DOI 10.1007/s10958-020-05067-7
Journal of Mathematical Sciences, Vol. 251, No. 1, November, 2020

ON ASYMPTOTICALLY MINIMAX NONPARAMETRIC DETECTION OF SIGNAL IN GAUSSIAN WHITE NOISE

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UDC 519.2

For the problem of nonparametric detection of signal in Gaussian white noise, strong asymptotically minimax tests are found. The sets of alternatives are balls in the Besov space $\mathbb{B}^s_{2\infty}$ with "small" balls in $\mathbb{L}_2$ removed. The balls in the Besov space are defined in terms of orthogonal expansions of functions in trigonometrical basis. Similar result is also obtained for nonparametric hypothesis testing on a solution of ill-posed linear inverse problem with Gaussian random noise. Bibliography: 19 titles.

1. Introduction

In the problem of nonparametric signal detection in Gaussian white noise, the rate of consistency of nonparametric tests has been studied for wide class of functional spaces and completely different setups (see [2, 9, 10, 14, 15] and references therein). At the same time asymptotically minimax tests with strong asymptotics of type I and type II error probabilities are known only in the following special cases:

if a priori information that the signal belongs to either ellipsoid in $\mathbb{L}_2$ (see [3, 4]) or bodies in Besov spaces defined in terms of orthogonal expansions with respect to wavelets (see [10]) is provided,

or if the signal satisfies the Lipshitz conditions (see [16]).

A goal of the present paper is to pay attention to the fact that asymptotically minimax nonparametric tests with strong asymptotics of type I and type II error probabilities can be obtained also for another sets of alternatives. For orthogonal trigonometrical system of functions, these sets are, for some norm, the balls $\mathbb{B}^s_{2\infty}(P_0)$, $s > 0$, $P_0 > 0$, in the Besov spaces.

Using the balls $\mathbb{B}^s_{2\infty}(P_0)$ in the problems of signal detection is rather natural.

The balls $\mathbb{B}^s_{2\infty}(P_0)$ provide reasonable information on the smoothness of signal.

For the most widespread nonparametric tests, these balls are the largest sets (maxisets) with given rate of consistency [6].

Maxisets are intensively explored in nonparametric estimation (see [12,13,18] and references therein). In particular, Kerkyacharian and Picard [12] have shown that the balls $\mathbb{B}^s_{2\infty}(P_0)$ in a Besov space are maxisets for linear estimators. For the balls $\mathbb{B}^s_{2\infty}(P_0)$, asymptotically minimax estimators [5] are the estimators of the Tikhonov regularizing algorithm. In nonparametric hypothesis testing, the maxisets have been explored in [1, 6].

In the present paper, the statement of the problem is as follows.

We observe a realization of a random process $Y_n(t)$, $t \in [0, 1)$, defined by the stochastic differential equation

$$dY_n(t) = f(t)\,dt + \frac{\sigma}{\sqrt{n}}\,dw(t), \quad t \in [0, 1], \quad \sigma > 0, \tag{1.1}$$

where $f \in \mathbb{L}_2(0, 1)$ is unknown signal and $dw(t)$ is Gaussian white noise.

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 474, 2018, pp. 124–138. Original article submitted November 7, 2018.

We have a priori information that

$$f \in \bar{\mathbb{B}}^s_{2\infty}(P_0) = \Big\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j,\ \sup_{\lambda>0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 \le P_0,\ \theta_j \in \mathbb{R}^1 \Big\}.$$

Here, the ϕ_j form an orthonormal system of functions in $\mathbb{L}_2(0,1)$ and $s > 0$.

We need to test the hypothesis $\mathbf{H_0}:\quad f(t) = 0,\ t \in [0,1)$, against the alternatives

$$\mathbf{H_n}: \quad \int_0^1 f^2(t)\,dt \ge \rho_n,$$

where $\rho_n \asymp n^{-\frac{4s}{1+4s}}$ as $n \to \infty$. Under some conditions on the basis of ϕ_j, $1 \le j < \infty$, the space

$$\bar{\mathbb{B}}^s_{2\infty} = \Big\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j, \quad \sup_{\lambda>0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 < \infty, \quad \theta_j \in \mathbb{R}^1 \Big\}$$

is the Besov space $\mathbb{B}^s_{2\infty}$ (see [18]). In particular, $\bar{\mathbb{B}}^s_{2\infty}$ is the Besov space $\mathbb{B}^s_{2\infty}$ if the ϕ_j, $1 \le j < \infty$, form a trigonometrical basis.

The balls in the Nikolskii class (see [19]),

$$\int (f^{(l)}(x+t) - f^{(l)}(x))^2 dx \le C|t|^{2(s-l)}, \qquad \int_0^1 f^2(t)\,dt < C_1,$$

with $l = [s]$, are balls in $\mathbb{B}^s_{2\infty}$.

Here and below, we use C and c as generic notation of positive constants.

We put $V_n = \{f : \|f\|^2 \ge \rho_n,\ f \in \bar{\mathbb{B}}^s_{2\infty}(P_0)\}$.

For any test K_n, we denote by $\alpha(K_n)$ and $\beta_f(K_n)$ the type I and type II error probabilities of K_n for the alternative $f \in V_n$.

Put

$$\beta(K_n, V_n) = \sup_{f \in V_n} \beta_f(K_n).$$

A sequence of tests L_n is said to be asymptotically minimax if for any sequence of tests K_n, $\alpha(K_n) \le \alpha(L_n)$, we have

$$\lim \sup_{n\to\infty} \beta(K_n, V_n) - \beta(L_n, V_n) \ge 0.$$

A goal of the present paper is to point out asymptotically minimax sequences of tests L_n for the sets of the alternatives V_n. When the sets of alternatives are ellipsoids with removed small balls in $\mathbb{L}_2$, asymptotically minimax tests has been found in [3]. For problems of nonparametric hypothesis testing, this result can be considered as an analog of the Pinsker theorem [11,17,19] in nonparametric estimation.

We note that for the bodies

$$\widetilde{\mathbb{B}}^s_{2\infty}(P_0) = \Big\{ f : f = \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \theta_{kj} \phi_{kj}, \quad \sup_k 2^{2ks} \sum_{j=1}^{2^k} \theta_{kj}^2 \le P_0 \Big\},$$

defined in terms of the wavelets ϕ_{kj}, asymptotically minimax tests have been found in [9]. In this setup, a completely different extremal problem arises in construction of asymptotically minimax test statistics.

2. Main results

The results are given in terms of a sequence model.

Using an orthonormal system of functions ϕ_j, $1 \le j < \infty$, one can rewrite stochastic differential equation (1.1) in the form

$$y_j = \theta_j + \frac{\sigma}{\sqrt{n}}\xi_j, \quad 1 \le j < \infty, \tag{2.1}$$

where

$$y_j = \int_0^1 \phi_j dY_n(t), \quad \xi_j = \int_0^1 \phi_j\, dw(t), \quad \text{and} \quad \theta_j = \int_0^1 f\phi_j\, dt.$$

Put $\boldsymbol{\theta} = \{\theta_j\}_{j=1}^{\infty}$.

The vector $\boldsymbol{\theta}$ belongs to the Hilbert space $\mathbb{H}$ with norm $\|\boldsymbol{\theta}\| = \left(\sum_{i=1}^{\infty} \theta_i^2\right)^{1/2}$. In what follows, $\|\cdot\|$ denotes the norm in $\mathbb{H}$.

We need to test the hypothesis $\mathbf{H}_0 : \boldsymbol{\theta} = 0$ against the alternatives

$$\mathbf{H}_n : \boldsymbol{\theta} \in V_n = \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| \ge \rho_n,\, \boldsymbol{\theta} \in \bar{\mathbb{B}}^s_{2\infty}(P_0)\}.$$

We note that in Sec. 1, the sets V_n have been defined for functions. Here, an equivalent definition is given for the coefficients of the orthogonal expansions of the functions. In what follows, we use this notation.

Set $k = k_n$ and $\kappa^2 = \kappa_n^2$ to be a solution of the equations

$$\frac{1}{2s} k_n^{1+2s}\kappa_n^2 = P_0 \tag{2.2}$$

and

$$k_n\kappa_n^2 + k_n^{-2s}P_0 = \rho_n. \tag{2.3}$$

Put $\kappa_j^2 = \kappa_{jn}^2 = \kappa_n^2$ for $1 \le j \le k_n$, and $\kappa_j^2 = \kappa_{jn}^2 = 2sP_0 j^{-2s-1}$ for $j > k_n$.

We define the test statistics

$$T_n^a(Y_n) = \sigma^{-2} n \sum_{j=1}^{\infty} \kappa_j^2 y_j^2$$

and put

$$A_n = \sigma^{-4} n^2 \sum_{j=1}^{\infty} \kappa_j^4,$$

$$C_n = \sigma^{-2} n \rho_n.$$

For the type I error probabilities α, $0 < \alpha < 1$, we define the critical regions

$$S_n^a == \{\boldsymbol{y} : (T_n^a(\boldsymbol{y}) - C_n)(2A_n)^{-1/2} > x_\alpha, \boldsymbol{y} \in \mathbb{R}^\infty\},$$

where x_α is defined from the equation $\alpha = 1 - \Phi(x_\alpha)$. Here, $\Phi(x)$ is a value of standard normal distribution function at the point $x \in \mathbb{R}^1$.

Theorem 2.1. *Let*

$$0 < \liminf_{n\to\infty} A_n \le \limsup_{n\to\infty} A_n < \infty. \tag{2.4}$$

Then the tests L_n^a with critical regions S_n^a are asymptotically minimax, $\alpha(L_n^a) = \alpha(1 + o(1))$, and

$$\beta(L_n^a, V_n) = \Phi(x_\alpha - (A_n/2)^{1/2})(1 + o(1)) \tag{2.5}$$

as $n \to \infty$.

Example. Let $\rho_n = R(\sigma^2/n)^{\frac{4s}{1+4s}}(1+o(1))$ as $n \to \infty$. Then

$$A_n = \sigma^{-4} n^2 \rho_n^{\frac{1+4s}{2s}} \frac{8s^2}{(1+4s)(1+2s)}((1+2s)P_0)^{-1/2s}(1+o(1))$$
$$= R^2 \frac{8s^2}{(1+4s)(1+2s)}((1+2s)P_0)^{-1/2s}(1+o(1)).$$

Ingster, Sapatinas, Suslina [10] and Laurent, Loubes, Marteau [14] have explored the problem of signal detection for linear inverse ill-posed problems. The setup was explored in terms of the sequence model

$$y_j = \lambda_j \theta_j + \frac{\sigma}{\sqrt{n}} \xi_j, \quad 1 \le j < \infty,$$

where the ξ_j are i.i.d. random vectors having standard normal distribution and the λ_j are the eigenvalues of a linear operator.

It is easy to see (see [6]) that if $|\lambda_j| \asymp j^{-\gamma}$, then for the tests statistics defined as quadratic forms of y_j, $1 \le j < \infty$, the maxisets are the balls $\mathbb{B}^s_{2\infty}$ with $r = \frac{2s}{1+4s+4\gamma}$. Thus it is of interest to find asymptotically minimax test statistics for the problem of testing of the hypothesis $\mathrm{H}_0 : \boldsymbol{\theta} = 0$ against the alternatives $\mathrm{H_n} : \boldsymbol{\theta} \in V_n$.

We define the test statistics

$$T_n^a(Y_n) = \sigma^{-2} n \sum_{j=1}^{\infty} \kappa_j^2 y_j^2,$$

where $\kappa_j^2 = \kappa_{jn}^2$ is defined by the equations $\kappa_j^2 = a_n \lambda_j^{-2}$ for $j \le k_n$, and $\kappa_j^2 = 2sP_0 \lambda_j^2 j^{-1-2s}$ for $j > k_n$, and the constants a_n and k_n are the solutions of the equations

$$a_n \sum_{j=1}^{k_n} \lambda_j^{-4} + P_0 k_n^{-2s} = \rho_n \quad \text{and} \quad a_n \lambda_{k_n}^{-4} = 2sP_0 k_n^{-1-4s}.$$

In this notation, the sequences A_n and critical regions S_n^a are as in Theorem 2.1.

Theorem 2.2. *Let $|\lambda_j| \asymp j^{-\gamma}$. Then in the above notation and statement of the problem, the conclusion of Theorem 2.1 holds.*

Example. Let $\lambda_j^2 = Aj^{-2\gamma}$, and let $\rho_n \asymp n^{\frac{-4s}{1+4s+4\gamma}}$. Then

$$A_n = \sigma^{-4} n^2 \rho_n^{\frac{1+4s+4\gamma}{2s}} A^2 \frac{8s^2(1+4\gamma)}{(1+2s+4\gamma)(1+4s+4\gamma)} \Big(\frac{1+2s+4\gamma}{1+4\gamma} P_0\Big)^{-\frac{1+4\gamma}{2s}} (1+o(1)).$$

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1 and hence is omitted.

3. Proof of Theorem 2.1

Fix δ, $0 < \delta < 1$. Put $\kappa_j^2(\delta) = 0$ for $j > \delta^{-1} k_n(1+o(1))$, and define $\kappa_j^2(\delta)$, $1 \le j < k_{n\delta} = \delta^{-1} k_n$, by equations (2.2) and (2.3) with P_0 and ρ_ϵ replaced with $P_0(1-\delta)$ and $\rho_n(1+\delta)$, respectively. Similarly to [3], we define a Bayes test for a priori distribution $\theta_j = \eta_j = \eta_j(\delta)$, $1 \le j < \infty$, where the η_j are Gaussian independent random variables such that $\mathbf{E}\eta_j = 0$ and $\mathbf{E}\eta_j^2 = \kappa_j^2(\delta)$. We show that these tests are asymptotically minimax for some sequence $\delta = \delta_n \to 0$ as $n \to \infty$.

Lemma 3.1. *For every δ, $0 < \delta < 1$, we have*

$$\mathbf{P}(\eta(\delta) = \{\eta_j(\delta)\}_{j=1}^{\infty} \in V_n) = 1 + o(1) \tag{3.1}$$

as $n \to \infty$.

Put

$$A_{n,\delta} = \sigma^{-4} n^2 \sum_{j=1}^{\infty} \kappa_j^4(\delta).$$

A straightforward calculation shows that

$$\lim_{\delta\to 0} \lim_{n\to\infty} A_n A_n^{-1}(\delta) = 1. \tag{3.2}$$

Put $\gamma_j^2(\delta) = \kappa_j^2(\delta)(n^{-1}\sigma^2 + \kappa_j^2(\delta))^{-1}$.

By the Neymann–Pearson lemma, the Bayes critical region is defined by the inequality

$$\begin{aligned} C_1 &< \prod_{j=1}^{k_{n\delta}} (2\pi)^{-1/2} \kappa_j^{-1}(\delta) \\ &\times \int \exp\Big\{ -\sum_{j=1}^{k_{n\delta}} (2\gamma_j^2(\delta))^{-1} (u_j - \gamma_j^2(\delta) y_j)^2 \Big\} \, d\boldsymbol{u} \exp\{-T_{n\delta}(Y_n)\} \\ &= C \exp\{-T_{n\delta}(Y_n)\}(1 + o(1)), \end{aligned} \tag{3.3}$$

where $\boldsymbol{u} = \{u_j\}_{j=1}^{\infty}$ and

$$T_{n\delta}(Y_n) = n\sigma^{-2} \sum_{j=1}^{\infty} \gamma_j^2(\delta) y_j^2.$$

We define the test statistics $R_{n\delta}(Y_n) = (T_{n\delta}(Y_n) - C_{n\delta})(2A_n(\delta))^{-1/2}$ with

$$C_{n\delta} = \mathbf{E}_0 T_{n\delta}(Y_n) = \sigma^{-2} n \sum_{j=1}^{\infty} \gamma_j^2(\delta),$$

and the critical region $S_{n\delta} = \{\boldsymbol{y} : R_{n\delta}(\boldsymbol{y}) > x_\alpha, \boldsymbol{y} \in \mathbb{R}^\infty\}$.

Denote by $L_{n\delta}$ the test with critical region $S_{n\delta}$.

Let $\gamma_j^2 = \gamma_j^2(0)$, $1 \le j < \infty$. The test statistics T_n, R_n, critical regions S_n, and constants C_n are defined in the same way with $\gamma_j^2(\delta)$ replaced by γ_j^2 as the test statistics $T_{n\delta}, R_{n\delta}$, critical regions $S_{n\delta}$, and constants $C_{n\delta}$, respectively. Denote by L_n the test having critical region S_n, and put

$$A_n(\boldsymbol{\theta}) = \sigma^{-4} n^2 \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2.$$

Lemma 3.2. *Let the hypothesis H_0 hold. Then the distributions of the tests statistics $R_n^a(Y_n)$ and $R_n(Y_n)$ converge to the standard normal distribution. Moreover, for any family $\theta_n = \{\theta_{jn}\} \in V_n$,*

$$\mathbf{P}_{\theta_n}\Big(\Big(T_n^a(Y_n) - C_n - A_n(\boldsymbol{\theta})\Big)(2A_n)^{-1/2} < x_\alpha \Big) = \Phi(x_\alpha)(1 + o(1)) \tag{3.4}$$

and

$$\mathbf{P}_{\theta_n}\Big(\Big(T_n(Y_n) - C_n - A_n(\boldsymbol{\theta})\Big)(2A_n)^{-1/2} < x_\alpha \Big) = \Phi(x_\alpha)(1 + o(1)) \tag{3.5}$$

as $n \to \infty$.

From Lemma 3.1, we get the following statement.

Lemma 3.3. *We have*

$$\beta(L_n, V_n) = \beta(L_n^a, V_n)(1 + o(1)) \tag{3.6}$$

as $n \to \infty$.

Lemma 3.4. *Let the hypothesis* H_0 *hold. Then the distributions of the tests statistics* $(T_{n\delta}(Y_n) - C_{n\delta})(2A_n)^{-1/2}$ *converge to the standard normal distribution. Moreover,*

$$\mathbf{P}_{\eta(\delta)}((T_{n\delta}(Y_n) - C_{n\delta} - A_{n\delta}(\eta_n))(2A_{n\delta})^{-1/2} < x_\alpha) = \Phi(x_\alpha)(1 + o(1)) \tag{3.7}$$

as $n \to \infty$.

Lemma 3.5. *We have*

$$\lim_{\delta\to 0}\lim_{n\to\infty} \mathbf{E}_{\eta(\delta)}\beta_{\eta(\delta)}(L_{n\delta}) = \lim_{n\to\infty} \mathbf{E}_{\eta_0}\beta_{\eta_0}(L_n), \tag{3.8}$$

where $\eta_0 = \{\eta_{0j}\}_{j=1}^\infty$ *and* η_{0j} *are i.i.d. Gaussian random variables,* $\mathbf{E}\,[\eta_{0j}] = 0$, *and* $\mathbf{E}[\eta_{0j}^2] = \kappa_j^2$, $1 \le j < \infty$.

We define the Bayes a priori distribution $\mathbf{P}_y$ as the conditional distribution of η under the condition $\eta \in V_n$. Denote by $K_n = K_{n\delta}$ the Bayes test with a priori distribution P_y, and set $W_{n\delta}$ to be the critical region of $K_{n\delta}$.

For any sets A and B, put $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Lemma 3.6. *We have*

$$\lim_{\delta\to 0}\lim_{n\to\infty} \int\limits_{V_n} \mathbf{P}_\theta(S_{n\delta} \triangle W_{n\delta}) d\mathbf{P}_y = 0 \tag{3.9}$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty} \mathbf{P}_0(S_{n\delta} \triangle W_{n\delta}) = 0. \tag{3.10}$$

Lemma 3.7. *We have*

$$\mathbf{E}_{\eta_0}\beta_{\eta_0}(L_n) = \beta_n(L_n)(1 + o(1)). \tag{3.11}$$

Lemma 3.8. *We have*

$$\lim_{\delta\to 0}\lim_{n\to\infty} \int\limits_{V_n} \mathbf{P}_\theta(S_{n\delta} \triangle W_{n\delta}) d\mathbf{P}_y = 0 \tag{3.12}$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty} \mathbf{P}_0(S_{n\delta} \triangle W_{n\delta}) = 0. \tag{3.13}$$

In the proof of Lemma 3.8, we show that the integrals on the right hand-side of (3.3) with integration domain V_n converge to one in probability as $n \to \infty$. This statement is proved both for the hypothesis and the Bayes alternative (see [3]).

Lemmas 3.1–3.8 imply that if $\alpha(K_n) = \alpha(L_n)$, then

$$\int\limits_{V_n} \beta_\theta(K_n)\, d\mathbf{P}_y = \int\limits_{V_n} \beta_\theta(L_n)\, d\mathbf{P}_y\,(1 + o(1)) = \int \beta_{\eta_0}(L_n)\, d\mathbf{P}_{\eta_0}(1 + o(1)). \tag{3.14}$$

Lemma 3.9. *We have*

$$\mathbf{E}_{\eta_0}\beta_{\eta_0}(L_n) = \beta_n(L_n)(1 + o(1)). \tag{3.15}$$

Lemmas 3.2, 3.5, formulas (3.2), (3.14), and Lemma 3.9 imply Theorem 2.1.

3.1. Proof of lemmas. The proofs of Lemmas 3.2, 3.3, and 3.5 are similar to the proofs of the corresponding statements in [3] and are omitted.

Proof of Lemma 3.1. A straightforward calculation shows that

$$\sum_{j=1}^{\infty} \mathbf{E}\eta_j^2(\delta) \ge \rho_n(1+\delta/2) \tag{3.16}$$

and

$$\mathbf{Var}\Big(\sum_{j=1}^{\infty} \eta_j^2(\delta)\Big) < Cn^2 A_n \asymp \rho_n^2 k_n^{-1}. \tag{3.17}$$

By the Chebyshev inequality, we get

$$\mathbf{P}\Big(\sum_{j=1}^{\infty} \eta_j^2(\delta) > \rho_n\Big) = 1 + o(1) \tag{3.18}$$

as $n \to \infty$. □

It remains to note that

$$\mathbf{P}_\mu(\eta \notin \mathbb{B}^s_{2\infty}(P_0)) = \mathbf{P}\Big(\max_{1\le i\le k_{n\delta}} i^{2s} \sum_{j=i}^{k_{n\delta}} \eta_j^2 - P_0(1-\delta_1/2) > P_0\delta_1/2\Big) \le \sum_{i=1}^{k_{n\delta}} J_i, \tag{3.19}$$

where

$$J_i = \mathbf{P}\Big(i^{2s} \sum_{j=i}^{k_{n\delta}} \eta_j^2 - P_0(1-\delta_1/2) > P_0\delta_1/2\Big).$$

To estimate J_i, we make use of the following statement (see [7]).

Proposition 3.1. *Let $\xi = \{\xi_i\}_{i=1}^l$ be a Gaussian random vector with i.i.d. random variables ξ_i such that $\mathbf{E}[\xi_i] = 0$ and $\mathbf{E}[\xi_i^2] = 1$. Let $A \in \mathbb{R}^l \times \mathbb{R}^l$ and $\Sigma = A^T A$. Then*

$$\mathbf{P}(||A\xi||^2 > \mathrm{tr}(\Sigma) + 2\sqrt{\mathrm{tr}(\Sigma^2)t} + 2\|\Sigma\|t) \le \exp\{-t\}. \tag{3.20}$$

We put $\Sigma_i = \{\sigma_{lj}\}_{l,j=i}^{k_{\epsilon\delta}}$ where $\sigma_{jj} = j^{-2s-1} i^{2s} \frac{P_0-\delta}{2s}$ and $\sigma_{lj} = 0$ if $l \ne j$.
Let $i \le k_n$. Then

$$\mathrm{tr}(\Sigma_i^2) = i^{4s} \sum_{j=i}^{\infty} \kappa_j^4(\delta) < i^{4s}((k_n - i)\kappa^4(\delta) + k_n^{-4s-1} P_0) < Ck_n^{-1} \tag{3.21}$$

and

$$\|\Sigma_i\| \le i^{2s}\kappa^2 < Ck_n^{-1}. \tag{3.22}$$

Therefore,

$$2\sqrt{\mathrm{tr}(\Sigma_i^2)t} + 2\|\Sigma_i\|t \le C(\sqrt{k_n^{-1}t} + k_n^{-1}t). \tag{3.23}$$

Hence, putting $t = k_n^{1/2}$ and using Proposition 3.1, we get

$$\sum_{i=1}^{k_n} J_i \le Ck_n \exp\{-Ck_n^{1/2}\}. \tag{3.24}$$

Let $i \ge k_n$. Then

$$\mathrm{tr}(\Sigma_i^2) < Ci^{-1} \quad \text{and} \quad ||\Sigma_i|| \le Ci^{-1}. \tag{3.25}$$

Hence, putting $t = i^{1/2}$ and using Proposition 3.1, we get

$$\sum_{i=k_n+1}^{k_{n\delta}} J_i \le \sum_{i=k_n+1}^{k_{n\delta}} \exp\{-Ci^{1/2}\} < \exp\{-C_1 k_n^{1/2}\}. \tag{3.26}$$

Now (3.19), (3.24), and (3.26) together imply Lemma 3.1.

Proof of Lemma 3.8. Using the same argument as in the proof of Lemma 4 in [3], we see that Lemma 3.8 follows if

$$\mathbf{P}\Big(\sum_{j=1}^{\infty}(\eta_j(\delta) + y_j\gamma_j(\delta)\sigma^{-1}n^{1/2})^2 > \rho_n\Big) = 1 + o(1) \tag{3.27}$$

and

$$\mathbf{P}\Big(\sup_i i^{2s}\sum_{j=i}^{\infty}(\eta_j(\delta) + y_j\gamma_j(\delta)\sigma^{-1}n^{1/2})^2 > \rho_n\Big) = 1 + o(1), \tag{3.28}$$

where the $y_j, 1 \le j < \infty$, are distributed according to the hypothesis or Bayes alternative. □

We prove (3.28) in the case of the Bayes alternative only. In the other cases, the argument is similar.

We have

$$\begin{aligned} i^{2s}\sum_{j=i}^{\infty}(\eta_j(\delta) + y_j\gamma_j(\delta)\sigma^{-1}n^{1/2})^2 &= i^{2s}\sum_{j=i}^{\infty}\eta_j^2(\delta) \\ + i^{2s}\sum_{j=i}^{\infty}\eta_j(\delta)y_j\gamma_j(\delta)\sigma^{-1}n^{1/2} + i^{2s}\sum_{j=i}^{\infty}y_j^2\gamma_j^2(\delta)\sigma^{-2}n &= J_{1i} + J_{2i} + J_{3i}. \end{aligned} \tag{3.29}$$

The required probability for J_{1n} is provided by Lemma 3.1.

We have

$$J_{2i} \le J_{1i}^{1/2}J_{3i}^{1/2}. \tag{3.30}$$

Thus it remains to show that for every C,

$$\mathbf{P}_{\eta(\delta)}\Big(\sup_i i^{2s}\sum_{j=i}^{\infty}y_j^2\gamma_j^4(\delta)\sigma^{-2}n > C\delta\Big) = o(1) \tag{3.31}$$

as $n \to \infty$.

We note that $y_j = \zeta_j + \sigma n^{-1/2}\xi_j$, where ζ_j and ξ_j, $1 \le j < \infty$, are i.i.d. Gaussian random variables such that $\mathbf{E}\zeta_j = 0$, $\mathbf{E}\zeta_j^2 = \kappa_j^2(\delta)$, $\mathbf{E}\xi_j = 0$, and $\mathbf{E}\xi_j^2 = 1$. Hence,

$$\begin{aligned} \sigma^{-2}n\sum_{j=i}^{\infty}y_j^2\gamma_j^4(\delta) &= \sigma^{-2}n\sum_{j=i}^{\infty}\gamma_j^4(\delta)\zeta_j^2 + \sigma^{-1}n^{1/2}\sum_{j=i}^{\infty}\gamma_j^4(\delta)\zeta_j\xi_j \\ &+ \sum_{j=i}^{\infty}\gamma_j^4(\delta)\xi_j^2 = I_{1i} + I_{2i} + I_{3i}. \end{aligned} \tag{3.32}$$

Since $n\gamma_j^2 = o(1)$, the estimates for the probability of $i^{2s}I_{1i}$ are obvious. It suffices to follow estimates (3.19). We have $I_{2i} \le I_{1i}^{1/2}I_{3i}^{1/2}$. Thus it remains to show that for every C,

$$\mathbf{P}_{\eta(\delta)}\Big(\sup_i i^{2s}\sum_{j=i}^{\infty}\gamma_j^4(\delta)\xi_j^2 > \delta/C\Big) = o(1) \tag{3.33}$$

as $n \to \infty$. Since $\gamma_j^2 = \kappa_j^2(1 + o(1)) = o(1)$, this estimate is obtained in the same way as estimates (3.19).

Proof of Lemma 3.9. By Lemmas 3.2, 3.3, and 3.5, it suffices to show that

$$\inf_{\boldsymbol{\theta}\in V_n} \sigma^4 n^{-2} A_n(\boldsymbol{\theta}) = \inf_{\boldsymbol{\theta}\in V_n} \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2 = \sum_{j=1}^{\infty} \kappa_j^4. \tag{3.34}$$

Put $u_k = k^{2s} \sum\limits_{j=k}^{\infty} \theta_j^2$. We note that $u_k \le P_0$. Then $\theta_j^2 = u_j j^{-2s} - u_{j+1}(j+1)^{-2s}$. Hence,

$$\begin{aligned}\sigma^4 n^{-2} A_n(\theta) &= \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2 = \kappa^2 \sum_{j=1}^{k_n} \theta_j^2 + \sum_{j=k_n}^{\infty} \kappa_j^2 (u_j j^{-2s} - u_{j+1}(j+1)^{-2s}) \\ &= \kappa^2 \sum_{j=1}^{k_n} \theta_j^2 + \kappa^2 u_{k_n} k_n^{-2s} + 2sP_0 \sum_{j=k_n+1}^{\infty} u_j (j^{-4s-1} - (j-1)^{-2s-1} j^{-2s}) \qquad (3.35) \\ &= \kappa^2 \rho_n + 2sP_0 \sum_{j=k_n+1}^{\infty} u_j (j^{-4s-1} - (j-1)^{-2s-1} j^{-2s}).\end{aligned}$$

Since $j^{-4s-1} - (j-1)^{-2s-1} j^{-2s}$ is negative, the inf $A(\theta)$ is attained for $u_j = P_0$. Therefore, $\theta_j^2 = \kappa_j^2$ for $j > k_\epsilon$. □

Thus the problem is reduced to finding

$$\kappa^2 \inf_{\theta_j} \sum_{j=1}^{k_n} \theta_j^2 + \sum_{j=k_n+1}^{\infty} \kappa_j^4 \tag{3.36}$$

under the constraints

$$\sum_{j=1}^{k_n} \theta_j^2 + \sum_{j=k_n+1}^{\infty} \kappa_j^2 = \rho_n$$

and

$$k_n^{2s} \sum_{j=k_n}^{\infty} \theta_j^2 < P_0, \quad 1 \le j < \infty,$$

with $\theta_j^2 = \kappa_j^2$ for $j \ge k_n$.

It is easily seen that this infimum is attained if $\theta_j^2 = \kappa_j^2 = \kappa^2$ for $j \le k_n$.

The research was supported by the RFBR grant No. 17-01-00828.

Translated by the author.

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