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SUFFICIENT CONDITIONS FOR A MULTIDIMENSIONAL SYSTEM OF PERIODIC WAVELETS TO BE A FRAME

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We study multidimensional periodic wavelet systems with matrix dilations. We obtain conditions sufficient for such a system to be Bessel. The conditions are given in terms of Fourier coefficients. We propose a method for constructing a wavelet Riesz basis that starts with a suitable sequence of trigonometric polynomials. Bibliography: 19 titles.

1. Introduction

A natural way to define periodic systems of wavelets is periodization of wavelets from $L_2(\mathbb{R}^d)$; this is possible if the wavelet functions decay sufficiently fast at infinity. Such wavelet systems are widely studied in the literature (see [5–9, 11–14], [10, Sec. 2.6, Sec. 3.1]). However, many periodic objects that "deserve" to be called wavelet systems, cannot be obtained by periodization, so there are other approaches to define wavelets on a period, in a more general sense. As in the non-periodic case, the wavelets can be constructed on the basis of multiresolution analyses. Namely, orthogonal bases and tight frames are constructed on the basis of a periodic multiresolution analysis (PMRA, in brief); and biorthogonal bases and dual frames are constructed on the basis of two PMRA (see [1–4, 17]). In the present work, we use the definition of multidimensional PMRA given by I. Maksimenko and M. Skopina [19] (see also [18, Chap. 9]). N. Atreas [16] showed that the Bessel property and certain technical conditions are sufficient for the dual wavelet systems to be frames. In the present work, we obtain conditions sufficient for the Bessel property of a multidimensional wavelet system. A one-dimensional analogue of this result was obtained in [15]. Also, based the result obtained, we provide a method for constructing biorthogonal dual wavelet bases for any suitable sequence of trigonometric polynomials.

2. Notation and auxiliary results

We use the following standard notation: $\mathbb{N}$ is the set of natural numbers, $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ are elements (vectors) of the d-dimensional Euclidean space $\mathbb{R}^d$, $(x, y) = x_1y_1 + \dots x_dy_d$, $\mathbf{0}=(0, \dots, 0)\in\mathbb{R}^d$, $|x| = \sqrt{(x,x)}$, $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, $\mathbb{Z} = \mathbb{Z}^1$, $\mathbb{Z}_+ = \{0, 1, \dots\}$, $\mathbb{T}^d = (-\frac{1}{2}; \frac{1}{2}]^d$ is the d-dimensional unit torus, $\delta_{n,k}$ is the Kronecker symbol, $\widehat{f}(k) = \int_{\mathbb{T}^d} f(t)e^{-2\pi i(k,t)}dt$ is the Fourier coefficient of $f \in L_2(\mathbb{T}^d)$ with number k, $\langle f, g\rangle$ is the scalar product in $L_2(\mathbb{T}^d)$.

If A is a $d \times d$ matrix, then $\|A\|$ is its Euclidean operator norm from $\mathbb{R}^d$ to $\mathbb{R}^d$, A^* is its hermitian conjugate matrix, $A^{*j} = (A^*)^j$, I_d is the unit $d \times d$ matrix. Given a nondegenerate integer $d \times d$ matrix A, we say that vectors k, n are congruent modulo A and we write $k \equiv n \pmod A$ if $k - n = Al$, where $l \in \mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ splits into cosets with respect to this congruency relation. The number of these cosets is $|\det A|$ (see, e.g., [18, Proposition 2.2.1]). A set with exactly one element from each coset is called a set of digits of the matrix A. If the exact set of selected digits is not important, then we assume that this set is arbitrary and we denote it by $D(A)$. Observe that the set $H(A) := \mathbb{Z}^d \cap A\mathbb{T}^d$

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is a set of digits (see [18, Proposition 2.2.1]). Also, the following relation between the sets of digits of A, A^j and A^{j+1} is known.

Lemma 1 ([18], Lemma 2.2.3)**.** *Let A be a non-degenerate $d\times d$ matrix with integer entries, $|\det A|>1$. Then the set $\{r+A^jp\}$, $r\in D(A^j)$, $p\in D(A)$, is the set of digits of A^{j+1}.*

Throughout this work, M denotes a matrix from the class of square integer matrices with all eigenvalues greater than one in modulus. Also, put $m:=|\det M|$. Observe that all eigenvalues of the matrix M^{-1} are less than one in modulus, hence, the spectral radius of the matrix M^{-1} is also less than one. Therefore,

$$\lim_{n\to\infty}\|M^{-n}\|=0. \tag{1}$$

Given a sequence $\{f_j\}_{j\in\mathbb{Z}_+}\subset L_2(\mathbb{T}^d)$, we consider the shifts $f_{jk}:=f_j(\cdot+M^{-j}k)$. A wavelet system is the shift system $\{f_{jk}\}_{j\in\mathbb{Z}_+,k\in D(M^j)}$ associated with the sequence $\{f_j\}_{j\in\mathbb{Z}_+}\subset L_2(\mathbb{T}^d)$; we write $\{f_{jk}\}_{j,k}$. Given several sequences $\{f_j^{(\nu)}\}_{j\in\mathbb{Z}_+}$, $\nu=1,\dots,n$, $n\in\mathbb{N}$, the union of the corresponding wavelet systems is also called a wavelet system; we use notation $\{f_{jk}^{(\nu)}\}_{j,k,\nu}$. To clarify the index sets, we write $\{f_{jk}^{(\nu)}\}_{j\in\mathbb{Z}_+,k\in D(M^j),\nu=1,\dots,n}$.

Also, we need several auxiliary lemmas.

Lemma 2 ([18], Lemma 2.2.6)**.** *Let A be a non-degenerate $d\times d$ matrix with integer entries, $|\det A|>1$. Then*

$$\sum_{s\in D(A^*)} e^{2\pi i(A^{-1}r,s)}=\begin{cases}|\det A|, & \text{if } r\equiv 0 \pmod{A},\\ 0, & \text{in all other cases.}\end{cases} \tag{2}$$

Lemma 3. *Let f, g, $\varphi,\widetilde{\varphi}_j\in L_2(\mathbb{T}^d)$ for all $j\in\mathbb{Z}_+$. Then*

$$\sum_{k\in D(M^j)}\langle f,\varphi_{jk}\rangle\langle\widetilde{\varphi}_{jk},g\rangle=\sum_{s\in D(M^{*j})}\Bigg(\sum_{n\in\mathbb{Z}^d}\widehat{f}(M^{*j}n+s)m^{j/2}\overline{\widehat{\varphi_j}(M^{*j}n+s)}$$
$$\times\overline{\sum_{n'\in\mathbb{Z}^d}\widehat{g}(M^{*j}n'+s)m^{j/2}\overline{\widehat{\widetilde{\varphi}_j}(M^{*j}n'+s)}}\Bigg).$$

Proof. Firstly, observe that

$$\widehat{\varphi_{jk}}(n)=\widehat{\varphi_j}(n)e^{2\pi i(M^{*-j}n,k)}. \tag{3}$$

Since M is a non-degenerate matrix with integer entries, the equality $p=Mk+s$ defines a one-to-one correspondence between the set of $p\in\mathbb{Z}^d$ and the set of pairs (k,s), $k\in\mathbb{Z}^d$, $s\in D(M)$. Using Parseval's equality and replacing the summation index by $\mathbb{Z}^d$, we have

$$\sum_{k\in D(M^j)}\langle f,\varphi_{jk}\rangle\langle\widetilde{\varphi}_{jk},g\rangle=\sum_{k\in D(M^j)}\Bigg(\sum_{l\in\mathbb{Z}^d}\widehat{f}(l)\overline{\widehat{\varphi_j}(l)e^{2\pi i(M^{*-j}l,k)}}\overline{\sum_{l'\in\mathbb{Z}^d}\widehat{g}(l')\overline{\widehat{\widetilde{\varphi}_j}(l')e^{2\pi i(M^{*-j}l',k)}}}\Bigg)$$
$$=\sum_{k\in D(M^j)}\Bigg(\sum_{n\in\mathbb{Z}^d}\sum_{s\in D(M^{*j})}\widehat{f}(M^{*j}n+s)\overline{\widehat{\varphi_j}(M^{*j}n+s)e^{2\pi i(M^{*-j}s,k)}}$$
$$\times\overline{\sum_{n'\in\mathbb{Z}^d}\sum_{s'\in D(M^{*j})}\widehat{g}(M^{*j}n'+s')\overline{\widehat{\widetilde{\varphi}_j}(M^{*j}n'+s')e^{2\pi i(M^{*-j}s',k)}}}\Bigg)$$
$$=\sum_{n\in\mathbb{Z}^d}\sum_{s\in D(M^{*j})}\sum_{n'\in\mathbb{Z}^d}\sum_{s'\in D(M^{*j})}\widehat{f}(M^{*j}n+s)\overline{\widehat{\varphi_j}(M^{*j}n+s)}$$
$$\times\overline{\widehat{g}(M^{*j}n'+s')\overline{\widehat{\widetilde{\varphi}_j}(M^{*j}n'+s')}}\sum_{k\in D(M^j)}\overline{e^{2\pi i(M^{*-j}s,k)}}e^{2\pi i(M^{*-j}s',k)}.$$

By Lemma 2,

$$\sum_{k\in D(M^j)} e^{2\pi i(M^{*-j}(s-s'),k)} = m^j\delta_{s,s'}, \tag{4}$$

since the definition of the set $D(M^{*j})$ implies that the property $s–s'\equiv 0 \pmod{M^{*j}}$ holds only if $s = s'$. Therefore, all summands with $s \neq s'$ are equal to zero; hence, the lemma is proved. □

3. Main theorem

Theorem 1. *Assume that the Fourier coefficients of functions $\psi_j \in L_2(\mathbb{T}^d)$, $j\in\mathbb{Z}_+$, have the following properties:*

$$\forall j \in \mathbb{Z}_+, l \in \mathbb{Z}^d \quad |m^{j/2}\widehat{\psi_j}(l)| \le C \min\left\{|M^{*-j}l|^{-(\frac{d}{2}+\varepsilon)}, |M^{*-j}l|^{\alpha}\right\} \tag{5}$$

for some $C > 0$, $\varepsilon > 0$, $\alpha > 0$. Then $\{\psi_{jk}\}_{j,k}$ is a Bessel wavelet system.

Proof. Firstly, observe that

$$\forall j \in \mathbb{Z}_+, l \in \mathbb{Z}^d \quad |m^{j/2}\widehat{\psi_j}(l)| \le C \tag{6}$$

by the assumptions of the theorem. Now, choose $f \in L_2(\mathbb{T}^d)$, $j \in \mathbb{Z}_+$, $\delta \in (0; \frac{2\varepsilon}{d+2\varepsilon})$. Applying Lemma 3 and the Cauchy–Bunyakovsky inequality, we obtain

$$\begin{aligned}\sum_{k\in H(M^j)} |\langle f,\psi_{jk}\rangle|^2 &= \sum_{s\in H(M^{*j})} \Big|\sum_{n\in\mathbb{Z}^d} \widehat{f}(M^{*j}n+s)\overline{m^{j/2}\widehat{\psi_j}(M^{*j}n+s)}\Big|^2 \\ &\le \sum_{s\in H(M^{*j})} \Big(\sum_{n\in\mathbb{Z}^d} |\widehat{f}(M^{*j}n+s)|^2 |m^{j/2}\widehat{\psi_j}(M^{*j}n+s)|^{2\delta} \\ &\qquad \times \sum_{n'\in\mathbb{Z}^d} |m^{j/2}\widehat{\psi_j}(M^{*j}n'+s)|^{2(1-\delta)}\Big).\end{aligned} \tag{7}$$

Consider that sum over $n' \in \mathbb{Z}^d$. For $n' = \mathbf{0}$, we use estimate (6). For $n' \neq \mathbf{0}$, applying the first estimate from (5), we have

$$|m^{j/2}\widehat{\psi_j}(M^{*j}n'+s)|^{2(1-\delta)} \le C^{2(1-\delta)}\Big(\frac{1}{|n'+M^{*-j}s|}\Big)^{2(1-\delta)(\frac{d}{2}+\varepsilon)}.$$

It is easy to verify that $2(1-\delta)(\frac{d}{2}+\varepsilon) > d$ for δ under consideration. Using the property $M^{*-j}s \in (-\frac{1}{2};\frac{1}{2}]^d$ for $s \in H(M^{*j})$, we obtain

$$\begin{aligned}&\sup_{j\in\mathbb{Z}_+}\Big(\sum_{n'\in\mathbb{Z}^d} |m^{j/2}\widehat{\psi_j}(M^{*j}n'+s)|^{2(1-\delta)}\Big) \\ &\quad \le \sup_{j\in\mathbb{Z}_+}\Big(C^{2(1-\delta)} + C^{2(1-\delta)}\sum_{\substack{n'\in\mathbb{Z}^d\\ n'\neq 0}}\Big(\frac{1}{|n'+M^{*-j}s|}\Big)^{2(1-\delta)(\frac{d}{2}+\varepsilon)}\Big) \le C',\end{aligned}$$

where C' depends on the matrix M only.

Summing inequalities (7) over j, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+}\sum_{k\in H(M^j)} |\langle f,\psi_{jk}\rangle|^2 &\le C'\sum_{j\in\mathbb{Z}_+}\sum_{s\in D(M^{*j})}\sum_{n\in\mathbb{Z}^d} |\widehat{f}(M^{*j}n+s)|^2|m^{j/2}\widehat{\psi_j}(M^{*j}n+s)|^{2\delta} \\ &= C'\sum_{j\in\mathbb{Z}_+}\sum_{l\in\mathbb{Z}^d} |\widehat{f}(l)|^2 |m^{j/2}\widehat{\psi_j}(l)|^{2\delta}\end{aligned}$$

$$\le C' \sup_{l\in\mathbb{Z}^d} \Big(\sum_{j\in\mathbb{Z}_+} |m^{j/2}\widehat{\psi_j}(l)|^{2\delta}\Big) \|f\|^2. \tag{8}$$

Consider the expression under the above supremum. Choose the minimal $J \in \mathbb{Z}_+$ such that $\|M^{*-J}\| < 1$ (such a J exists by (1)) and fix an $l \in \mathbb{Z}^d$. We split the sum over j into J parts as follows:

$$\sum_{j\in\mathbb{Z}_+} |m^{j/2}\widehat{\psi_j}(l)|^{2\delta} = \sum_{k=0}^{J-1}\sum_{s\in\mathbb{Z}_+} |m^{\frac{(sJ+k)}{2}}\widehat{\psi}_{sJ+k}(l)|^{2\delta}. \tag{9}$$

Now, choose the maximal $j' \in \mathbb{Z}_+$ such that $|M^{*-j'J}l| \ge 1$, fix an index k and split the sum over $s \in \mathbb{Z}_+$ as

$$\sum_{s\in\mathbb{Z}_+} = \sum_{\substack{s\ge 0\\ s\le j'}} + \sum_{s>j'} := \sigma_1 + \sigma_2.$$

For σ_2, applying the second estimate from (5), we obtain

$$\sigma_2 \le C^{2\delta}\|M^{*-k}\|^{2\delta\alpha} \sum_{s>j'} |M^{*-sJ}l|^{2\delta\alpha} = C'_k \sum_{s=1}^{\infty} |M^{*-(s+j')J}l|^{2\delta\alpha}$$
$$\le C'_k |M^{*-(j'+1)J}l|^{2\delta\alpha} \sum_{s=1}^{\infty} \|M^{*-J}\|^{(s-1)2\delta\alpha} \le C'',$$

where C'' depends on the matrix M only, since $|M^{*-(j'+1)J}l|^{2\delta\alpha} < 1$ for J and j' under consideration, the number of constants C'_k is finite and also depends on the matrix M only.

For σ_1, we apply the first estimate from (5):

$$\sigma_1 \le C^{2\delta}\|M^{*-k}\|^{-(\frac{d}{2}+\varepsilon)2\delta} \sum_{\substack{s\ge 0\\ s\le j'}} |M^{*-sJ}l|^{-(\frac{d}{2}+\varepsilon)2\delta}.$$

Put $r = M^{*-j'J}l$. Observe that $|r| \ge 1$ by the choice of j'. Next,

$$\sum_{\substack{s\ge 0\\ s\le j'}} |M^{*-(s+j'-j')J}l|^{-(\frac{d}{2}+\varepsilon)2\delta} = \sum_{\substack{s\ge 0\\ s\le j'}} |M^{*-(s-j')J}r|^{-(\frac{d}{2}+\varepsilon)2\delta}$$
$$= \sum_{i=0}^{j'} |M^{*iJ}r|^{-(\frac{d}{2}+\varepsilon)2\delta} = \sum_{i=0}^{j'} \Big(\frac{|M^{*iJ}r|}{|r|}|r|\Big)^{-(\frac{d}{2}+\varepsilon)2\delta}$$
$$= \sum_{i=0}^{j'} \Big(\frac{|M^{*-iJ}M^{*iJ}r|}{|M^{*iJ}r|}\frac{1}{|r|}\Big)^{(\frac{d}{2}+\varepsilon)2\delta} \le \sum_{i=0}^{j'} \|M^{*-iJ}\|^{(\frac{d}{2}+\varepsilon)2\delta}$$
$$\le \sum_{i=0}^{\infty} \|M^{*-J}\|^{i(\frac{d}{2}+\varepsilon)2\delta} \le C''',$$

where C''' depends on M only.

Therefore, the sums σ_1 and σ_2 are uniformly bounded in l; hence, the expression on the right-hand side of (9) has the same property. Finally, returning to inequality (8), we have

$$\sum_{j\in\mathbb{Z}_+}\sum_{k\in H(M^j)} |\langle f, \psi_{jk}\rangle|^2 \le \widetilde{C}\|f\|^2,$$

where $\widetilde{C}$ depends on M only. $\square$

4. Construction of polynomial wavelet bases

Definition 1 ([18, Definition 9.1.1]). *Let $V_j \subset L_2(\mathbb{T}^d)$, $j \in \mathbb{Z}_+$. The collection $\{V_j\}_{j=0}^{\infty}$ is called a* periodic multiresolution analysis (PMRA, in brief) *if the following properties (axioms) hold*:
MR1. $V_j \subset V_{j+1}$;
MR2. $\overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\mathbb{T}^d)$;
MR3. $\dim V_j = m^j$;
MR4. $\dim\{f \in V_j : f(\cdot + M^{-j}n) = \lambda_n f \ \forall n \in \mathbb{Z}^d\} \le 1$, $\forall \{\lambda_n\}_{n\in\mathbb{Z}^d}, \lambda_n \in \mathbb{C}$;
MR5. $f \in V_j \Leftrightarrow f(\cdot + M^{-j}n) \in V_j \ \forall n \in \mathbb{Z}^d$;
MR6. $f\in V_j \Rightarrow f(M\cdot)\in V_{j+1}$; $f\in V_{j+1} \Rightarrow \sum_{s\in D(M)} f(M^{-1}\cdot + M^{-1}s)\in V_j$.

Definition 2 ([18, Definition 9.1.3]). *Let $\{V_j\}_{j=0}^{\infty}$ be a* PMRA *in $L_2(\mathbb{T}^d)$. A functional sequence $\{\varphi_j\}_{j\in\mathbb{Z}_+}$, $\varphi_j \in V_j$, is called* scaling *if the functions φ_{jk}, $k \in D(M^j)$, form a basis of V_j.*

Theorem 2 ([18, Theorem 9.1.4]). *Functions $\{\varphi_j\}_{j=0}^{\infty} \subset L_2(\mathbb{T}^d)$ form a scaling sequence for a* PMRA *if and only if*
S1. $\widehat{\varphi_0}(k) = 0$ *for all* $k \ne 0$;
S2. *for any $j \in \mathbb{Z}_+$ and $n \in \mathbb{Z}^d$, there exists $k \equiv n \pmod{M^{*j}}$ such that $\widehat{\varphi_j}(k) \ne 0$;*
S3. *for any $k \in \mathbb{Z}^d$, there exists $j \in \mathbb{Z}_+$ such that $\widehat{\varphi_j}(k) \ne 0$;*
S4. *for any $j \in \mathbb{Z}_+$, $n \in \mathbb{Z}^d$, there exists $\gamma_n^j \ne 0$ such that $\gamma_n^j\widehat{\varphi_j}(k) = \widehat{\varphi_{j+1}}(M^*k)$ for all $k \equiv n \pmod{M^{*j}}$;*
S5. *for any $j \in \mathbb{N}$ and $n \in \mathbb{Z}^d$, there exists μ_n^j such that $\widehat{\varphi_{j-1}}(k) = \mu_n^j\widehat{\varphi_j}(k)$ for all $k \equiv n \pmod{M^{*j}}$.*

Observe that the numerical sequences $\{\gamma_k^j\}_{k\in\mathbb{Z}^d}$, $\{\mu_k^j\}_{k\in\mathbb{Z}^d}$ in Theorem 2 are M^{*j}-periodic in k for any $j \in \mathbb{Z}_+$.

Assume that $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ are scaling sequences, s_k are arbitrarily enumerated digits of the matrix M^*, and matrices $A^{(r)} = \{a_{nk}^{(r)}\}_{n,k=0}^{m-1}$, $\widetilde{A}^{(r)} = \{\widetilde{a}_{nk}^{(r)}\}_{n,k=0}^{m-1}$ have the following properties:

$$a_{0k}^{(r)} = \mu_{r+M^{*j}s_k}^{j+1}, \quad \widetilde{a}_{0k}^{(r)} = \widetilde{\mu}_{r+M^{*j}s_k}^{j+1} \tag{10}$$

and $A^{(r)}\widetilde{A}^{(r)*} = mI_m$ for any $r \in D(M^{*j})$. For $\nu = 1,\dots,m-1$, put

$$\alpha_{r+M^{*j}s_k}^{\nu,j} = a_{\nu k}^{(r)}, \quad \widetilde{\alpha}_{r+M^{*j}s_k}^{\nu,j} = \widetilde{a}_{\nu k}^{(r)}. \tag{11}$$

By Lemma 1, the vectors $r + M^{*j}s_k$ cover the whole set $D(M^{*j+1})$, i.e., these sequences are extendable M^{*j+1}-periodically to $\mathbb{Z}^d$. Define the functions $\psi_j^{(\nu)}$, $\widetilde{\psi}_j^{(\nu)}$ by means of their Fourier coefficients as

$$\widehat{\psi_j^{(\nu)}}(l) = \alpha_l^{\nu,j}\widehat{\varphi_{j+1}}(l), \quad \widehat{\widetilde{\psi}_j^{(\nu)}}(l) = \widetilde{\alpha}_l^{\nu,j}\widehat{\widetilde{\varphi}_{j+1}}(l). \tag{12}$$

The systems $\{\varphi_0\} \cup \{\psi_{jk}^{(\nu)}\}_{j\in\mathbb{Z}_+,k\in D(M^j),\nu=1,\dots,m-1}$ and $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}^{(\nu)}\}_{j,k,\nu}$ are called *the dual wavelet systems generated by the scaling sequences* $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$. Now, we formulate a theorem on the frameness conditions for such systems. Observe that the Bessel condition is important in the theorem; properties sufficient for this condition are given in the previous section.

Theorem 3 (see [16]). *Assume that* $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ *are scaling sequences such that*

$$\lim_{j\to+\infty} m^j \widehat{\varphi_j}(k)\overline{\widehat{\widetilde{\varphi}_j}(k)} = 1 \quad \forall k \in \mathbb{Z}^d, \tag{13}$$

$\{\varphi_0\} \cup \{\psi_{jk}^{(\nu)}\}_{j,k,\nu}$ *and* $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}^{(\nu)}\}_{j,k,\nu}$ *are the corresponding Bessel dual wavelet systems. Then these systems are dual frames.*

Put $\chi_S = \sum_{k\in S} e^{ik\cdot}$, where $S \subset \mathbb{Z}^d$.

Theorem 4. *Suppose that* M *is a matrix such that* $\mathbb{T}^d \subset M^*\mathbb{T}^d$ *and* $\{\varphi_j\}_{j=0}^{\infty}$ *is a sequence of trigonometric polynomials such that*

$$\begin{cases} A \le |m^{j/2}\widehat{\varphi_j}(k)| \le B & \text{for } k \in H(M^{*j}), \\ \widehat{\varphi_j}(k) = 0 & \text{for } k \notin H(M^{*j}), \end{cases} \tag{14}$$

where $A, B > 0$. *Then*
1. $\{\varphi_j\}_{j=0}^{\infty}$ *is a scaling sequence.*
2. *For any* $j \in \mathbb{Z}_+$, *the set* $H(M^{*j+1}) \setminus H(M^{*j})$ *splits into subsets* $N_j^{(\nu)}$ *such that the wavelet system* $\{\varphi_0\} \cup \{\psi_{jk}^{(\nu)}\}_{j\in\mathbb{Z}_+,\, k\in D(M^j),\, \nu=1,\dots m-1}$ *with*

$$\psi_j^{(\nu)} = \sqrt{m}(\chi_{N_j^{(\nu)}} * \varphi_{j+1})$$

is a Riesz basis in $L_2(\mathbb{T}^d)$.
3. *There exists a wavelet basis* $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}^{(\nu)}\}_{j,k,\nu}$, *biorthogonal with basis* $\{\varphi_0\} \cup \{\psi_{jk}^{(\nu)}\}_{j,k,\nu}$, *which also consists of trigonometric polynomials.*

Proof. To verify the first claim, recall that $H(M^{*j})$ is the set of digits of the matrix M^{*j}. Properties S1 and S2 follow immediately from (14). By (1), for any $k \in \mathbb{Z}^d$, there exists $j \in \mathbb{Z}_+$ such that $M^{*-j}k \in \mathbb{T}^d$, hence, $k \in H(M^{*j})$ and $\widehat{\varphi_j}(k) \ne 0$; this implies S3. Setting

$$\mu_k^j = \frac{\widehat{\varphi_{j-1}}(k)}{\widehat{\varphi_j}(k)}, \qquad \gamma_k^j = \frac{\widehat{\varphi_{j+1}}(M^*k)}{\widehat{\varphi_j}(k)} \tag{15}$$

for $j \in \mathbb{Z}_+$, $k \in H(M^{*j})$ and extending these sequences M^{*j}-periodically to $\mathbb{Z}^d$ by the lower index, we easily see that properties S4, S5 also hold.

Now, we prove the second and the third claims.

For $j \in \mathbb{Z}_+$, put

$$\widehat{\widetilde{\varphi}_j}(k) = \begin{cases} \frac{m^{-j}}{\overline{\widehat{\varphi_j}(k)}}, & k \in H(M^{*j}), \\ 0, & k \notin H(M^{*j}). \end{cases} \tag{16}$$

It is easy to check that the assumptions of the theorem hold for the sequence $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ with constants $\widetilde{A} = \frac{1}{B}$, $\widetilde{B} = \frac{1}{A}$; hence, this sequence is also scaling. Clearly, we have

$$\lim_{j\to+\infty} m^j \widehat{\varphi_j}(k)\overline{\widehat{\widetilde{\varphi}_j}(k)} = 1 \quad \forall k \in \mathbb{Z}^d. \tag{17}$$

Now, we construct the dual wavelet systems generated by the scaling sequences under consideration. It is convenient to represent the sets of digits of the matrices M^{*j} in terms of Lemma 1, namely,

$$D(M^{*j}) = \bigcup_{\substack{r\in D(M^{*j-1}) \\ p\in D(M^*)}} \{r + M^{*j-1}p\}. \tag{18}$$

Observe that this set does not necessarily coincide with $H(M^{*j})$. Nevertheless, the coefficients μ_n^j are M^{*j}-periodic, thus they are canonically defined on any set of digits (in particular, on $H(M^{*j})$) once they are defined on at least one set of digits.

By (15),

$$\begin{aligned} \mu_k^{j+1} &\neq 0 \quad \text{for } k \in H(M^{*j}), \\ \mu_k^{j+1} &= 0 \quad \text{for } k \in H(M^{*j+1}) \setminus H(M^{*j}). \end{aligned} \tag{19}$$

Putting $D(M^{*j}) = H(M^{*j})$ and $D(M^*) = H(M^*)$ in Lemma 1, we rewrite (19) as

$$\forall r \in H(M^{*j}) \qquad \mu_{r+M^{*j}p}^{j+1} \begin{cases} \neq 0 & \text{for } p = \mathbf{0}, \\ = 0 & \text{for } p \neq \mathbf{0},\ p \in H(M^*). \end{cases} \tag{20}$$

Now, we construct $A^{(r)}$ and $\widetilde{A}^{(r)}$. We renumerate the digits $p \in H(M^*)$ in such a way that $p_0 = \mathbf{0}$. Define the first string as follows:

$$a_{0k}^{(r)} = \mu_{r+M^{*j}p_k}^{j+1}, \quad \widetilde{a}_{0k}^{(r)} = \widetilde{\mu}_{r+M^{*j}p_k}^{j+1}, \quad k = 0, 1, \ldots, m-1. \tag{21}$$

Starting from the second string, we put $\sqrt{m}$ on the diagonal and zero elsewhere, and we obtain square matrices.

By (20), the matrices under consideration are diagonal, namely,

$$A^{(r)} = \begin{bmatrix} \mu_r^{j+1} & 0 & \ldots & 0 \\ 0 & \sqrt{m} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{m} \end{bmatrix}, \quad \widetilde{A}^{(r)} = \begin{bmatrix} \widetilde{\mu}_r^{j+1} & 0 & \ldots & 0 \\ 0 & \sqrt{m} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{m} \end{bmatrix}.$$

It is easy to verify that $A^{(r)}\widetilde{A}^{(r)*} = mI_m$. Put

$$\alpha_{r+M^{*j}p_k}^{\nu,j} = a_{\nu k}^{(r)}, \quad \widetilde{\alpha}_{r+M^{*j}p_k}^{\nu,j} = \widetilde{a}_{\nu k}^{(r)}.$$

Since $r \in H(M^{*j})$ and $p_k \in H(M^{*j})$, the vectors $r + M^{*j}p_k$, $k = 1, \ldots, m-1$, cover the set of digits $D(M^{*j+1})$. Therefore, the coefficients $\alpha_l^{\nu,j}$, $\widetilde{\alpha}_l^{\nu,j}$ are M^{*j+1}-periodically extendable to $\mathbb{Z}^d$.

Now, for $\nu = 1, \ldots, m-1$, put

$$\widehat{\psi_j^{(\nu)}}(l) = \alpha_l^{\nu,j}\widehat{\varphi_{j+1}}(l), \quad \widehat{\widetilde{\psi}_j^{(\nu)}}(l) = \alpha_l^{\nu,j}\widehat{\widetilde{\varphi}_{j+1}}(l).$$

By the assumptions of the theorem,

$$\widehat{\psi_j^{(\nu)}}(l) = \begin{cases} \sqrt{m}\widehat{\varphi_{j+1}}(l) & \text{for } l \in H(M^{*j+1}), l \equiv r + M^{*j}p_\nu \\ & \ (\text{mod } M^{*j+1}), \\ & r \in H(M^{*j}), p_\nu \in H(M^*); \\ 0 & \text{otherwise}. \end{cases} \tag{22}$$

Hence, we have the following upper estimate:

$$|m^{j/2}\widehat{\psi_j^{(\nu)}}(l)| \leq |m^{j/2}\sqrt{m}\widehat{\varphi_{j+1}}(l)| \leq \sqrt{m}B. \tag{23}$$

Put

$$\begin{aligned} N_j^{(\nu)} = \{&l \in H(M^{*j+1}) : l \equiv r + M^{*j}p_\nu \ (\text{mod } M^{*j+1}), \\ &r \in H(M^{*j}), p_\nu \in H(M^*)\}. \end{aligned}$$

We claim that the set $N_j^{(\nu)}$ is in $H(M^{*j+1}) \setminus H(M^{*j})$ for any $j \in \mathbb{Z}_+$, $\nu = 1, \dots, m-1$. By the definition of the set $N_j^{(\nu)}$, it is sufficient to show that $N_j^{(\nu)} \not\subset H(M^{*j})$. Indeed, if $r + M^{*j}p_\nu \in H(M^{*j+1})$ for all $\nu = 1, \dots, m-1$, then the required property is obvious, since p_ν are non-zero digits. If $r + M^{*j}p_\nu \notin H(M^{*j+1})$, then we have to prove that there is no $r' \in H(M^{*j})$ such that $r' = r + M^{*j}p_\nu + M^{*j+1}k$ for $p_\nu, k \neq \mathbf{0}$, $k \in \mathbb{Z}^d$. Indeed, if not, then we have the representation $r' - r = M^{*j}(M^*k + p_\nu)$; this property holds for $r, r' \in H(M^{*j})$ only if $M^*k + p_\nu = \mathbf{0}$. The last equality never holds, since $p_\nu \in H(M^*)$. Also, by Lemma 1,

$$\{r + M^{*j}p_\nu\}_{r \in H(M^{*j}), \nu=1,\dots,m-1} = D(M^{*j+1}) \setminus H(M^{*j}). \tag{24}$$

The sets $N_j^{(\nu)}$ consist of vectors from $H(M^{*j+1})$ congruent modulo M^{*j+1} to vectors from (24); the elements of the last set are not pairwise congruent modulo M^{*j+1} by the definition of the set of digits. Thus, a vector from $N_j^{(\nu)}$ is not congruent modulo M^{*j+1} to two different vectors from (24). Hence, the union of all sets $N_j^{(\nu)}$ with fixed j and set (24) have the same number of elements. Since the set $H(M^{*j+1}) \setminus H(M^{*j})$ has $m^j(m-1)$ elements, we conclude that the sets $N_j^{(\nu)}$ are pairwise disjoint and their union is $H(M^{*j+1}) \setminus H(M^{*j})$. Therefore, the set $H(M^{*j+1}) \setminus H(M^{*j})$ indeed splits into the sets $N_j^{(\nu)}$.

Therefore, l is in the set $H(M^{*j+1}) \setminus H(M^{*j})$ for every non-zero coefficient $\widehat{\psi_j^{(\nu)}}(l)$. For such an l, we have

$$M^{*-j}l \in H(M^*) \setminus \mathbb{T}^d$$

and, accordingly, we have the inequality

$$\frac{1}{2} \leq |M^{*-j}l| \leq \|M^*\|\sqrt{d}. \tag{25}$$

Obviously, we have to check (5) for non-zero coefficients only. For $l \in H(M^{*j+1}) \setminus H(M^{*j})$, by (23) and (25),

$$|m^{j/2}\widehat{\psi_j^{(\nu)}}(l)| \leq \sqrt{m}B \leq 2\sqrt{m}B|M^{*-j}l|,$$
$$|m^{j/2}\widehat{\psi_j^{(\nu)}}(l)| \leq \sqrt{m}B \leq \|M^*\|^d d^{d/2}\sqrt{m}B|M^{*-j}l|^{-d},$$

thus, (5) holds for $\varepsilon = d/2$, $\alpha = 1$.

Now, observe that $\{\widetilde{\varphi}_0\} \cup \{\widetilde{\psi}_{jk}^{(\nu)}\}_{j,k,\nu}$ is also a Bessel system; in fact, this system has essentially the same properties as $\{\varphi_0\} \cup \{\psi_{jk}^{(\nu)}\}_{j,k,\nu}$. Also, by construction, these systems are dual wavelet systems generated by a pair of scaling sequences. Using this property and equality (17), we conclude that all assumptions of Theorem 3 hold, thus these wavelet systems are dual frames.

Now, we verify that the systems $\{\varphi_{jn}\}_{n \in D(M^j)}$ and $\{\widetilde{\varphi}_{jk}\}_{k \in D(M^j)}$ are biorthonormal for all $j \in \mathbb{Z}_+$. The Fourier coefficients of the functions in question are equal to zero outside of $H(M^{*j})$, hence, by formula (3),

$$\langle \varphi_{jk}, \widetilde{\varphi}_{jl} \rangle = \sum_{n \in \mathbb{Z}^d} \widehat{\varphi_{jk}}(n)\overline{\widehat{\widetilde{\varphi}_{jl}}(n)} = \sum_{s \in H(M^{*j})} \widehat{\varphi_j}(s)\overline{\widehat{\widetilde{\varphi}_j}(s)}e^{2\pi i(M^{*-j}s, k-l)}.$$

Thus, Lemma 2 and formula (16) imply the required property. Now, biorthogonality of the wavelets follows from [18, Sec. 9.2, Theorem 9.2.4]; biorthogonality of the dual frames implies that both frames are Riesz bases (see [10, Sec. 1.2]). □

Remark 1. The condition $\mathbb{T}^d \subset M^*\mathbb{T}^d$ in Theorem 4 is essential: without this condition, property **S5** from Theorem 2 may be missing for the sequence $\{\varphi_j\}_j$. To give a concrete example, put

$$M = \begin{bmatrix} 2 & 0 \\ 9 & 3 \end{bmatrix}, \quad k = (3, 1).$$

Obviously, the eigenvalues of the above matrix are greater than one in modulus; it is easy to see that $\mathbb{T}^d \not\subset M\mathbb{T}^d$. Also, $k \in H(M^*)$ but $k \notin H(M^{*2})$. Whence $\widehat{\varphi_1}(k) \neq 0$ but $\widehat{\varphi_2}(k) = 0$; thus, there is no μ_k^2 such that $\widehat{\varphi_1}(k) = \mu_k^2 \widehat{\varphi_2}(k)$.
Remark 2. The condition $\mathbb{T}^d \subset M^*\mathbb{T}^d$ definitely holds if $\|M^{*-1}\| \leq \frac{1}{\sqrt{d}}$.

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REFERENCES

1. C. K. Chui and J. Z. Wang, "A general framework of compact supported splines and wavelets," *J. Approx. Theory*, **71** (1992), 263–304.
2. S. S. Gon, S. Z. Lee, Z. Shen, and W. S. Tang, "Construction of Schauder decomposition on Banach spaces of periodic functions," in: *Proceedings of the Edinburgh Mathematical Society*, **41**, No. 1 (1998), pp. 61–91.
3. A. P. Petukhov, "Periodic wavelets," *Mat. Sb.*, **188**, No. 10 (1997), 69–94.
4. V. A. Zheludev, "Periodic splines and wavelets," in: *Proc. of the Conference "Math. Analysis and Signal Processing*," Cairo, Jan. 2–9 (1994).
5. B. Han, "On dual wavelet tight frames," *ACHA*, **4** (1997), 380–413.
6. I. Daubechies, B. Han, A. Ron, and Z. Shen, "Framelets: MRA-based constructions of wavelet frames," *ACHA*, **14**, No. 1 (2003), 1–46.
7. B. Han, "Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix," *J. Comput. Appl. Math.*, **155** (2003), 43–67.
8. M. Skopina, "Wavelet approximation of periodic functions," *J. Approx. Theory*, **104**, No. 2 (2000), 302–329.
9. M. A. Skopina, "Local convergence of Fourier series with respect to periodized wavelets," *J. Approx. Theory*, **94**, No. 2 (1998), 191–202.
10. A. Krivoshein, V. Protasov, and M. Skopina, *Multivariate Wavelet Frames*, Springer Singapore (2016).
11. A. Ron, Z. Shen, "Gramian analysis of affine bases and affine frames," in: *Approximation Theory* VIII, Vol. 2, *Wavelets* (C. K. Chui and L. Schumaker, eds.) World Scientific Publishing Co. Inc (Singapore) (1995), pp. 375–382.
12. A. Ron and Z. Shen, "Frame and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$," *Canad. J. Math.*, **47**, No. 5 (1995), 1051–1094.
13. A. Ron and Z. Shen, "Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator," *J. Funct. Anal.*, **148** (1997), 408–447.
14. A. Ron and Z. Shen, "Affine systems in $L_2(\mathbb{R}^d)$: dual systems," *J. Fourier Anal. Appl.*, **3** (1997), 617–637.
15. P. Andrianov, "On sufficient frame conditions for periodic wavelet systems," *Intern. J. Wavelets*, **16**, No. 1 (2018) 1850002.
16. N. D. Atreas, "Characterization of dual multiwavelet frames of periodic functions," *Intern. J. Wavelets*, **14**, No. 03 (2016), 1650012.

17. M. Skopina, "Multiresolution analysis of periodic functions," *East J. Approx.*, **3**, No. 2 (1997), 203–224.
18. I. Y. Novikov, V. Y. Protasov, and M. A. Skopina, *Wavelet Theory*, American Mathematical Society (2011).
19. I. Maksimenko and M. Skopina, "Multivariate periodic wavelets," *St. Petersburg Math. J.*, **15** (2004), 165–190.