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# EXAMPLES OF INTEGRABLE SYSTEMS WITH DISSIPATION ON THE TANGENT BUNDLES OF FOUR-DIMENSIONAL MANIFOLDS

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***Abstract.*** In this paper, we prove the integrability of certain classes of dynamical systems on the tangent bundles of four-dimensional manifolds (systems with four degrees of freedom). The force field considered possessed so-called variable dissipation; they are generalizations of fields studied earlier. This paper continues earlier works of the author devoted to systems on the tangent bundles of two- and three-dimensional manifolds.

***Keywords and phrases***: multidimensional dynamical system, nonconservative force field, integrability, transcendental first integral.

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**Introduction.** Configuration spaces of of many dynamical systems are four-dimensional smooth manifolds; naturally, their phase spaces are tangent bundles of these manifolds. For example, the motion of a five-dimensional generalized spherical pendulum in a nonconservative force field is described by a dynamical system on the tangent bundle of the four-dimensional sphere whose metric is induced by an additional symmetry group (see [9, 12, 13]). In this case, dynamical systems that describe the motion of such a pendulum possess variable dissipation, and a complete list of first integrals consists of transcendental functions that can be expressed as finite combinations of elementary functions.

Another class of problems consists of problems on the motion of a particle on a four-dimensional surface whose metric is induced by the Euclidean metric of the ambient space. In some cases, one manages to find a complete list of transcendental first integrals for systems with variable dissipation. These results are especially important for systems in nonconservative force fields (see [1, 4, 7, 19]).

In this paper, we prove the integrability of certain classes of dynamical systems on tangent bundles of smooth four-dimensional manifolds in the case of systems with variable dissipation (see [1, 4, 19]), which are generalizations of systems studied earlier. Similar results for manifolds of dimensions 2 and 3 were obtained by the author in [10, 11, 17]).

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**1. Equations of geodesics, changes of coordinates, and first integrals.** As is well known, in the case of a four-dimensional smooth Riemannian manifold $M^4$ with coordinates $(\alpha, \beta)$, where $\beta = (\beta_1, \beta_2, \beta_3)$, and an affine connection $\Gamma^i_{jk}(x)$, the equations of geodesic lines on the tangent bundle $T_*M^4\{\dot\alpha, \dot\beta_1, \dot\beta_2, \dot\beta_3;\ \alpha, \beta_1, \beta_2, \beta_3\}$ have the following form (differentiation is performed with respect to the natural parameter):

$$\ddot{x}^i + \sum_{j,k=1}^{4} \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \quad i = 1, \dots, 4, \tag{1}$$

where $\alpha = x^1$, $\beta_1 = x^2$, $\beta_2 = x^3$, $\beta_3 = x^4$, and $x = (x^1, x^2, x^3, x^4)$. We examine the behavior of Eqs. (1) under the change of coordinates on the tangent bundle $T_*M^4$. We perform the following

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change of coordinates on the tangent space depending on a point $x$ of the manifold:

$$\dot{x}^i = \sum_{j=1}^{4} R^{ij}(x) z_j \tag{2}$$

and assume that it can be inverted almost everywhere:

$$z_j = \sum_{i=1}^{4} T_{ji}(x)\dot{x}^i;$$

here $R^{ij}$ and $T_{ji}$, $i,j = 1,\dots,4$, are functions of $x^1$, $x^2$, $x^3$, and $x^4$ such that $RT = E$, where $R = (R^{ij})$, and $T = (T_{ji})$. Equations (2) are called the *new kinematic relations*, i.e., they are relations on the tangent bundle $T_*M^4$.

The following identities hold:

$$\dot{z}_j = \sum_{i=1}^{4} \dot{T}_{ji}\dot{x}^i + \sum_{i=1}^{4} T_{ji}\ddot{x}^i, \quad \dot{T}_{ji} = \sum_{k=1}^{4} T_{ji,k}\dot{x}^k, \tag{3}$$

where

$$T_{ji,k} = \frac{\partial T_{ji}}{\partial x^k}, \quad i,j,k = 1,\dots,4.$$

Substituting Eqs. (1) to (3) we obtain

$$\dot{z}_i = \sum_{j,k=1}^{4} T_{ij,k}\dot{x}^j\dot{x}^k - \sum_{j,p,q=1}^{4} T_{ij}\Gamma^j_{pq}\dot{x}^p\dot{x}^q; \tag{4}$$

where one must substitute the formulas (2) instead of $\dot{x}^i$, $i = 1,\dots,4$.

In other words, Eq. (4) can be rewritten in the form

$$\dot{z}_i + \sum_{j,k=1}^{4} Q_{ijk}\dot{x}^j\dot{x}^k\big|_{(2)} = 0, \tag{5}$$

where

$$Q_{ijk}(x) = \sum_{s=1}^{4} T_{is}(x)\Gamma^s_{jk}(x) - T_{ij,k}(x). \tag{6}$$

**Proposition 1.** *In a domain where* $\det R(x) \neq 0$, *the system* (1) *is equivalent to the composite system* (2), (4).

Thus, the result of the transition from the equations of geodesic (1) to an equivalent system of Eqs. (2), (4) depends on the change of variables (2) on the tangent space (i.e., on the kinematic relations introduced) and on the affine connection $\Gamma^i_{jk}(x)$.

**2. Almost general case.** Further, we consider the following form of kinematic relations:

$$\dot{\alpha} = -z_4, \quad \dot{\beta}_1 = z_3 f_1(\alpha), \quad \dot{\beta}_2 = z_2 f_2(\alpha) g_1(\beta_1), \quad \dot{\beta}_3 = z_1 f_3(\alpha) g_2(\beta_1) h(\beta_2), \tag{7}$$

where $f_k(\alpha)$, $k = 1,2,3$, $g_l(\beta_1)$, $l = 1,2$, and $h(\beta_2)$ are certain smooth functions. Such coordinates $z_1$, $z_2$, $z_3$, and $z_4$ on the tangent space can be introduced if one considers the following equations of geodesics (in particular, on spheres and more general surfaces of revolution):

$$\begin{cases} \ddot{\alpha} + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{\alpha}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_1 + 2\Gamma^1_{\alpha 1}(\alpha,\beta)\dot{\alpha}\dot{\beta}_1 + \Gamma^1_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_2 + 2\Gamma^2_{\alpha 2}(\alpha,\beta)\dot{\alpha}\dot{\beta}_2 + 2\Gamma^2_{12}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_2 + \Gamma^2_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_3 + 2\Gamma^3_{\alpha 3}(\alpha,\beta)\dot{\alpha}\dot{\beta}_3 + 2\Gamma^3_{13}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_3 + 2\Gamma^3_{23}(\alpha,\beta)\dot{\beta}_2\dot{\beta}_3 = 0, \end{cases} \tag{8}$$

(see [4, 5, 18]); the other connection coefficients vanish. In the case (7), Eqs. (4) become

$$\dot z_1=\left[2\Gamma^3_{\alpha3}(\alpha,\beta)+\frac{d\ln|f_3(\alpha)|}{d\alpha}\right]z_1z_4-\left[2\Gamma^3_{13}(\alpha,\beta)+\frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right]f_1(\alpha)z_1z_3 \\ -\left[2\Gamma^3_{23}(\alpha,\beta)+\frac{d\ln|h(\beta_2)|}{d\beta_2}\right]f_2(\alpha)g_1(\beta_1)z_1z_2,$$

$$\dot z_2=\left[2\Gamma^2_{\alpha2}(\alpha,\beta)+\frac{d\ln|f_2(\alpha)|}{d\alpha}\right]z_2z_4-\left[2\Gamma^2_{12}(\alpha,\beta)+\frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right]f_1(\alpha)z_2z_3 \\ -\Gamma^2_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)}h^2(\beta_2)z_1^2, \quad (9)$$

$$\dot z_3=\left[2\Gamma^1_{\alpha1}(\alpha,\beta)+\frac{d\ln|f_1(\alpha)|}{d\alpha}\right]z_3z_4-\Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)}g_1^2(\beta_1)z_2^2 \\ -\Gamma^1_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)}g_2^2(\beta_1)h^2(\beta_2)z_1^2,$$

$$\dot z_4=\Gamma^\alpha_{11}f_1^2(\alpha)z_3^2+\Gamma^\alpha_{22}f_2^2(\alpha)g_1^2(\beta_1)z_2^2+\Gamma^\alpha_{33}f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)z_1^2,$$

and Eqs. (8) are equivalent almost everywhere to the composite system (7), (9) on the tangent bundle $T_*M^4\{z_4,z_3,z_2,z_1;\alpha,\beta_1,\beta_2,\beta_3\}$.

For complete integration of the system (7), (9), we need, generally speaking, seven independent first integrals. However, in the case considered, the number of needed first integrals is less; we prove this below for dissipative systems.

**Proposition 2.** *If the system of equalities*

$$\begin{cases} 2\Gamma^1_{\alpha1}(\alpha,\beta)+\dfrac{d\ln|f_1(\alpha)|}{d\alpha}+\Gamma^\alpha_{11}(\alpha,\beta)f_1^2(\alpha)\equiv 0,\\ 2\Gamma^2_{\alpha2}(\alpha,\beta)+\dfrac{d\ln|f_2(\alpha)|}{d\alpha}+\Gamma^\alpha_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1)\equiv 0,\\ \left[2\Gamma^2_{12}(\alpha,\beta)+\dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right]f_1^2(\alpha)+\Gamma^1_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1)\equiv 0,\\ 2\Gamma^3_{\alpha3}(\alpha,\beta)+\dfrac{d\ln|f_3(\alpha)|}{d\alpha}+\Gamma^\alpha_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\equiv 0,\\ \left[2\Gamma^3_{13}(\alpha,\beta)+\dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right]f_1^2(\alpha)+\Gamma^1_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\equiv 0,\\ \left[2\Gamma^3_{23}(\alpha,\beta)+\dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right]f_2^2(\alpha)g_1^2(\beta_1)+\Gamma^2_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\equiv 0, \end{cases} \quad (10)$$

*is valid, then the system* (7), (9) *has an analytic first integral of the form*

$$\Phi_1(z_4,\ldots,z_1)=z_1^2+\ldots+z_4^2=C_1^2=\text{const}. \quad (11)$$

At first glance, the problem on the existence of a sufficiently simple first integral of the form (11) can be solved by an easier method than the solution of the sufficiently complicate system of quasilinear equations (10) (which contains, in general, partial differential equations that can be reduced to ordinary equations). In this paper, we apply another method, which allows one to find complete sets of first integrals for dissipative systems by solving the system (10).

One can prove a separate theorem on the existence of a solution $f_k(\alpha)$, $k=1,2,3$, $g_l(\beta_1)$, $l=1,2$, $h(\beta_2)$ to the system (10) of quasilinear equations and hence prove the existence of an analytical first integral (11) for the system (7), (9) of the equations of geodesics (8). However, in the study of dissipative dynamical systems we do not use the conditions (10). Nevertheless, we assume below that for Eqs. (7) the following conditions hold:

$$f_1(\alpha) = f_2(\alpha) = f_3(\alpha) = f(\alpha); \tag{12}$$

the functions $g_l(\beta_1)$, $l = 1, 2$, and $h(\beta_2)$ must satisfy the transformed equations from (10), namely,

$$\begin{cases} 2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1} + \Gamma^1_{22}(\alpha,\beta)g_1^2(\beta_1) \equiv 0, \\ 2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1} + \Gamma^1_{33}(\alpha,\beta)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ 2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2} + \Gamma^2_{33}(\alpha,\beta)h^2(\beta_2) \equiv 0. \end{cases} \tag{13}$$

Thus, the function $g_l(\beta_1)$, $l = 1, 2$, and $h(\beta_2)$ depend on the connection coefficients due to the system (13); restrictions for the function $f(\alpha)$ will be specified below.

**Proposition 3.** *If the properties* (12) *and* (13) *are fulfilled and the equalities*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma^3_{\alpha 3}(\alpha,\beta) = \Gamma_1(\alpha) \tag{14}$$

*hold, then the system* (7), (9) *has the following smooth first integral*:

$$\begin{gathered} \Phi_2(z_3, z_2, z_1; \alpha) = \sqrt{z_1^2 + z_2^2 + z_3^2}\,\Phi_0(\alpha) = C_2 = \text{const}, \\ \Phi_0(\alpha) = f(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}. \end{gathered} \tag{15}$$

**Proposition 4.** *If the conditions of Proposition* 3 *and the condition*

$$g_1(\beta_1) = g_2(\beta_1) = g(\beta_1) \tag{16}$$

*are fulfilled and, moreover, the equalities*

$$\Gamma^2_{12}(\alpha,\beta) = \Gamma^3_{13}(\alpha,\beta) = \Gamma_2(\beta_1) \tag{17}$$

*hold, then the system* (7), (9) *has the following smooth first integral*:

$$\begin{gathered} \Phi_3(z_2, z_1; \alpha, \beta_1) = \sqrt{z_1^2 + z_2^2}\,\Phi_0(\alpha)\Psi_1(\beta_1) = C_3 = \text{const}, \\ \Psi_1(\beta_1) = g(\beta_1)\exp\left\{2\int\limits_{\beta_{10}}^{\beta_1}\Gamma_2(b)db\right\}. \end{gathered} \tag{18}$$

**Proposition 5.** *If the conditions of Propositions* 3 *and* 4 *are fulfilled and the equality*

$$\Gamma^3_{23}(\alpha,\beta) = \Gamma_3(\beta_2), \tag{19}$$

*holds, then the system* (7), (9) *has the following smooth first integral*:

$$\begin{gathered} \Phi_4(z_1; \alpha, \beta_1, \beta_2) = z_1\Phi_0(\alpha)\Psi_1(\beta_1)\Psi_2(\beta_2) = C_4 = \text{const}, \\ \Psi_2(\beta_2) = h(\beta_2)\exp\left\{2\int\limits_{\beta_{20}}^{\beta_2}\Gamma_3(b)db\right\}. \end{gathered} \tag{20}$$

**Proposition 6.** *If the conditions of Propositions* 3, 4, *and* 5 *are fulfilled, then the system* (7), (9) *has the following first integral*:

$$\Phi_5(z_2, z_1; \alpha, \beta) = \beta_3 \pm \int\limits_{\beta_{20}}^{\beta_2} \frac{C_4 h(b)}{\sqrt{C_3^2 \Phi_2^2(b) - C_4^2}}\, db = C_5 = \text{const}, \tag{21}$$

*after the calculation of the integral in* (21), *one must substitute the left-hand sides of Eqs.* (18) *and* (20) *instead of the constants* $C_3$ *and* $C_4$, *respectively.*

The set of the first integrals (11), (15), (18), (20), and (21) is a complete set of independent first integrals of the system (7), (9) under the conditions listed above. We explain below that the complete set consists of five (not of seven) first integrals.

The problem on the smoothness of the first integrals (21) is quite difficult. Generally speaking, it can be expressed as a finite combination of elementary functions, for example, rational functions. However, since the dynamical system considered does not possess asymptotic limit sets, the function (21) cannot be transcendental (from the standpoint of complex analysis): indeed, it has no essential singular points. But from the standpoint of the theory of elementary functions, it can be transcendental (see also [6, 8]).

**3. Equations of motion on a potential force field and their first integrals.** Now we modify the system (7), (9) under the conditions (12)–(14), (16), (17), and (19) and obtain a conservative system. Namely, the presence of a force field is described by a sufficiently smooth coefficient $F(\alpha)$ in the second equation of the system (22). The system on the tangent bundle $T_*M^4\{z_4, z_3, z_2, z_1; \alpha, \beta_1, \beta_2, \beta_3\}$ takes the form

$$\dot\alpha = -z_4, \tag{22a}$$

$$\dot z_4 = F(\alpha) + \Gamma^\alpha_{11} f_1^2(\alpha) z_3^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^\alpha_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \tag{22b}$$

$$\dot z_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \Gamma^1_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \tag{22c}$$

$$\dot z_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \Gamma^2_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \tag{22d}$$

$$\dot z_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \left[2\Gamma^3_{23}(\alpha,\beta) + \frac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \tag{22e}$$

$$\dot\beta_1 = z_3 f(\alpha), \tag{22f}$$

$$\dot\beta_2 = z_2 f(\alpha) g(\beta_1), \tag{22g}$$

$$\dot\beta_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2); \tag{22h}$$

it is equivalent to the following system almost everywhere:

$$\begin{cases} \ddot{\alpha} + F(\alpha) + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{\alpha}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_1 + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_1 + \Gamma^{1}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{1}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_2 + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_2 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_2 + \Gamma^{2}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_3 + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_3 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_3 + 2\Gamma_3(\beta_2)\dot{\beta}_2\dot{\beta}_3 = 0. \end{cases}$$

**Proposition 7.** *If the condition of Proposition* 2 *are fulfilled, then the system* (22) *possesses the following smooth first integral*:

$$\Phi_1(z_4,\ldots,z_1;\alpha) = z_1^2 + \ldots + z_4^2 + F_1(\alpha) = C_1 = \text{const}, \quad F_1(\alpha) = 2\int\limits_{\alpha_0}^{\alpha} F(a)da. \tag{23}$$

**Proposition 8.** *If the conditions of Propositions* 3, 4, *and* 5 *are fulfilled, then the system* (22) *possesses three smooth first integrals* (15), (18), *and* (20).

**Proposition 9.** *If the conditions of Proposition* 6 *are fulfilled, then the system* (22) *possesses the first integral* (21).

The set of first integrals (23), (15), (18), (20), and (21) is a complete set of independent first integrals of the system (22) under the conditions specified above; below we explain that the complete set consists of five (not seven) first integrals.

As above, the problem on the smoothness of the first integrals (21) is quite difficult. Since the dynamical system considered does not possess asymptotic limit sets even under the action of a smooth conservative force field, the function (21) cannot be transcendental from the standpoint of complex analysis since it has no essential singular points. But from the standpoint of the theory of elementary functions, it can be transcendental (see also [8]).

**4. Equations of motion in a dissipative force field and their first integrals.** Now we complicate the system (22) and obtain a dissipative system. The dissipation (generally speaking, sign-alternating dissipation) is described by the sufficiently smooth coefficient $b\delta(\alpha)$ in the first equation of the following system:

$$\dot{\alpha} = -z_4 + b\delta(\alpha), \tag{24a}$$

$$\dot{z}_4 = F(\alpha) + \Gamma^{\alpha}_{11}f_1^2(\alpha)z_3^2 + \Gamma^{\alpha}_{22}f_2^2(\alpha)g_1^2(\beta_1)z_2^2 + \Gamma^{\alpha}_{33}f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)z_1^2, \tag{24b}$$

$$\dot{z}_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)}g_1^2(\beta_1)z_2^2 - \Gamma^1_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)}g_2^2(\beta_1)h^2(\beta_2)z_1^2, \tag{24c}$$

$$\dot{z}_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha)z_2 z_3 - \Gamma^2_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)}h^2(\beta_2)z_1^2, \tag{24d}$$

$$\dot{z}_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha)z_1 z_3 - \left[2\Gamma^3_{23}(\alpha,\beta) + \frac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha)g_1(\beta_1)z_1 z_2, \tag{24e}$$

$$\dot{\beta}_1 = z_3 f(\alpha), \tag{24f}$$

$$\dot{\beta}_2 = z_2 f(\alpha) g(\beta_1), \tag{24g}$$

$$\dot{\beta}_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2). \tag{24h}$$

This system is equivalent to the following system almost everywhere:

$$\begin{cases} \ddot{\alpha} - b\dot{\alpha}\delta'(\alpha) + F(\alpha) + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{\alpha}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_1 - b\dot{\beta}_1\delta(\alpha)f(\alpha) + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_1 + \Gamma^{1}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{1}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_2 - b\dot{\beta}_2\delta(\alpha)f(\alpha) + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_2 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_2 + \Gamma^{2}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_3 - b\dot{\beta}_3\delta(\alpha)f(\alpha) + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_3 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_3 + 2\Gamma_3(\beta_2)\dot{\beta}_2\dot{\beta}_3 = 0. \end{cases}$$

Now we integrate the eighth-order system (24) under the conditions (12), (13), and (16), and the equalities

$$\Gamma^{\alpha}_{11}(\alpha,\beta) = \Gamma^{\alpha}_{22}(\alpha,\beta)g^2(\beta_1) = \Gamma^{\alpha}_{33}(\alpha,\beta)g^2(\beta_1)h^2(\beta_2) = \Gamma_4(\alpha). \tag{25}$$

Similarly to (13), we impose the following restriction for the function $f(\alpha)$: it must satisfy the first equality from (10) transformed as follows:

$$2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha} + \Gamma_4(\alpha)f^2(\alpha) \equiv 0. \tag{26}$$

For complete integration of the system (24), one needs, in general, seven independent first integrals. However, after the substitution

$$w_4 = z_4, \quad w_3 = \sqrt{z_1^2 + z_2^2 + z_3^2}, \quad w_2 = \frac{z_2}{z_1}, \quad w_1 = \frac{z_3}{\sqrt{z_1^2 + z_2^2}},$$

the system (24) splits as follows:

$$\begin{cases} \dot{\alpha} = -w_4 + b\delta(\alpha), \\ \dot{w}_4 = F(\alpha) + \Gamma_4(\alpha)f^2(\alpha)w_3^2, \\ \dot{w}_3 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] w_3 w_4, \end{cases} \tag{27}$$

$$\begin{cases} \dot{w}_2 = \pm w_3\sqrt{1+w_2^2}\, f(\alpha)g(\beta_1)\left[2\Gamma_3(\beta_2) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right], \\ \dot{\beta}_2 = \pm \dfrac{w_2 w_3}{\sqrt{1+w_2^2}} f(\alpha)g(\beta_1), \end{cases} \tag{28}$$

$$\begin{cases} \dot{w}_1 = \pm w_3\sqrt{1+w_1^2}\, f(\alpha)\left[2\Gamma_2(\beta_1) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right], \\ \dot{\beta}_1 = \pm \dfrac{w_1 w_3}{\sqrt{1+w_1^2}} f(\alpha), \end{cases} \tag{29}$$

$$\dot{\beta}_3 = \pm \frac{w_3}{\sqrt{1+w_2^2}} f(\alpha)g(\beta_1)h(\beta_2). \tag{30}$$

We see that for the complete integrability of the system (27)–(30), it suffices to find two independent first integrals of the system (27), one first integral for each of the systems (28) and (29) (changing independent variables in them), and an additional first integral, which "links" Eq. (30). So, the total number of first integrals is *five*.

**Theorem 1.** *Assume that for certain $\kappa, \lambda \in \mathbf{R}$, the equalities*

$$\Gamma_4(\alpha)f^2(\alpha) = \kappa\frac{d}{d\alpha}\ln|\delta(\alpha)|, \quad F(\alpha) = \lambda\frac{d}{d\alpha}\frac{\delta^2(\alpha)}{2} \tag{31}$$

*are fulfilled. Then under the conditions* (12), (13), (16), (25), *and* (26), *the system* (24) *possesses a complete set* (*namely, five*) *independent* (*generally speaking, transcendental*) *first integrals.*

First, to the third-order system (27), we assign the following nonautonomous second-order system:

$$\begin{cases} \dfrac{dw_4}{d\alpha} = \dfrac{F(\alpha) + \Gamma_4(\alpha) f^2(\alpha) w_3^2}{-w_4 + b\delta(\alpha)}, \\ \dfrac{dw_3}{d\alpha} = \dfrac{\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] w_3 w_4}{-w_3 + b\delta(\alpha)}. \end{cases} \tag{32}$$

Further, introducing the homogeneous variables by the formulas

$$w_4 = u_4\delta(\alpha), \quad w_3 = u_3\delta(\alpha), \tag{33}$$

we reduce the system (32) to the following form:

$$\begin{cases} \delta(\alpha)\dfrac{du_4}{d\alpha} + \delta'(\alpha)u_4 = \dfrac{F(\alpha) + \Gamma_4(\alpha) f^2(\alpha)\delta^2(\alpha) u_3^2}{-u_4\delta(\alpha) + b\delta(\alpha)}, \\ \delta(\alpha)\dfrac{du_3}{d\alpha} + \delta'(\alpha)u_3 = \dfrac{\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] \delta^2(\alpha) u_3 u_4}{-u_4\delta(\alpha) + b\delta(\alpha)}. \end{cases} \tag{34}$$

This system is equivalent to the following system almost everywhere:

$$\begin{cases} \delta(\alpha)\dfrac{du_4}{d\alpha} = \dfrac{F_3(\alpha) + \Gamma_4(\alpha) f^2(\alpha)\delta(\alpha) u_3^2 + \delta'(\alpha)u_4^2 - b\delta'(\alpha)u_4}{-u_4 + b}, \\ \delta(\alpha)\dfrac{du_3}{d\alpha} = \dfrac{-\Gamma_4(\alpha) f^2(\alpha)\delta(\alpha) u_3 u_4 + \delta'(\alpha)u_3u_4 - b\delta'(\alpha)u}{-u_4 + b}; \end{cases} \tag{35}$$

here $F_3(\alpha) = F(\alpha)/\delta(\alpha)$.

If the conditions (31) hold, then the system (35) is reduced to the first-order equation

$$\frac{du_4}{du_3} = \frac{\lambda + \kappa u_3^2 + u_4^2 - bu_4}{(1-\kappa)u_3u_4 - bu_3}. \tag{36}$$

Equation (36) is an Abel equation (see [6]). In particular, for $\kappa = -1$ it possesses the first integral

$$\frac{u_4^2 + u_3^2 - bu_4 + \lambda}{u_3} = C_1 = \text{const}, \tag{37}$$

which in the initial variables has the form

$$\Theta_1(w_4, w_3; \alpha) = G_1\left(\frac{w_4}{\delta(\alpha)}, \frac{w_3}{\delta(\alpha)}\right) = \frac{w_4^2 + w_3^2 - bw_4\delta(\alpha) + \lambda\delta^2(\alpha)}{w_3\delta(\alpha)} = C_1 = \text{const}\,. \tag{38}$$

Further, we find an explicit form of the additional first integral of the third-order system (27) for $\kappa = -1$. We transform the invariant relation (37) for $u_3 \neq 0$ as follows:

$$\left(u_4 - \frac{b}{2}\right)^2 + \left(u_3 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - \lambda. \tag{39}$$

Clearly, the parameters of this invariant relations must satisfy the condition

$$b^2 + C_1^2 - 4\lambda \geq 0, \tag{40}$$

and the phase space of the system (23) splits into surfaces determined by Eq. (39).

Thus, due to the relation (37), the first equation of the system (35) for $\kappa = -1$ has the form

$$\frac{\delta(\alpha)}{\delta'(\alpha)}\frac{du_4}{d\alpha} = \frac{2(\lambda - bu_4 + u_4^2) - C_1U_1(C_1, u_4)}{-u_4 + b}, \tag{41}$$

where

$$U_1(C_1,u_4)=\frac{1}{2}\left\{C_1\pm\sqrt{C_1^2-4(u_4^2-bu_4+\lambda)}\right\}; \tag{42}$$

the integration constant $C_1$ is determined by the condition (40). Then the additional first integral for the system (27) has the form

$$\Theta_2(w_4,w_3;\alpha)=G_2\left(\delta(\alpha),\frac{w_4}{\delta(\alpha)},\frac{w_3}{\delta(\alpha)}\right)=C_2=\text{const}; \tag{43}$$

for $\kappa=-1$, it can be found from the quadrature

$$\ln|g(\alpha)|=\int\frac{(b-u_4)du_4}{2(\lambda-bu_4+u_4^2)-C_1\left\{C_1\pm\sqrt{C_1^2-4(u_4^2-bu_4+\lambda)}\right\}/2},$$

where $u_4=w_4/\delta(\alpha)$. After calculating this integral, one must substitute the left-hand side of Eq. (38) instead of $C_1$. The right-hand side of this equality is expressed as a finite combination of elementary functions, whereas the left-hand side depends on the function $\delta(\alpha)$. Therefore, the expression of the first integrals (38) and (43) as finite combinations of elementary functions depends on quadratures and on the explicit form of the function $\delta(\alpha)$.

The first integrals (28) and (29) for the system have the form

$$\Theta_{s+2}(w_s;\beta_s)=\frac{\sqrt{1+w_s^2}}{\Psi_s(\beta_s)}=C_{s+2}=\text{const},\quad s=1,2; \tag{44}$$

the functions $\Psi_s(\beta_s)$, $s=1,2$, are defined by (18) and (20). The additional first integral, which "links" Eq. (30), is similar to (21):

$$\Theta_5(w_2,w_1;\alpha,\beta)=\beta_3\pm\int\limits_{\beta_{20}}^{\beta_2}\frac{C_4h(b)}{\sqrt{C_3^2\Psi_2^2(b)-C_4^2}}db=C_5=\text{const};$$

after calculating this integral, one must substitute the corresponding left-hand sides of Eqs. (44) instead of the constants $C_3$ and $C_4$.

**5. Remark on the structure of first integrals of dissipative systems.** If $\alpha$ is a $2\pi$-periodic coordinate, then the system (27) becomes a dynamical system with variable dissipation with zero mean (see [14–16]). Moreover, for $b=0$ it turns into a conservative system possessing two smooth first integrals (23) and (15). Due to (31), we have

$$\Phi_1(z_4,\ldots,z_1;\alpha)=z_1^2+\ldots+z_4^2+2\int\limits_{\alpha_0}^{\alpha}F(a)da\cong w_4^2+w_3^2+\lambda\delta^2(\alpha), \tag{45}$$

where $\cong$ means equality up to an additive constant. Moreover, due to (26) and (31), we have

$$\Phi_2(z_3,z_2,z_1;\alpha)=\sqrt{z_1^2+z_2^2+z_3^2}\;f(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}\simeq w_3\delta(\alpha)=C_2=\text{const}, \tag{46}$$

where $\simeq$ means equality up to a multiplicative constant.

Obviously, the ratio of the first integrals (45) and (46) (or (23) and (15)) is also a first integral of the system (27) for $b=0$. However, for $b\neq0$, the functions

$$w_4^2+w_3^2-bw_4\delta(\alpha)+\lambda\delta^2(\alpha) \tag{47}$$

and (46) are not first integrals of the system (27), but the ratio of the functions (47) and (46) is a first integral of the system (27) (for $\kappa=-1$) for any $b$.

In general, for dissipative systems, the transcendence of first integrals (in the sense of the presence of essential singularities) follows from the presence of attracting or repulsive limit sets in the system (see [8]).

**6. Conclusion.** Similarly to low-dimensional cases, we indicate the following two important cases for the function $f(\alpha)$, which determines the metric on the sphere:

$$\text{(a)} \quad f(\alpha) = \frac{\cos\alpha}{\sin\alpha}, \qquad \text{(b)} f(\alpha) = \frac{1}{\cos\alpha\sin\alpha}. \tag{48}$$

The case (48)(a) corresponds to the class of systems that describe the motion of a dynamically symmetric five-dimensional rigid body on zero levels of cyclic integral, generally speaking, in a nonconservative force field (see [14–16]). The case (48)(b) corresponds to the class of systems that describe the motion of a particle on the four-dimensional sphere, also, generally speaking, in a nonconservative force field. In particular, for $\delta(\alpha) \equiv F(\alpha) \equiv 0$, the system considered describes a geodesic flow on the four-dimensional sphere. In the case (48), if $\delta(\alpha) = F(\alpha)\cos\alpha$, then the system describes the spatial motion of a five-dimensional rigid body in a force field $F(\alpha)$ under the action of a tracing force (see [2, 3, 9]). In particular, if $F(\alpha) = \sin\alpha\cos\alpha$ and $\delta(\alpha) = \sin\alpha$, then the system also describes a generalized five-dimensional spherical pendulum in a nonconservative force field and possesses a complete set of transcendental first integrals, which can be expressed as finite combinations of elementary functions (see [14–16]).

If the function $\delta(\alpha)$ is not periodic, then the dissipative system considered is a system with variable dissipation with nonzero mean (i.e., it is properly dissipative). However, in this case one can also obtain an explicit form of transcendental first integrals that can be expressed as finite combinations of elementary functions. This fact is a new nontrivial result on the explicit integrability of dissipative systems.

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## REFERENCES

1. O. I. Bogoyavlenskii, "Some integrable cases of Euler equation," *Dokl. Akad. Nauk SSSR*, **287**, No. 5, 1105–1108 (1986).
2. S. A. Chaplygin, "On motion of heavy bodies in an incompressible fluid," in: *Complete Collection of Works* [in Russian], Vol. 1, Izd. Akad. Nauk SSSR, Leningrad (1933), pp. 133–135.
3. S. A. Chaplygin, *Selected Works* [in Russian], Nauka, Moscow (1976).
4. B. A. Dubrovin and S. P. Novikov, "On Poisson brackets of hydrodynamic type," *Dokl. Akad. Nauk SSSR*, **279**, No. 2, 294–297 (1984).
5. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Geometry. Theory and Applications* [in Russian], Nauka, Moscow (1979).
6. E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen.* I. *Gewöhnliche Differentialgleichungen*, Teubner, Stuttgart (1977).
7. S. V. Manakov, "Note on the integration of Euler's equations of the dynamics of an $n$-dimensional rigid body," *Funkts. Anal. Prilozh.*, **10**, No. 4, 93–94 (1976).
8. M. V. Shamolin, "On integrability in transcendental functions," *Usp. Mat. Nauk*, **53**, No. 3, 209–210 (1998).
9. M. V. Shamolin, "Dynamical systems with variable dissipation: Approaches, methods, and applications," *Fundam. Prikl. Mat.*, **14**, No. 3, 3–237 (2008).
10. M. V. Shamolin, "A new case of integrability in dynamics of a four-dimensional rigid body in a nonconservative field," *Dokl. Ross. Akad. Nauk*, **437**, No. 2, 190–193 (2011).
11. M. V. Shamolin, "A new case of integrability in spatial dynamics of a rigid solid interacting with a medium under assumption of linear damping," *Dokl. Ross. Akad. Nauk*, **442**, No. 4, 479–481 (2012).

12. M. V. Shamolin, "Variety of integrable cases in dynamics of low- and multi-dimensional rigid bodies in nonconservative force fields," *J. Math. Sci.*, **204**, No. 4, 379–530 (2015).
13. M. V. Shamolin, "Integrable systems with variable dissipation on the tangent bundle of a multidimensional sphere and their applications," *Fundam. Prikl. Mat.*, **20**, No. 4, 3–231 (2015).
14. M. V. Shamolin, "Multidimensional pendulum in a nonconservative force field," *Dokl. Akad. Nauk SSSR*, **460**, No. 2, 165–169 (2015).
15. M. V. Shamolin, "Complete set of first integrals of dynamical equations of motions of a multidimensional rigid body in a nonconservative field," *Dokl. Ross. Akad. Nauk*, **461**, No. 5, 533–536 (2015).
16. M. V. Shamolin, "Complete list of first integrals of equations of motion of a multidimensional rigid body in a nonconservative field under assumption of linear damping," *Dokl. Ross. Akad. Nauk*, **464**, No. 6, 688–692 (2015).
17. M. V. Shamolin, "New examples of integrable systems with dissipation on the tangent bundles of the two-dimensional and three-dimensional spheres," *Dokl. Ross. Akad. Nauk*, **471**, No. 5, 547–551 (2016).
18. V. V. Trofimov and M. V. Shamolin, "Geometrical and dynamical invariants of integrable Hamiltonian and dissipative systems," *Fundam. Prikl. Mat.*, **16**, No. 4, 3–229 (2010).
19. A. P. Veselov, "On integrability conditions for the Euler equations on $so(4)$," *Dokl. Akad. Nauk SSSR*, **270**, No. 6, 1298–1300 (1983).

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