tensor power of quantum mappings Φ, in particular, of $\Phi^{\otimes 2}$. We presented a survey of positivity properties of unital qubit mappings $\Phi^{\otimes 2}$ and reduced the study of nonunital mappings to unital mappings. For arbitrary finite-dimensional systems, we proved the equivalence of the positive and completely positive divisibility of the dynamical mappings $\Phi_t^{\otimes 2}$. Thus, the positive divisibility of the dynamical mapping $\Phi_t^{\otimes 2}$ is equivalent to the completely positive divisibility (the Markov property) of the mapping Φ_t. For arbitrary qubit mappings Φ and Φ', a criterion of annihilation of entanglement by the mapping $\Phi \otimes \Phi'$ is found. The results are demonstrated by a particular case of a nonunital qubit channel (5).

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REFERENCES

1. G. Aubrun and S. J. Szarek, *Two proofs of Størmer's theorem*, e-print `arXiv:1512.03293[math.FA]`.
2. F. Benatti, R. Floreanini, and R. Romano, "Complete positivity and dissipative factorized dynamics," *J. Phys. A: Math. Gen.*, **35**, L351 (2002).
3. F. Benatti, D. Chruściński, and S. Filippov, "Tensor power of dynamical maps and positive versus completely positive divisibility," *Phys. Rev. A*, **95**, 012112 (2017).
4. G. Birkhoff, "Tres observaciones sobre el algebra lineal," *Univ. Nac. Tucumán Rev. Ser. A*, **5**, 147 (1946).
5. H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, "Colloquium: Non-Markovian dynamics in open quantum systems," *Rev. Mod. Phys.*, **88**, 021002 (2016).
6. H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford Univ. Press, New York (2002).
7. M.-D. Choi, "Completely positive linear maps on complex matrices," *Linear Algebra Appl.*, **10**, 285 (1975).
8. D. Chruściński and S. Maniscalco, "Degree of non-Markovianity of quantum evolution," *Phys. Rev. Lett.*, **112**, 120404 (2014).
9. J. De Pillis, "Linear transformations which preserve Hermitian and positive semidefinite operators," *Pac. J. Math.*, **23**, 129 (1967).
10. I. De Vega and D. Alonso, "Dynamics of non-Markovian open quantum systems," *Rev. Mod. Phys.*, **89**, 015001 (2017).
11. S. N. Filippov, "PPT-inducing, distillation-prohibiting, and entanglement-binding quantum channels," *J. Russ. Laser Research.*, **35**, 484 (2014).
12. S. N. Filippov, "Influence of deterministic attenuation and amplification of optical signals on entanglement and distillation of Gaussian and non-Gaussian quantum states," *EPJ Web Conf.*, **103**, 03003 (2015).
13. S. N. Filippov, "Quantum mappings and characterization of entangled quantum states," *Itogi Nauki Tekhn. Ser. Sovr. Mat. Prilozh. Temat. Obz.*, **138**, 99–124 (2017).
14. S. N. Filippov, V. V. Frizen, and D. V. Kolobova, *Ultimate entanglement robustness of two-qubit states to general local noises*, e-print `arXiv:1708.08208[quant-ph]`.

15. S. N. Filippov and K. Yu. Magadov, “Positive tensor products of qubit maps and n-tensor-stable positive qubit maps,” *J. Phys. A: Math. Theor.*, **50**, 055301 (2017).
16. S. N. Filippov, K. Yu. Magadov, and M. A. Jivulescu, “Absolutely separating quantum maps and channels,” *New J. Phys.*, **19**, 083010 (2017).
17. S. N. Filippov, A. A. Melnikov, and M. Ziman, “Dissociation and annihilation of multipartite entanglement structure in dissipative quantum dynamics,” *Phys. Rev. A*, **88**, 062328 (2013).
18. S. N. Filippov, J. Piilo, S. Maniscalco, and M. Ziman, *Divisibility of quantum dynamical maps and collision models*, e-print `arXiv:1708.04994[quant-ph]`.
19. S. N. Filippov, T. Rybár, and M. Ziman, “Local two-qubit entanglement-annihilating channels,” *Phys. Rev. A*, **85**, 012303 (2012).
20. S. N. Filippov and M. Ziman, “Bipartite entanglement-annihilating maps: Necessary and sufficient conditions,” *Phys. Rev. A*, **88**, 032316 (2013).
21. S. N. Filippov and M. Ziman, “Entanglement sensitivity to signal attenuation and amplification,” *Phys.Rev. A*, **90**, 010301(R) (2014).
22. V. Giovannetti and G. M. Palma, “Master equations for correlated quantum channels,” *Phys. Rev. Lett.*, **108**, 040401 (2012).
23. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, “Completely positive dynamical semigroups of N-level systems,” *J. Math. Phys.*, **17**, 821 (1976).
24. M. J. W. Hall, “Complete positivity for time-dependent qubit master equations,” *J. Phys. A: Math. Theor.*, **41**, 205302 (2008).
25. A. S. Holevo, “Quantum coding theorems,” *Russ. Math. Surv.*, **53**, No. 6, 1295–1331 (1998).
26. A. S. Holevo, “Entanglement-breaking channels in infinite dimensions,” *Probl. Peredachi Inform.*, **44**, 3 (2008).
27. A. S. Holevo, *Quantum Systems, Channels, Information*, Walter de Gruyter, Berlin (2012).
28. A. S. Holevo and V. Giovannetti, “Quantum channels and their entropic characteristics,” *Repts. Progr. Phys.*, **75**, 046001 (2012).
29. M. Horodecki, P. Horodecki, and R. Horodecki, “Separability of mixed states: necessary and sufficient conditions,” *Phys. Lett. A*, **223**, 1 (1996).
30. M. Horodecki, P. W. Shor, and M. B. Ruskai, “Entanglement breaking channels,” *Rev. Math. Phys.*, **15**, 629 (2003).
31. R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” *Rev. Mod. Phys.*, **81**, 865 (2009).
32. A. Jamiołkowski, “Linear transformations which preserve trace and positive semidefiniteness of operators,” *Repts. Math. Phys.*, **3**, 275 (1972).
33. C. King, “Maximization of capacity and l_p norms for some product channels,” *J. Math. Phys.*, **43**, 1247 (2002).
34. G. Lindblad, “On the generators of quantum dynamical semigroups,” *Commun. Math. Phys.*, **48**, 119 (1976).
35. S. Lorenzo, F. Ciccarello, and G. M. Palma, *Composite quantum collision models*, e-print `arXiv:1705.03215[quant-ph]`.
36. I. A. Luchnikov and S. N. Filippov, “Quantum evolution in the stroboscopic limit of repeated measurements,” *Phys. Rev. A*, **95**, 022113 (2017).
37. L. Moravčíková and M. Ziman, “Entanglement-annihilating and entanglement-breaking channels,” *J. Phys. A: Math. Theor.*, **43**, 275306 (2010).
38. A. Müller-Hermes, D. Reeb, and M. M. Wolf, “Positivity of linear maps under tensor powers,” *J. Math. Phys.*, **57**, 015202 (2016).

39. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge (2000).
40. A. Peres, “Separability criterion for density matrices,” *Phys. Rev. Lett.*, **77**, 1413 (1996).
41. J. Rau, “Relaxation phenomena in spin and harmonic oscillator systems,” *Phys. Rev.*, **129**, 1880 (1963).
42. Á. Rivas, S. F. Huelga, and M. B. Plenio, “Entanglement and non-Markovianity of quantum evolutions,” *Phys. Rev. Lett.*, **105**, 050403 (2010).
43. Á. Rivas, S. F. Huelga, and M. B. Plenio, “Quantum non-Markovianity: characterization, quantification and detection,” *Repts. Progr. Phys.*, **77**, 094001 (2014).
44. M. B. Ruskai, “Qubit entanglement breaking channels,” *Rev. Math. Phys.*, **15**, 643 (2003).
45. M. B. Ruskai, S. Szarek, and E. Werner, “An analysis of completely-positive trace-preserving maps on M_2,” *Linear Algebra Appl.*, **347**, 159 (2002).
46. T. Rybár, S. N. Filippov, M. Ziman, and V. Bužek, “Simulation of indivisible qubit channels in collision models,” *J. Phys. B: Atom. Molec. Opt. Phys.*, **45**, 154006 (2012).
47. V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin, and V. Bužek, “Thermalizing quantum machines: Dissipation and entanglement,” *Phys. Rev. Lett.*, **88**, 097905 (2002).
48. E. Schrödinger, “Discussion of probability relations between separated systems,” *Math. Proc. Cambridge Phil. Soc.*, **31**, 555–563 (1935).
49. E. Schrödinger, “Probability relations between separated systems,” *Math. Proc. Cambridge Phil. Soc.*, **32**, 446–452 (1936).
50. P. W. Shor, “Additivity of the classical capacity of entanglement-breaking quantum channels,” *J. Math. Phys.*, **43**, 4334 (2002).
51. E. Størmer, “Positive linear maps of operator algebras,” *Acta Math.*, **110**, 233 (1963).
52. J. von Neumann, “A certain zero-sum two-person game equivalent to an optimal assignment problem,” *Ann. Math. Stud.*, **28**, 5 (1953).
53. R. F. Werner, “Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model,” *Phys. Rev. A*, **40**, 4277 (1989).
54. M. M. Wolf and J. I. Cirac, “Dividing quantum channels,” *Commun. Math. Phys.*, **279**, 147 (2008).
55. M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, “Assessing non-Markovian quantum dynamics,” *Phys. Rev. Lett.*, **101**, 150402 (2008).
56. M. Ziman and V. Bužek, “Open system dynamics of simple collision models,” in: *Quantum Dynamics and Information* (R. Olkiewicz et al., eds.), World Scientific, Singapore (2011), pp. 199–227.
57. M. Ziman, P. Štelmachovič, V. Bužek, M. Hillery, V. Scarani, and N. Gisin, “Diluting quantum information: An analysis of information transfer in system-reservoir interactions,” *Phys. Rev. A*, **65**, 042105 (2002).

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