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FIXED POINTS AND COMPLETENESS IN METRIC AND GENERALIZED METRIC SPACES

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ABSTRACT. The famous Banach contraction principle holds in complete metric spaces, but completeness is not a necessary condition: there are incomplete metric spaces on which every contraction has a fixed point. The aim of this paper is to present various circumstances in which fixed point results imply completeness. For metric spaces, this is the case of Ekeland's variational principle and of its equivalent, Caristi's fixed point theorem. Other fixed point results having this property will also be presented in metric spaces, in quasi-metric spaces, and in partial metric spaces. A discussion on topology and order and on fixed points in ordered structures and their completeness properties is included as well.

All roads lead to Rome.
An old saying

All topologies come from generalized metrics.
Ralph Kopperman
Am. Math. Mon., ***95****, No. 2, 89–97 (1988)*

Introduction

The famous Banach contraction principle holds in complete metric spaces, but completeness is not a necessary condition – there are incomplete metric spaces on which every contraction has a fixed point (see, e.g., [54]). The aim of the present paper is to present various circumstances in which fixed point results imply completeness. For metric spaces this is the case of Ekeland's variational principle (and of its equivalent, Caristi's fixed point theorem) (see, for instance, [30,112,171]) but this is also true in quasi-metric spaces [39,89] and in partial metric spaces [3,151]. Other fixed point results having this property will also be presented. Various order completeness conditions of some ordered structures implied by fixed point properties will be considered as well.

Concerning proofs, in several cases we give proofs, mainly to the converse results, i.e., completeness implied by fixed point results. In Sec. 3, we give full proofs to results relating topology and order as well as in Sec. 5 in what concerns the properties of partial metric spaces.

1. Banach Contraction Principle in Metric Spaces

The Banach contraction principle was proved by S. Banach in his thesis from 1920, published in 1922 [24]. Although the idea of successive approximations in some concrete situations (solving differential and integral equations) appears in some works of P. L. Chebyshev, E. Picard, R. Caccioppoli, *et al.*, it was Banach who placed it in the right abstract setting, making it suitable for a wide range of applications (see the expository paper [99]).

1.1. Contractions and Weakly Contractive Mappings. Let (X,ρ) and (Y,d) be metric spaces. A mapping $f\colon X\to Y$ is called *Lipschitz* if there exists a number $\alpha\ge 0$ such that

$$\forall\, x,y\in X\quad d\big(f(x),f(y)\big)\le \alpha\rho(x,y). \tag{1.1}$$

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The number α is called a *Lipschitz constant* for f, and one says sometimes that the mapping f is α-*Lipschitz*. If $\alpha = 0$, then the mapping f is constant $f(x) = f(x_0)$ for some point $x_0 \in X$. If $\alpha = 1$, i.e.,

$$\forall\, x, y \in X \quad d\big(f(x), f(y)\big) \le \rho(x, y), \tag{1.2}$$

then the mapping f is called *nonexpansive*. If

$$\forall\, x, y \in X \quad d\big(f(x), f(y)\big) = \rho(x, y), \tag{1.3}$$

then f is called an *isometry*.

Suppose now $Y = X$. An α-Lipschitz mapping $f\colon X \to X$ with $0 \le \alpha < 1$ is called a *contraction*. A mapping $f\colon X \to X$ satisfying the relation

$$\forall\, x, y \in X,\ x \ne y, \quad \rho\big(f(x), f(y)\big) < \rho(x, y) \tag{1.4}$$

is called *weakly contractive*.

A point $x_0 \in X$ such that $f(x_0) = x_0$ is called a *fixed point* of the mapping $f\colon X \to X$. The study of the fixed points of mappings is one of the most important branches of mathematics, with numerous applications to the solution of various kinds of equations (differential, integral, partial differential, operator), optimization, game theory, etc.

The following theorem is, perhaps, the most known fixed point result.

Theorem 1.1 (Banach contraction principle)**.** *Any contraction on a complete metric space has a fixed point.*

More precisely, suppose that for some α, $0 \le \alpha < 1$, f is an α-contraction on a complete metric space (X, ρ). Then, for an arbitrary point $x_1 \in X$, the sequence (x_n) defined by the recurrence relation

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N}, \tag{1.5}$$

converges to a fixed point x_0 of the mapping f, and the following estimations hold:

(a) $\forall\, n \in \mathbb{N} \quad \rho(x_n, x_{n+1}) \le \alpha^{n-1} \rho(x_1, x_2)$;

(b) $\forall\, n \in \mathbb{N}\ \forall\, k \in \mathbb{N} \quad \rho(x_n, x_{n+k}) \le \dfrac{1 - \alpha^k}{1 - \alpha} \alpha^{n-1} \rho(x_1, x_2)$;

(c) $\forall\, n \in \mathbb{N} \quad \rho(x_n, x_0) \le \dfrac{\alpha^{n-1}}{1 - \alpha} \rho(x_1, x_2)$.

Under a supplementary condition, weakly contractive mappings also have fixed points.

Theorem 1.2 (M. Edelstein (1962) [50, 51])**.** *Let (X, ρ) be a metric space and $f\colon X \to X$ be a weakly contractive mapping. If there exists $x \in X$ such that the sequence of iterates $\big(f^n(x)\big)$ has a limit point $\xi \in X$, then ξ is the unique fixed point of f.*

Theorem 1.2 has the following important consequence.

Corollary 1.3 (Nemytskiĭ (1936) [131])**.** *If the metric space (X, ρ) is compact, then every weakly contractive mapping $f\colon X \to X$ has a unique fixed point in X.*

Moreover, for any $x_1 \in X$ the sequence defined by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$, converges to the fixed point of the mapping f.

Fixed point results for isometries were proved by Edelstein in [52].

1.2. Converses of Banach's Contraction Principle. Supposing that a function f acting on a metric space (X, ρ) has a unique fixed point, one looks for conditions ensuring the existence of a metric $\bar{\rho}$ on X, topologically equivalent to ρ such that f is a contraction on $(X, \bar{\rho})$. The first result of this kind was obtained by Bessaga [32]. Good presentations of various aspects of fixed points for contraction mappings and their generalizations as well as converse-type results are contained in [73, 99, 108, 138, 155, 157, 160].

We say that a metric d on a set X is *complete* if (X, d) is a complete metric space.

Theorem 1.4 (Bessaga (1959) [32])**.** *Let X be a nonempty set, $f\colon X \to X$, and $\alpha \in (0,1)$.*

(1) *If for every $n \in \mathbb{N}$, f^n has at most one fixed point, then there exists a metric ρ on X such that f is an α-contraction with respect to ρ.*

(2) *If, in addition, some f^n has a fixed point, then there exists a complete metric ρ on X such that f is an α-contraction with respect to ρ.*

A different proof of Theorem 1.4 was given Wong [199], a version of which is included in [45, p. 191-192]. Other proofs as well as some extensions were given by Babu [23], Jachymski [78] (see also [74]), Palczewski and Miczko [140, 141], and Wang *et al.* [194] (cf. the MR review). Angelov [16, 17] proved a converse result in the context of uniform spaces.

In the case of compact metric spaces, Janoš [84] proved the following result.

Theorem 1.5. *Let (X, ρ) be a compact metric space and $f\colon X \to X$ be a continuous mapping such that, for some $\xi \in X$,*

$$\bigcap_{n=1}^{\infty} f^n(X) = \{\xi\}. \tag{1.6}$$

Then for every $\alpha \in (0,1)$, there exists a metric ρ_α on X, topologically equivalent to ρ, such that f is an α-contraction with respect to ρ_α (with ξ as the unique fixed point).

A mapping f satisfying (1.6) is called *squeezing*.

Another proof of Janoš' theorem was given by Edelstein [53].

Kasahara [91] showed that compactness is also necessary for the validity of Janoš' result.

Theorem 1.6. *Let (X, ρ) be a metric space. If for every squeezing mapping $f\colon X \to X$ and every $\alpha \in (0,1)$ there exists a metric ρ_α on X, topologically equivalent to ρ, such that f is an α-contraction with respect to ρ_α, then the space X is compact.*

Janoš extended in [85] this result to uniform spaces (more precisely, to completely regular spaces whose topology is generated by a family of semimetrics); see also [15–18]. Rus [156] extended Janoš' result to weakly Picard mappings. An operator f on a metric space (X, ρ) is called *weakly Picard* if, for every $x \in X$, the sequence $\big(f^n(x)\big)$ of iterates converges to a fixed point of f. Further, if the limit is independent of x (i.e., f has a unique fixed point), then f is called a *Picard operator* (see [158] or [160]).

Other extensions of Janoš' result were given by Leader [109] (see also [110, 111, 128–130]. For a metric space (X, ρ) and $\xi \in X$ consider the following properties:

(i) $f^n(x) \to \xi$ for every $x \in X$;

(ii) the convergence in (i) is uniform on some neighborhood U of ξ.

The condition (ii) means that

$$\forall\, \varepsilon > 0\ \exists\, n_0 = n_0(\varepsilon) \text{ such that } \forall\, n \geq n_0 \quad f^n(U) \subset B[\xi, \varepsilon]. \tag{1.7}$$

To designate the uniform convergence on a subset A of X of the sequence (f^n) to a point ξ, one uses the notation

$$f^n(A) \rightrightarrows \xi.$$

Leader [109] proved the following results.

Theorem 1.7. *Let (X, ρ) be a metric space and $f\colon X \to X$.*

(1) *There exists a metric $\bar\rho$ topologically equivalent to ρ on X such that f is a Banach contraction under $\bar\rho$ with fixed point ξ if and only if f is continuous and both* (i) *and* (ii) *hold.*

(2) *There exists a bounded metric $\bar\rho$ topologically equivalent to ρ on X such that f is a Banach contraction under $\bar\rho$ with fixed point ξ if and only if f is continuous and $f^n(X) \rightrightarrows \xi$.*

(3) *There exists a bounded metric $\bar\rho$ uniformly equivalent to ρ on X such that f is a Banach contraction under $\bar\rho$ if and only if f is uniformly continuous and*

$$\operatorname{diam}_\rho\big(f^n(X)\big) \to 0 \quad \text{as} \quad n \to \infty. \tag{1.8}$$

In the case of an ultrametric space, the situation is simpler. An *ultrametric space* is a metric space (X, ρ) such that ρ satisfies the so called *strong triangle* (or *ultrametric*) inequality

$$\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}, \tag{1.9}$$

for all $x, y, z \in X$.

Below, we present some specific properties of these spaces.

Proposition 1.8. *Let (X, ρ) be an ultrametric space. Then for all $x, y, z \in X$ and $r > 0$,*

(i) $\rho(x, y) \neq \rho(y, z) \implies \rho(x, z) = \max\{\rho(x, y), \rho(y, z)\}$;
(ii) $y \in B[x, r] \implies B[x, r] = B[y, r]$;
(iii) $r_1 \leq r_2$ *and* $B[x, r_1] \cap B[x, r_2] \neq \varnothing \implies B[x, r_1] \subset B[x, r_2]$.

Similar relations hold for the open balls $B(x, r)$.

An ultrametric space (X, ρ) is called *spherically complete* if for every collection $B_i = B[x_i, r_i]$, $i \in I$, of closed ball in X such that $B_i \cap B_j \neq \varnothing$ for all $i, j \in I$, has nonempty intersection, $\bigcap_{i \in I} B_i \neq \varnothing$. It is obvious that a spherically complete ultrametric space is complete. In an arbitrary metric space, this property is called the *binary intersection property* (or the property (2.∞.I.P.)).

Prieß-Crampe [149] proved the following converse to Edelstein's theorem on weakly contractive mappings.

Theorem 1.9. *An ultrametric space (X, ρ) is spherically complete if and only if every weakly contractive mapping on X has a (unique) fixed point.*

Remark 1.10. In fact, Prieß-Crampe [149] proved this result in the more general context of an ultrametric ρ taking values in a totally ordered set Γ having a least element 0 such that $0 < \gamma$ for all $\gamma \in \Gamma$.

Fixed point theorems for weakly contractive and for nonexpansive mappings on spherically complete non-Archimedean normed spaces were proved by Petalas and Vidalis [145].

Concerning contractions, we mention the following result obtained by Hitzler and Seda [66].

Theorem 1.11. *Let (X, τ) be a T_1 topological space and $f \colon X \to X$ be a function on X. The following are equivalent*:

(1) (i) *The mapping f has a unique fixed point $\xi \in X$ and*
 (ii) *for every $x \in X$ the sequence $\big(f^n(x)\big)$ converges to ξ with respect to the topology τ.*
(2) *There exists a complete ultrametric ρ on X such that $\rho\big(f(x), f(y)\big) \leq 2^{-1} d(x, y)$ for all $x, y \in X$.*

For applications of these fixed point results to logic programming, see [67].

Remark 1.12. We are not sure that the metric ρ from (2) generates the topology τ, but for every $x \in X$ the sequence $\big(f^n(x)\big)$ converges to ξ with respect to the topology τ and the metric ρ.

1.3. Neither Completeness Nor Compactness Is Necessary. In this subsection, we shall provide some examples of peculiar topological spaces having the fixed point property (FPP) for various classes of mappings. Let $\mathcal{F}$ be a class of mappings on a set X. One says that X has the fixed point property for the class $\mathcal{F}$ if every mapping $f \in \mathcal{F}$ has a fixed point in X.

Examples 1.13 (Elekes [54]).

(1) The space $X = \big\{\big(x, \sin(1/x)\big) \colon x \in (0, 1]\big\}$ is a nonclosed (hence incomplete) subset of $\mathbb{R}^2$ having the FPP for contractions.
(2) For every $n \in \mathbb{N}$ every open subset of $\mathbb{R}^n$ possessing the Banach fixed point property coincides with $\mathbb{R}^n$, hence is closed.
(3) Every simultaneously F_σ and G_δ subset of $\mathbb{R}$ with the Banach fixed point property is closed.
(4) There exists a nonclosed G_δ set $X \subset \mathbb{R}$ with the Banach fixed point property. Moreover, $X \subset [0, 1]$ and every contraction mapping of X into itself is constant.

(5) There exists a nonclosed F_σ subset of $[0,1]$ with the Banach fixed point property.
(6) There is a bounded Borel (even F_σ) subset of $\mathbb{R}$ with the Banach fixed point property that is not complete with respect to every equivalent metric.
(7) For every integer $n > 0$ there exists a nonmeasurable set in $\mathbb{R}^n$ with the Banach fixed point property.

We give the proof only for (1), following [54]. A proof based on some similar ideas was given by Borwein [34].

Proof of the assertion (1). Let $X = \{(x, \sin(1/x)) : x \in (0,1]\}$ and $f : X \to X$ be a contraction with constant $0 < \alpha < 1$. For $H \subset (0,1]$ put $X_H := \{(x,y) \in X : x \in H\}$.

Let $0 < \varepsilon < 1$ be such that $\alpha\sqrt{\varepsilon^2 + 4} < 2$. Then for all $z = (x,y)$ and $z' = (x', y')$ in X with $0 < x, x' < \varepsilon$,

$$\|f(z) - f(z')\| \le \alpha\sqrt{(x-x')^2 + (y-y')^2} < \alpha\sqrt{\varepsilon^2+4} < 2.$$

Consequently, $X_{(0,\varepsilon)}$ does not contain both a local minimum and a local maximum of the graph. Since $X_{(0,\varepsilon)}$ is connected, it follows that it is contained in at most two consecutive monotone parts of the graph of $\sin(1/x)$. Therefore, there exists $\delta_1 > 0$ such that $f(X_{(0,\varepsilon)}) \subset X_{[\delta_1,1]}$ for some $\delta_1 > 0$. By compactness $f(X_{[\varepsilon,1]}) \subset X_{[\delta_2,1]}$ for some $\delta_2 > 0$.

Taking $\delta = \min\{\delta_1, \delta_2\}$, it follows that $f(X) \subset X_{[\delta,1]}$ and so $f(X_{[\delta,1]}) \subset X_{[\delta,1]}$. Applying the Banach fixed point theorem to $X_{[\delta,1]}$, it follows that f has a fixed point. □

Some examples of spaces having the FPP for continuous mappings were given by Connell [42]. These examples show that, in the author's words: "in the general case, compactness and the FPP are only vaguely related."

We first mention the following result of Klee.

Theorem 1.14 (Klee [102])**.** *A locally connected, locally compact metric space with the FPP for continuous mappings is compact.*

Examples 1.15. (Connell [42])

(1) There exists a Hausdorff topological space X having the FPP for continuous mappings such that the only compact subsets of X are the finite ones.
(2) There exists a metric space X having the FPP for continuous mappings such that X^2 does not have the FPP for continuous mappings.
(3) There exists a separable, locally contractible metric space that has the FPP for continuous mappings, yet it is not compact.
(4) There exists a compact metric space X that does not have the FPP for continuous mappings, yet it contains a dense subset Y that does have the FPP for continuous mappings.

1.4. Completeness and Other Properties Implied by FPP. We shall present some fixed point results that imply the completeness of the underlying space. The papers [30, 112, 171] contain surveys on this topic. A good analysis is given in the Master Thesis of Nicolae [132].

We first mention the following characterization of the field of real numbers among totally ordered fields.

Suppose R is an ordered field. Call a continuous map $f : R \to R$ a contraction if there exists $r < 1$ (in R) such that $|f(x) - f(y)| \le r|x - y|$ for all $x, y \in R$ (where $|x| := \max\{x, -x\}$).

The following result is taken from `http://mathoverflow.net/questions/65874/converse-to-banach-s-fixed-point-theorem-for-ordered-fields`.

Asking a question posed by James Propp, George Lowther proved the following result.

Theorem 1.16. *If R is an ordered field such that every contraction on R has a fixed point, then $R \cong \mathbb{R}$.*

The proof is done in two steps.

(I) One shows first that the order of R is Archimedean.

(II) One proves that every Cauchy sequence is convergent (i.e., the completeness of R).

These are two properties that characterize the field $\mathbb{R}$ among the ordered fields.

The first characterization of completeness in terms of contraction was done by Hu [70].

Theorem 1.17. *A metric space (X,ρ) is complete if and only if, for every nonempty closed subset Y of X, every contraction on Y has a fixed point in Y.*

Proof. The idea of the proof is simple. One takes a Cauchy sequence (x_n) in X. If it has a convergent subsequence, then it converges. Supposing that this is not the case, then

$$\beta(x_n) := \inf\{\rho(x_n, x_m) \colon m > n\} > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

For a given α with $0 < \alpha < 1$, one constructs inductively a subsequence (x_{n_k}) such that $\rho(x_i, x_j) \leq \alpha\beta(x_{n_{k-1}})$ for all $i, j \geq n_k$. Then $Y = \{x_{n_k} \colon k \in \mathbb{N}\}$ is a closed subset of X and the function $f(x_{n_k}) = x_{n_{k+1}}$, $k \in \mathbb{N}$, is an α-contraction on Y without fixed points. □

Subrahmanyam [169] proved the following completeness result.

Theorem 1.18. *A metric space (X,ρ) in which every mapping $f \colon X \to X$ satisfying the conditions*

(i) *there exists $\alpha > 0$ such that $\rho\big(f(x), f(y)\big) \leq \alpha \max\big\{\rho\big(x, f(x)\big), \rho\big(y, f(y)\big)\big\}$ for all $x, y \in X$,*

(ii) *$f(X)$ is countable,*

has a fixed point, is complete.

The condition (i) in this theorem is related to the Kannan and Chatterjea conditions: there exists $\alpha \in (0, 1/2)$ such that for all $x, y \in X$,

$$\rho\big(f(x), f(y)\big) \leq \alpha\big[\rho\big(x, f(x)\big) + \rho\big(y, f(y)\big)\big], \tag{K}$$

respectively,

$$\rho\big(f(x), f(y)\big) \leq \alpha\big[\rho\big(x, f(y)\big) + \rho\big(y, f(x)\big)\big]. \tag{Ch}$$

Kannan and Chatterjea proved that any mapping f on a complete metric space satisfying (K) or (Ch) has a fixed point (see, for instance, [160]). As is remarked in [169], Theorem 1.18 provides completeness of metric spaces on which every Kannan, or every Chatterjea map, has a fixed point.

Another case where the fixed point property for contractions implies completeness was discovered by Borwein [34].

A metric space (X,ρ) is called *uniformly Lipschitz connected* if there exists $L \geq 0$ such that for any pair x_0, x_1 of points in X there exists a mapping $g \colon [0,1] \to X$ such that $g(0) = x_0$, $g(1) = x_1$, and

$$\rho\big(g(s), g(t)\big) \leq L|s-t|\rho\big(g(0), g(1)\big), \tag{1.10}$$

for all $s, t \in [0,1]$.

Obviously, a convex subset C of a normed space X is uniformly Lipschitz connected, the mapping g connecting $x_0, x_1 \in C$ being given by $g(t) = (1-t)x_0 + tx_1$, $t \in [0,1]$. In this case,

$$\|g(s) - g(t)\| = |s-t|\,\|x_1 - x_0\|,$$

for all $s, t \in [0,1]$.

From the following theorem it follows that a convex subset C of a normed space X is complete if and only if any contraction on C has a fixed point. In particular, this holds for the normed space X.

Theorem 1.19. *Let C be a uniformly Lipschitz connected subset of a complete metric space (X,ρ). Then the following conditions are equivalent.*

(1) *The set C is closed.*

(2) *Every contraction on C has a fixed point.*

(3) *Any contraction on X that leaves C invariant has a fixed point in C.*

Proof. The implication (1) $\Longrightarrow$ (2) is the Banach fixed point theorem, and (2) $\Longrightarrow$ (3) is obvious.

It remains to prove (3) $\Longrightarrow$ (1). Supposing that C is not closed, there exists a point $\bar{x} \in \bar{C} \setminus C$. Let $(x_k)_{k\in\mathbb{N}_0}$ be a sequence of pairwise distinct points in C such that

$$\rho(x_k, \bar{x}) \leq \min\left\{\frac{1}{2^{k+4}}, \frac{L}{2^{k+4}}\right\} \tag{1.11}$$

for $k = 0, 1, \dots$, where $L > 0$ is the constant given by the uniform Lipschitz connectedness of C. It follows that

$$\rho(x_k, x_{k+1}) \leq \min\left\{\frac{1}{2^{k+3}}, \frac{L}{2^{k+3}}\right\} \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{1.12}$$

Let $g_k\colon [0,1] \to C$ be such that $g_k(0) = x_k$, $g_k(1) = x_{k+1}$, and

$$\rho\big(g_k(s), g_k(t)\big) \leq L|s-t|\rho(x_k, x_{k+1}) \tag{1.13}$$

for all $s, t \in [0,1]$. Define $g\colon (0,\infty) \to C$ by

$$g(t) = \begin{cases} x_0 & \text{for } 1 < t < \infty, \\ g_k(2^{k+1}t - 1) & \text{for } 1/2^{k+1} < t \leq 1/2^k. \end{cases} \tag{1.14}$$

It follows that $g(2^{-k}) = g_k(1) = x_{k+1}$.

Let $\Delta_k = (2^{-(k+1)}, 2^{-k}]$. Then for $s, t \in \Delta_k$, taking into account (1.13) and (1.12), one obtains

$$\rho\big(g(s), g(t)\big) \leq L \cdot 2^{k+1}|s-t|\rho(x_k, x_{k+1}) \leq L \cdot 2^{k+1} \cdot |s-t| \cdot \frac{1}{2^{k+3}} = \frac{L}{4} \cdot |s-t| \leq L \cdot |s-t|, \quad s, t \in \Delta_k.$$

Since $|s-t| < 1/2^{k+1}$, it follows also that

$$\rho\big(g(s), g(t)\big) \leq L \cdot 2^{k+1} \cdot \frac{1}{2^{k+1}} \cdot \frac{1}{2^{k+3}} = \frac{L}{2^{k+3}} \quad \text{for all} \quad s, t \in \Delta_k.$$

If $s \in \Delta_k$ and $t \in \Delta_p$ with $k \leq p$, then the above inequality and (1.11) yield

$$\begin{aligned} &\rho\big(g(s), g(2^{-k})\big) \leq \frac{L}{2^{k+3}}, \\ &\rho(x_{k+1}, x_{p+1}) \leq \rho(x_{k+1}, \bar{x}) + \rho(\bar{x}, x_{p+1}) \leq L\left(\frac{1}{2^{k+5}} + \frac{1}{2^{p+5}}\right), \\ &\rho\big(g(2^{-p}), g(t)\big) \leq \frac{L}{2^{p+3}}, \end{aligned}$$

so that

$$\rho\big(g(s), g(t)\big) \leq \rho\big(g(s), g(2^{-k})\big) + \rho(x_{k+1}, x_{p+1}) + \rho\big(g(2^{-p}), g(t)\big) \leq L \cdot \left(\frac{1}{2^{k+3}} + \frac{1}{2^{k+5}} + \frac{1}{2^{p+5}} + \frac{1}{2^{p+3}}\right).$$

Observe that

$$s - t > \frac{1}{2^k} - \frac{1}{2^{p+1}},$$

and so if we show that

$$\frac{1}{2^{k+3}} + \frac{1}{2^{k+5}} + \frac{1}{2^{p+5}} + \frac{1}{2^{p+3}} \leq \frac{1}{2^k} - \frac{1}{2^{p+1}}, \tag{1.15}$$

then

$$\rho\big(g(s), g(t)\big) \leq L|s-t|. \tag{1.16}$$

Since all the fractions with p at the denominator are less than or equal to the corresponding ones with k at the denominator, it follows that

$$\frac{1}{2^{k+3}} + \frac{1}{2^{k+5}} + \frac{1}{2^{p+5}} + \frac{1}{2^{p+3}} + \frac{1}{2^{p+1}} \leq \frac{1}{2^{k+2}} + \frac{1}{2^{k+4}} + \frac{1}{2^{k+1}} = \frac{13}{2^{k+4}} < \frac{1}{2^k},$$

so that (1.15) holds.

Now put $g(0) = \bar{x}$. If $t \in \Delta_k$, then

$$\rho\big(g(0), g(t)\big) \le \rho(\bar{x}, x_{k+1}) + \rho\big(x_{k+1}, g(t)\big) \le L\left(\frac{1}{2^{k+5}} + \frac{1}{2^{k+3}}\right) < L \cdot \frac{1}{2^{k+1}} < L \cdot t,$$

showing that g satisfies (1.16) for all $s, t \in [0, \infty)$. Let $h\colon X \to [0, \infty)$ and $f\colon X \to X$ be defined for $x \in X$ by

$$h(x) := (2L)^{-1}\rho(x, \bar{x}) \quad \text{and} \quad f(x) := (g \circ h)(x),$$

respectively. Then, for all $x, x' \in X$,

$$\rho\big(f(x), f(x')\big) = \rho\left(g\left(\frac{1}{2L}\rho(x, \bar{x})\right), g\left(\frac{1}{2L}\rho(x', \bar{x})\right)\right) \le L \cdot \frac{1}{2L}\left|\rho(x, \bar{x}) - \rho(x', \bar{x})\right| \le \frac{1}{2} \cdot \rho(x, x'),$$

i.e., f is a $(1/2)$-contraction on X. Since

$$f(C) = g\big(h(C)\big) \subset g\big((0, \infty)\big) \subset C,$$

it follows that C is invariant for f. Since

$$\bar{x} = g(0) = g\big(h(\bar{x})\big) = f(\bar{x}),$$

it follows that the only fixed point of f is $\bar{x}$, which does not belong to C, in contradiction to the hypothesis. □

We mention the following consequences.

Corollary 1.20.

(1) *A uniformly Lipschitz connected metric space (X, ρ) is complete if and only if it has the fixed point property for contractions.*

(2) *A convex subset C of a normed space X is complete if and only if any contraction on C has a fixed point. In particular, this holds for the normed space X.*

Proof. For (1) consider X as a uniformly Lipschitz connected subset of its completion $\tilde{X}$. The results in (2) were discussed before Theorem 1.19. □

Example 1.21 (Borwein [34])**.** There is a starshaped nonclosed subset of $\mathbb{R}^2$ having the fixed point property for contractions, but not for continuous functions.

One takes

$$L_k = \operatorname{co}\left(\left\{(0,0), \left(1, \frac{1}{2^k}\right)\right\}\right), \quad k \in \mathbb{N},$$

and

$$C = \bigcup\{L_k\colon k \in \mathbb{N}\}.$$

Then C is starshaped with respect to $(0, 0)$ and nonclosed, because $\operatorname{co}(\{(0, 0), (1, 0)\}) \subset \overline{C} \setminus C$. One shows that C has the required properties (see [34] for details).

Xiang [200] completed and extended Borwein's results results. Let (X, ρ) be a metric space. By an arc we mean a continuous function $g\colon \Delta \to X$, where Δ is an interval in $\mathbb{R}$. An arc $g\colon (0, 1] \to X$ is called *semi-closed* if

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \rho\big(g(s), g(t)\big) < \varepsilon \text{ for all } s, t \in (0, \delta). \tag{1.17}$$

The arc g is called *Lipschitz semi-closed* if the mapping g is Lipschitz and satisfies (1.17).

The metric space (X, ρ) is called *arcwise complete* if for every semi-closed arc $g\colon (0, 1] \to X$ there exists the limit $\lim_{t \searrow 0} g(t)$. If this holds for every Lipschitz semi-closed arc $g\colon (0, 1] \to X$, then X is called *Lipschitz complete*.

Some examples, [200, Examples 1.1, 1.2, and 2.3], show that the arcwise completeness is weaker than the usual completeness even in an arcwise connected space, and so is Lipschitz completeness. It is obvious from the definitions that Lipschitz completeness is weaker than arcwise completeness.

A metric space (X, ρ) is called *locally arcwise connected* (*locally Lipschitz connected*) if there exists $\delta > 0$ such that any pair x_0, x_1 of points in X with $\rho(x_0, x_1) \le \delta$ can be linked by an arc (respectively, by a Lipschitz arc).

Theorem 1.22 ([200, Theorems 3.1 and 3.2]). *Let (X, ρ) be a metric space.*

(1) *If (X, ρ) has the fixed point property for contractions, then X is Lipschitz complete.*
(2) *If (X, ρ) is locally Lipschitz connected, then X has the fixed point property for contractions if and only if it is Lipschitz complete.*

One says that the metric space (X, ρ) has the *strong contraction property* if every mapping $f\colon X \to X$ that is a contraction with respect to some metric $\bar{\rho}$ on X, uniformly equivalent to ρ, has a fixed point.

Theorem 1.23 ([200, Theorems 4.1 and 4.4]). *Let (X, ρ) be a metric space.*

(1) *If (X, ρ) has the strong contraction property, then X is arcwise complete.*
(2) *If (X, ρ) is locally arcwise connected, then X has the strong contraction property if and only if it is arcwise complete.*

Suzuki [177] found an extension of Banach contraction principle that implies completeness. He considered the function $\theta\colon [0,1) \to (1/2, 1]$

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1 \end{cases} \tag{1.18}$$

and proved the following fixed point result.

Theorem 1.24. *Let (X, ρ) be a complete metric space and $f\colon X \to X$.*

(1) *If there exists $r \in [0,1)$ such that*
$$\theta(r)\rho\big(x, f(x)\big) \le d(x,y) \implies \rho\big(f(x), f(y)\big) \le r\rho(x,y), \tag{1.19}$$
for all $x, y \in X$, then f has a fixed point $\bar{x}$ in X and $\lim_n f^n(x) = \bar{x}$ for every point $x \in X$.
(2) *Moreover, $\theta(r)$ is the best constant in (1.19) for which the result holds, in the sense that for every $r \in [0,1)$, there exist a complete metric space (X, ρ) and a function $f\colon X \to X$ without fixed points and such that*
$$\theta(r)\rho\big(x, f(x)\big) < \rho(x,y) \implies \rho\big(f(x), f(y)\big) \le r\rho(x,y), \tag{1.20}$$
for all $x, y \in X$.

Extensions of the Suzuki fixed point theorem to partial metric spaces and to partially ordered metric spaces were given by Paesano and Vetro [139].

The converse result is the following one.

Theorem 1.25 ([177, Corollary 1]). *For a metric space (X, ρ), the following are equivalent.*

(1) *The space (X, ρ) is complete.*
(2) *There exists $r \in (0,1)$ such that every mapping $f\colon X \to X$ satisfying*
$$\frac{1}{10000}\,\rho\big(x, f(x)\big) \le \rho(x,y) \implies \rho\big(f(x), f(y)\big) \le r\rho(x,y), \tag{1.21}$$
for all $x, y \in X$, has a fixed point.

It is clear that the function $\theta(r)$ given by (1.18) satisfies the equality $\lim_{r \nearrow 1} \theta(r) = 1/2$. The critical case of functions acting on a subset X of a Banach space E satisfying the condition

$$\frac{1}{2}\|x - f(x)\| \le \|x - y\| \implies \|f(x) - f(y)\| \le \|x - y\|, \tag{1.22}$$

for all $x, y \in X$ was examined by Suzuki [176]. Condition (1.22) was called condition (C) and the functions satisfying this condition are called *generalized nonexpansive*. It is clear that every nonexpansive mapping satisfies (1.22), but there are discontinuous functions satisfying (1.22), so that the class of generalized nonexpansive mappings is strictly larger than that of nonexpansive ones. The term generalized nonexpansive is justified by the fact that the generalized nonexpansive mappings share with nonexpansive mappings several properties concerning fixed points; in some Banach spaces E they have fixed points on every weakly compact convex subset of E, and for every closed bounded convex subset X of E and every generalized nonexpansive mapping f on X there exists an almost fixed point sequence, i.e., a sequence (x_n) in X such that $\|x_n - f(x_n)\| \to 0$ as $n \to \infty$ (see [176]). Also a generalized nonexpansive mapping f is quasi-nonexpansive, in the sense that $\|f(x) - y\| \le \|x - y\|$ for all $x \in X$ and $y \in \operatorname{Fix}(f)$ (the set of fixed points of f). It is known that every nonexpansive mapping having a fixed point is quasi-nonexpansive (for fixed points of nonexpansive mappings and other fixed point results see [60, 101]).

For further results and extensions, see [47–49, 56, 117, 118].

Amato [11–13] proposed another approach to study the connections between fixed points and completeness in metric spaces. For a metric space (E, d) he considers a pair (Y, Ψ), where Y is a subset of X and Ψ is a class of mappings on Y. The pair (Y, Ψ) is said to be a completion class for E if Ψ/ρ is a completion of (E, d), where ρ is a semimetric on Ψ (defined in a concrete manner) and Ψ/ρ is the quotient space with respect to the equivalence relation

$$f \equiv g \iff \rho(f, g) = 0.$$

Among other results, he proves that if E is an infinite dimensional normed space and K is a compact subset of E, then it is possible to take $Y = E \setminus K$ and Ψ the class of all compact contractions of Y.

We mention also the following characterization of completeness in terms of fixed points of set-valued mappings. For a metric space (X, ρ) denote by $\mathcal{P}_{\mathrm{cl}}(X)$ the family of all nonempty closed subsets of X.

For a mapping $F\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ consider the following two properties:

(J1) $F\big(F(x)\big) \subset F(x)$ for every $x \in X$;
(J2) $\forall\, x \in X\ \forall\, \varepsilon > 0\ \exists y \in F(x)$ with $\operatorname{diam} F(y) < \varepsilon$.

For $F\colon X \to 2^X$, a point $\bar{x} \in X$ is called

- a *fixed point* of F if $\bar{x} \in F(\bar{x})$;
- a *stationary* point of F if $F(\bar{x}) = \{\bar{x}\}$.

Theorem 1.26 ([82, Corollary 1])**.** *For any metric space (X, ρ), the following conditions are equivalent.*

(1) *The space (X, ρ) is complete.*
(2) *Every set-valued mapping $F\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ satisfying* (J1) *and* (J2) *has a fixed point.*
(3) *Every set-valued mapping $F\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ satisfying* (J1) *and* (J2) *has a stationary point.*

Characterizations of the completeness of a metric space in terms of the existence of fixed points for various classes of set-valued mappings acting on them were done by Jiang [86] and Liu [116].

We present the results from Jiang [86]. Let (X, ρ) be a metric space. For a bounded subset Y of X denote by $\alpha(Y)$ the Kuratowski measure of noncompactness of the set Y defined by

$$\alpha(Y) := \inf\{\varepsilon > 0\colon Y \text{ can be covered by the union of a finite family of subsets of } X \text{, each of diameter } \le \varepsilon\}. \quad (1.23)$$

For a set-valued mapping $F\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ one considers the following conditions:

(a) $F\big(F(x)\big) \subset F(x)$ for every $x \in X$;
(b) there exists a sequence (x_n) in X such that $x_{n+1} \in F(x_n)$ for all $n \in \mathbb{N}$ and $\lim_n \operatorname{diam}\big(F(x_n)\big) = 0$;
(c) there exists a sequence (x_n) in X such that $x_{n+1} \in F(x_n)$ for all $n \in \mathbb{N}$ and $\lim_n \alpha\big(F(x_n)\big) = 0$;
(d) $\lim \rho(x_n, x_{n+1}) = 0$ for each sequence (x_n) in X such that $x_{n+1} \in F(x_n)$ for all $n \in \mathbb{N}$.

Remark 1.27. Condition (a) is identical to (J1) and it is easy to check that (J2) implies (b). Condition (d) is condition (iv) from Theorem 2.12.

One considers also the following classes of set-valued mappings $F\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$:

$$\begin{aligned} \mathrm{AB}(X) &:= \{F\colon F \text{ satisfies (a) and (b)}\};\\ \mathrm{AC}(X) &:= \{F\colon F \text{ satisfies (a) and (c)}\};\\ \mathrm{AD}(X) &:= \{F\colon F \text{ satisfies (a) and (d)}\}. \end{aligned}$$

Theorem 1.28 (Jiang [86, Theorems 3.1 and 3.2])**.** *For any metric space (X, ρ), the following conditions are equivalent.*

(1) *The metric space (X, ρ) is complete.*
(2) *Every F in $\mathrm{AB}(X)$ has a fixed point.*
(3) *Every F in $\mathrm{AC}(X)$ has a fixed point.*
(4) *Every F in $\mathrm{AD}(X)$ has a fixed point.*
(5) *Every F in $\mathrm{AB}(X)$ has a stationary point.*
(6) *Every F in $\mathrm{AD}(X)$ has a stationary point.*

2. Ekeland's Variational Principle and Completeness

This section is concerned with Ekeland's variational principle in metric and in quasi-metric spaces and its relations to the completeness of these spaces.

2.1. The Case of Metric Spaces.

The general form of Ekeland's variational principle is the following.

Theorem 2.1 (Ekeland's variational principle)**.** *Let (X, ρ) be a complete metric space and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (lsc) proper function bounded from below. Let $\varepsilon > 0$ and $x_\varepsilon \in X$ be such that*

$$f(x_\varepsilon) \le \inf f(X) + \varepsilon. \tag{2.1}$$

Then given $\lambda > 0$ there exists $z = z_{\varepsilon,\lambda} \in X$ such that

$$\begin{aligned} &\text{(a)} \quad f(z) + \frac{\varepsilon}{\lambda}\rho(z, x_\varepsilon) \le f(x_\varepsilon);\\ &\text{(b)} \quad \rho(z, x_\varepsilon) \le \lambda;\\ &\text{(c)} \quad f(z) < f(x) + \frac{\varepsilon}{\lambda}\rho(z, x) \ \ \textit{for all} \ \ x \in X,\ x \ne z. \end{aligned}$$

An important consequence is obtained by taking $\lambda = \sqrt{\varepsilon}$ in Theorem 2.1.

Corollary 2.2. *Under the hypotheses of Theorem 2.1, for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*

$$\begin{aligned} &\text{(a)} \quad f(y_\varepsilon) + \sqrt{\varepsilon}\rho(y_\varepsilon, x_\varepsilon) \le f(x_\varepsilon);\\ &\text{(b)} \quad \rho(y_\varepsilon, x_\varepsilon) \le \sqrt{\varepsilon};\\ &\text{(c)} \quad f(y_\varepsilon) < f(x) + \sqrt{\varepsilon}\rho(y_\varepsilon, x) \ \ \textit{for all} \ \ x \in X,\ x \ne y_\varepsilon. \end{aligned}$$

Taking $\lambda = 1$ in Theorem 2.1, one obtains the following form of the Ekeland's variational principle, known as the *weak form of the Ekeland's variational principle.*

Corollary 2.3 (Ekeland's variational principle — weak form)**.** *Let (X, ρ) be a complete metric space and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ an lsc proper function bounded from below. Then for every $\varepsilon > 0$ there exists an element $y_\varepsilon \in X$ such that*

$$f(y_\varepsilon) \le \inf f(X) + \varepsilon \tag{2.2}$$

and

$$f(y_\varepsilon) < f(y) + \varepsilon\rho(y, y_\varepsilon) \ \ \textit{for all} \ \ y \in X \setminus \{y_\varepsilon\}. \tag{2.3}$$

Note that the validity of Ekeland's variational principle (in its weak form) implies the completeness of the metric space X. This was discovered by Weston [196] in 1977 and rediscovered by Sullivan [170] in 1981 (see also the survey [171]). More precisely, the following result holds.

Proposition 2.4. *Let (X, ρ) be a metric space. Then X is complete if and only if for every lsc proper function $f\colon X \to \mathbb{R} \cup \{\infty\}$ bounded from below and $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that the conclusions* (2.2) *and* (2.3) *of Corollary* 2.3 *hold.*

Proof. If X is complete, one appeals to Corollary 2.3 to conclude.

The proof of the converse is simple. For a Cauchy sequence (x_n) in X, the inequality

$$|\rho(x, x_n) - \rho(x, x_{n+k})| \leq \rho(x_n, x_{n+k}),$$

shows that $\big(\rho(x, x_n)\big)$ is a Cauchy sequence in $\mathbb{R}$ for every $x \in X$.

Consequently, the function $f\colon X \to [0, \infty)$ given by $f(x) = \lim\limits_{n\to\infty} \rho(x_n, x)$, $x \in X$, is well defined. The inequalities $|\rho(x_n, x) - \rho(x_n, x')| \leq \rho(x, x')$, $n \in \mathbb{N}$, yield for $n \to \infty$, $|f(x) - f(x')| \leq \rho(x, x')$, showing that f is continuous. For every $\varepsilon > 0$ there exists n_0 such that $\rho(x_n, x_{n+k}) < \varepsilon$, for all $n \geq n_0$ and $k \in \mathbb{N}$. Letting $k \to \infty$, one obtains that $f(x_n) \leq \varepsilon$ for all $n \geq n_0$. Consequently, $\lim\limits_{n\to\infty} f(x_n) = 0$, implying $\inf f(X) = 0$. Let $0 < \varepsilon < 1$. By the hypothesis, there exists $y \in X$ such that

$$f(y) \leq f(x) + \varepsilon\rho(x, y), \tag{2.4}$$

for every $x \in X$. Putting $x = x_n$ in (2.4) and letting $n \to \infty$, one obtains $f(y) \leq \varepsilon f(y)$, implying $f(y) = 0$, which is equivalent to $\lim\limits_{n\to\infty} \rho(x_n, y) = 0$, i.e., (x_n) converges to y. ☐

Remark 2.5. The proof of Proposition 2.4 shows that it is sufficient to suppose that the conclusions of the weak form of Ekeland's variational principle hold only for Lipschitz (even nonexpansive) functions $f\colon X \to \mathbb{R}$.

Ekeland's variational principle is equivalent to many important fixed point and geometric results (the drop property, Caristi's fixed point theorem, the flower petal theorem, etc., see [144]). We mention here only Caristi's fixed point theorems — for both single-valued and set-valued mappings.

Theorem 2.6 (Caristi–Kirk fixed point theorem)**.** *Let (X, ρ) be a complete metric space and $\varphi\colon X \to \mathbb{R}$ an lsc function bounded from below. If the mapping $f\colon X \to X$ satisfies the condition*

$$\rho\big(x, f(x)\big) \leq \varphi(x) - \varphi\big(f(x)\big), \quad x \in X, \tag{2.5}$$

then f has a fixed point in X.

Another consequence of Ekeland's variational principle is a set-valued version of Caristi's fixed point theorem.

Theorem 2.7. *Let (X, ρ) be a complete metric space, $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be an lsc function bounded from below, and $F\colon X \rightrightarrows X$ be a set-valued mapping. If the mapping F satisfies the condition*

$$\forall\, x \in X\ \forall\, y \in F(x) \quad \rho(x, y) \leq \varphi(x) - \varphi(y) \tag{2.6}$$

then F has a fixed point, i.e., there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.

It follows that the validity of Caristi's fixed point theorem also implies the completeness of the underlying metric space.

Corollary 2.8. *Let (X, ρ) be a complete metric space. If every function $f\colon X \to X$ satisfying the hypotheses of Caristi's fixed point theorem for some lsc function $\varphi\colon X \to \mathbb{R}$ has a fixed point in X, then the metric space X is complete.*

Remark 2.9. Replacing in both Theorems 2.6 and 2.7 and in Corollary 2.8 the function φ by $\varphi - \inf \varphi(X)$, one can consider, without restricting the generality, that the function φ is lsc and takes values in $\mathbb{R}_+$.

Remark 2.10. Suzuki [181] proved that some forms of the strong Ekeland's variational principle, as proved by Georgiev [57], in a Banach space X, imply the reflexivity of X. In the case of a metric space X, one obtains the compactness of every bounded closed subset of X (such a metric space X is called *boundedly compact*).

A characterization of completeness of a metric space in terms of the existence of weak sharp minima of proper lsc functions bounded from below defined on it was done by Huang [71].

2.2. Other Principles. In this section, we shall present some results equivalent to Ekeland's variational principle. The first one was proved by Takahashi [184] (see also [87] and [186, Theorem 2.1.1]).

Theorem 2.11 (Takahashi principle)**.** *Let (X, ρ) be a complete metric space and $f\colon X \to \mathbb{R} \cup \{\infty\}$ be a proper lsc function bounded from below. If for every $x \in X$ with $\inf f(X) < f(x)$ there exists $y_x \in X \setminus \{x\}$ such that*

$$f(y_x) + \rho(x, y_x) \le f(x), \tag{2.7}$$

then there exists $x_0 \in X$ such that $f(x_0) = \inf f(X)$.

Another result, also equivalent to Ekeland's variational principle, was proved by Dancs, Hegedűs, and Medvegyev [43].

Theorem 2.12. *Let (X, ρ) be a complete metric space and $F\colon X \rightrightarrows X$ be a set-valued function satisfying the following conditions*:

(i) *$F(x)$ is closed for every $x \in X$;*
(ii) *$x \in F(x)$ for every $x \in X$;*
(iii) *$x_2 \in F(x_1) \implies F(x_2) \subset F(x_1)$ for all $x_1, x_2 \in X$;*
(iv) *$\lim_n \rho(x_n, x_{n+1}) = 0$ for every sequence (x_n) in X such that $x_{n+1} \in F(x_n)$ for all $n \in \mathbb{N}$.*

Then there exists $x_0 \in X$ such that $F(x_0) = \{x_0\}$. Moreover, for every $\bar{x} \in X$, there exists such a point in $F(\bar{x})$.

This result admits an equivalent formulation in terms of an order on X.

Theorem 2.13. *Let (X, ρ) be a complete metric space and $\preceq$ a continuous partial ordering on X. If $\lim_n \rho(x_n, x_{n+1}) = 0$ for every increasing sequence $x_1 \preceq x_2 \preceq \dots$ in X, then there is a maximal element in X. In fact, for every $\bar{x} \in X$ there exists a maximal element in the set $\{x \in X : \bar{x} \preceq x\}$.*

Remark 2.14. If $F\colon X \rightrightarrows X$ is a set-valued mapping, then for every $x_0 \in X$, a sequence (x_n) satisfying $x_n \in F(x_{n-1})$, $n \in \mathbb{N}$, is called a generalized Picard sequence. For the properties of set-valued Picard operators, defined in terms of the convergence of generalized Picard sequences, see the surveys [146, 147].

An order $\preceq$ on a metric space is said to be *closed* if $x_n \preceq y_n$, for all $n \in \mathbb{N}$, implies $\lim_n x_n \preceq \lim_n y_n$, provided both limits exist. This is equivalent to the fact that the graph of $\preceq$, $\operatorname{Graph}(\preceq) := \{(x, y) \in X \times X : x \preceq y\}$ is closed in $X \times X$ with respect to the product topology.

Remark 2.15. As each of these results in a metric space (X, ρ) is equivalent to Ekeland's variational principle, it follows that the validity of each of them implies the completeness of the underlying metric space (X, ρ). In fact, the converse completeness property is mentioned in [43, Theorem 3.3].

2.3. Ekeland's Variational Principle in Quasi-Metric Spaces. This subsection is concerned with Ekeland's variational principle and Caristi's fixed point theorem in the context of quasi-metric spaces.

Quasi-metric spaces. We shall briefly present the fundamental properties of quasi-metric spaces. Details and references can be found in [41].

Definition 2.16. A *quasi-semimetric* on an arbitrary set X is a mapping $\rho\colon X\times X\to[0;\infty)$ satisfying the following conditions:

$$\text{(QM1)}\quad \rho(x,y)\ge 0,\quad \rho(x,x)=0,$$
$$\text{(QM2)}\quad \rho(x,z)\le\rho(x,y)+\rho(y,z),$$

for all $x,y,z\in X$. If, further,

$$\text{(QM3)}\quad \rho(x,y)=\rho(y,x)=0 \implies x=y,$$

for all $x,y\in X$, then ρ is called a *quasi-metric*. The pair (X,ρ) is called a *quasi-semimetric space*, respectively, a *quasi-metric space*. The conjugate of the quasi-semimetric ρ is the quasi-semimetric $\bar\rho(x,y)=\rho(y,x)$, $x,y\in X$. The mapping $\rho^s(x,y)=\max\{\rho(x,y),\bar\rho(x,y)\}$, $x,y\in X$, is a semimetric on X that is a metric if and only if ρ is a quasi-metric.

If (X,ρ) is a quasi-semimetric space, then for $x\in X$ and $r>0$ we define the balls in X by the formulas

$$B_\rho(x,r)=\{y\in X\colon \rho(x,y)<r\}\quad\text{(the open ball)},$$
$$B_\rho[x,r]=\{y\in X\colon \rho(x,y)\le r\}\quad\text{(the closed ball)}.$$

The topology τ_ρ of a quasi-semimetric space (X,ρ) can be defined starting from the family $\mathcal{V}_\rho(x)$ of neighborhoods of an arbitrary point $x\in X$:

$$V\in\mathcal{V}_\rho(x)\iff \exists\, r>0 \text{ such that } B_\rho(x,r)\subset V$$
$$\iff \exists\, r'>0 \text{ such that } B_\rho[x,r']\subset V.$$

The convergence of a sequence (x_n) to x with respect to τ_ρ, called ρ-convergence and denoted by $x_n\xrightarrow{\rho}x$, can be characterized in the following way:

$$x_n\xrightarrow{\rho}x\iff\rho(x,x_n)\to 0. \tag{2.8}$$

Also

$$x_n\xrightarrow{\bar\rho}x\iff\bar\rho(x,x_n)\to 0\iff\rho(x_n,x)\to 0. \tag{2.9}$$

As a space equipped with two topologies, τ_ρ and $\tau_{\bar\rho}$, a quasi-metric space can be viewed as a bitopological space in the sense of Kelly [95]. The problem of quasi-metrizability of topologies is discussed in [107].

An important example of quasi-metric space is the following.

Example 2.17. On $X=\mathbb{R}$ let $q(x,y)=(y-x)^+$, where α^+ stands for the positive part of a real number α. Then $\bar q(x,y)=(x=y)^+$ and $q^s(x,y)=|y-x|$. The balls are given by

$$B_q(x,r)=(-\infty,x+r)\quad\text{and}\quad B_{\bar q}(x,r)=(x-r,\infty).$$

The following topological properties are true for quasi-semimetric spaces.

Proposition 2.18 (see [41])**.** *If (X,ρ) is a quasi-semimetric space, then*

(1) *the ball $B_\rho(x,r)$ is τ_ρ-open and the ball $B_\rho[x,r]$ is $\tau_{\bar\rho}$-closed. The ball $B_\rho[x,r]$ need not be τ_ρ-closed;*
(2) *if ρ is a quasi-metric, then the topology τ_ρ is T_0, but not necessarily T_1 (and so nor T_2 as in the case of metric spaces). The topology τ_ρ is T_1 if and only if $\rho(x,y)>0$ whenever $x\ne y$;*
(3) *for every fixed $x\in X$, the mapping $\rho(x,\cdot)\colon X\to(\mathbb{R},|\cdot|)$ is τ_ρ-usc and $\tau_{\bar\rho}$-lsc. For every fixed $y\in X$, the mapping $\rho(\cdot,y)\colon X\to(\mathbb{R},|\cdot|)$ is τ_ρ-lsc and $\tau_{\bar\rho}$-usc;*
(4) *if the mapping $\rho(x,\cdot)\colon X\to(\mathbb{R},|\cdot|)$ is τ_ρ-continuous for every $x\in X$, then the topology τ_ρ is regular. If $\rho(x,\cdot)\colon X\to(\mathbb{R},|\cdot|)$ is $\tau_{\bar\rho}$-continuous for every $x\in X$, then the topology $\tau_{\bar\rho}$ is semi-metrizable.*

Completeness in quasi-metric spaces. The lack of symmetry in the definition of quasi-metric and quasi-uniform spaces causes a lot of trouble, mainly concerning completeness, compactness, and total boundedness in such spaces. There are a lot of completeness notions in quasi-metric and in quasi-uniform spaces, all agreeing with the usual notion of completeness in the case of metric or uniform spaces, each of them having its advantages and weaknesses.

As in what follows we shall work only with one of these notions; we shall present only it, referring to [41] for other notions of Cauchy sequence and for their properties.

A sequence (x_n) in (X, ρ) is called *left ρ-K-Cauchy* if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall\, n, m \text{ with } n_\varepsilon \le n < m \;\; \rho(x_n, x_m) < \varepsilon \iff \forall\, n \ge n_\varepsilon \;\forall\, k \in \mathbb{N} \;\; \rho(x_n, x_{n+k}) < \varepsilon.$$

Similarly, a sequence (x_n) in (X, ρ) is called *right ρ-K-Cauchy* if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall\, n, m \text{ with } n_\varepsilon \le n < m \;\; \rho(x_m, x_n) < \varepsilon \iff \forall\, n \ge n_\varepsilon \;\forall\, k \in \mathbb{N} \;\; \rho(x_{n+k}, x_n) < \varepsilon.$$

Remarks 2.19. Let (X, ρ) be a quasi-semimetric space.

(1) Obviously, a sequence is left ρ-K-Cauchy if and only if it is right $\bar{\rho}$-K-Cauchy.

(2) Let (x_n) be a left ρ-K-Cauchy sequence. If (x_n) contains a subsequence that is $\tau(\rho)$- ($\tau(\bar{\rho})$-) convergent to some $x \in X$, then the sequence (x_n) is $\tau(\rho)$- (respectively, $\tau(\bar{\rho})$-) convergent to x [41, P. 1.2.4].

(3) If a sequence (x_n) in X satisfies $\sum\limits_{n=1}^{\infty} \rho(x_n, x_{n+1}) < \infty$ ($\sum\limits_{n=1}^{\infty} \rho(x_{n+1}, x_n) < \infty$), then it is left- (right-) ρ-K-Cauchy.

(4) There are examples showing that a ρ-convergent sequence need not be left ρ-K-Cauchy, showing that in the asymmetric case the situation is far more complicated than in the symmetric one (see [41, Sec. 1.2]).

(5) If each convergent sequence in a regular quasi-metric space (X, ρ) admits a left K-Cauchy subsequence, then X is metrizable (see [41, P. 1.2.1]).

A quasi-metric space (X, ρ) is called *left ρ-K-complete* if every left ρ-K-Cauchy sequence is ρ-convergent, with the corresponding definition of the *right ρ-K-completeness.* The quasi-metric space (X, ρ) is called *left* (*right*) *Smyth complete* if every left (right) ρ-K-Cauchy sequence is ρ^s-convergent and *bicomplete* if the associated metric space (X, ρ^s) is complete.

Remark 2.20. In spite of the obvious fact that left ρ-K-Cauchy is equivalent to right $\bar{\rho}$-K-Cauchy, left ρ-K- and right $\bar{\rho}$-K-completeness do not agree, due to the fact that right $\bar{\rho}$-completeness means that every left ρ-Cauchy sequence converges in $(X, \bar{\rho})$, while left ρ-completeness means the convergence of such sequences in the space (X, ρ). Also, it is easy to check that Smyth completeness (left or right) of a quasi-metric space (X, ρ) implies the completeness of the associated metric space (X, ρ^s) (i.e., the bicompleteness of the quasi-metric space (X, ρ)).

Example 2.21. The spaces $(\mathbb{R}, q)$ and $(\mathbb{R}, \bar{q})$ from Example 2.17 are not right q-K-complete. The sequence $x_n = n$, $n \in \mathbb{N}$, is right q-K-Cauchy and not convergent in $(\mathbb{R}, q)$, and the sequence $y_n = -n$, $n \in \mathbb{N}$, is right $\bar{q}$-K-Cauchy and not convergent in $(\mathbb{R}, \bar{q})$.

Indeed, $q(x_{n+k}, x_n) = (n - n - k)^+ = 0$ for all $n, k \in \mathbb{N}$. For $x \in \mathbb{R}$ let $n_x \in \mathbb{N}$ be such that $n_x > x$. Then $q(x, x_n) = n - x \ge n_x - x > 0$ for all $n \ge n_x$. The case of the space $(\mathbb{R}, \bar{q})$ and of the sequence $y_n = -n$, $n \in \mathbb{N}$, can be treated similarly.

The following version of Ekeland's variational principle in quasi-metric spaces was proved in [39].

Theorem 2.22 (Ekeland's variational principle)**.** *Suppose that (X, ρ) is a T_1 quasi-metric space and $f\colon X \to \mathbb{R} \cup \{\infty\}$ is a proper function bounded below. For given $\varepsilon > 0$, let $x_\varepsilon \in X$ be such that*

$$f(x_\varepsilon) \le \inf f(X) + \varepsilon. \tag{2.10}$$

(1) *If (X, ρ) is right ρ-K-complete and f is ρ-lsc, then for every $\lambda > 0$ there exists $z = z_{\varepsilon,\lambda} \in X$ such that*

(a) $f(z) + (\varepsilon/\lambda)\rho(z, x_\varepsilon) \le f(x_\varepsilon)$;
(b) $\rho(z, x_\varepsilon) \le \lambda$;
(c) $f(z) < f(x) + (\varepsilon/\lambda)\rho(x, z)$ *for all* $x \in X \setminus \{z\}$.

(2) *If* (X, ρ) *is right* $\bar{\rho}$-K*-complete and* f *is* $\bar{\rho}$*-lsc, then for every* $\lambda > 0$ *there exists* $z = z_{\varepsilon,\lambda} \in X$ *such that*
(a′) $f(z) + (\varepsilon/\lambda)\rho(x_\varepsilon, z) \le f(x_\varepsilon)$;
(b′) $\rho(x_\varepsilon, z) \le \lambda$;
(c′) $f(z) < f(x) + (\varepsilon/\lambda)\rho(z, x)$ *for all* $x \in X \setminus \{z\}$.

Again, taking $\lambda = 1$ in Theorem 2.22, one obtains the weak form of Ekeland's variational principle in quasi-metric spaces.

Corollary 2.23 (Ekeland's variational principle — weak form)**.** *Suppose that* (X, ρ) *is a* T_1 *quasi-metric space and* $f\colon X \to \mathbb{R} \cup \{\infty\}$ *is a bounded below proper function.*

(1) *If* X *is right* ρ-K*-complete and* f *is* ρ*-lsc, then for every* $\varepsilon > 0$ *there exists an element* $y_\varepsilon \in X$ *such that*
(i) $f(y_\varepsilon) \le \inf f(X) + \varepsilon$,
(ii) $\forall\, x \in X \setminus \{y_\varepsilon\} \quad f(y_\varepsilon) < f(x) + \varepsilon\rho(x, y_\varepsilon)$.

(2) *If* X *is right* $\bar{\rho}$-K*-complete and* f *is* $\bar{\rho}$*-lsc, then for every* $\varepsilon > 0$ *there exists an element* $y_\varepsilon \in X$ *such that*
(i) $f(y_\varepsilon) \le \inf f(X) + \varepsilon$,
(ii) $\forall\, x \in X \setminus \{y_\varepsilon\} \quad f(y_\varepsilon) < f(x) + \varepsilon\rho(y_\varepsilon, x)$.

Caristi's fixed point theorem version in quasi-metric spaces is the following.

Theorem 2.24 (Caristi–Kirk fixed point theorem [39])**.** *Let* (X, ρ) *be a* T_1 *quasi-metric space,* $f\colon X \to X$*, and* $\varphi\colon X \to \mathbb{R}$*.*

(1) *If* X *is right* ρ-K*-complete,* φ *is bounded below and* ρ*-lsc, and the mapping* f *satisfies the condition*

$$\rho(f(x), x) \le \varphi(x) - \varphi\big(f(x)\big), \quad x \in X, \tag{2.11}$$

then f *has a fixed point in* X*.*

(2) *If* X *is right* $\bar{\rho}$-K*-complete,* φ *is bounded below and* $\bar{\rho}$*-lsc, and the mapping* f *satisfies the condition*

$$\rho\big(x, f(x)\big) \le \varphi(x) - \varphi\big(f(x)\big), \quad x \in X, \tag{2.12}$$

then f *has a fixed point in* X*.*

In this case, we have also a set-valued version.

Theorem 2.25 (Caristi–Kirk fixed point theorem — set-valued version [39])**.** *Let* (X, ρ) *be a* T_1 *quasi-metric space,* $F\colon X \rightrightarrows X$ *be a set-valued mapping such that* $F(x) \neq \varnothing$ *for every* $x \in X$*, and* $\varphi\colon X \to \mathbb{R}$*.*

(1) *If* X *is right* ρ-K*-complete,* φ *is bounded below and* ρ*-lsc, and the mapping* F *satisfies the condition*

$$\forall\, x \in X \ \forall\, y \in F(x) \quad \rho(y, x) \le \varphi(x) - \varphi(y), \tag{2.13}$$

then F *has a fixed point in* X*.*

(2) *If* X *is right* $\bar{\rho}$-K*-complete,* φ *is bounded below and* $\bar{\rho}$*-lsc, and the mapping* F *satisfies the condition*

$$\forall\, x \in X \ \forall\, y \in F(x) \quad \rho(x, y) \le \varphi(x) - \varphi(y), \tag{2.14}$$

then F *has a fixed point in* X*.*

As in the symmetric case, the weak form of Ekeland's variational principle is equivalent to Caristi's fixed point theorem [39].

Proposition 2.26. *Let (X, ρ) be a T_1 quasi-metric space. Consider the following assertions.*

(wEk) *For any ρ-closed subset Y of X, for every bounded below ρ-lsc proper function $f\colon Y \to \mathbb{R} \cup \{\infty\}$, and for every $\varepsilon > 0$, there exists $x_\varepsilon \in Y$ such that*

$$\forall\, y \in Y \setminus \{x_\varepsilon\} \quad f(x_\varepsilon) < f(y) + \varepsilon\rho(y, x_\varepsilon). \tag{2.15}$$

(C) *For every ρ-closed subset Y of X and for any ρ-lsc function $\varphi\colon Y \to \mathbb{R}$, any function $g\colon Y \to Y$ satisfying* (2.11) *on Y has a fixed point.*

Then (wEk) $\iff$ (C).

As we have seen, in the case of a metric space X, the validity of the weak form of Ekeland's variational principle implies the completeness of X (Proposition 2.4). The following proposition contains some partial converse results in the quasi-metric case.

Proposition 2.27 ([39])**.** *Let (X, ρ) be a T_1 quasi-metric space.*

(1) *If for every ρ-lsc function $f\colon X \to \mathbb{R}$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*

$$\forall\, x \in X \quad f(y_\varepsilon) \le f(x) + \varepsilon\rho(y_\varepsilon, x), \tag{2.16}$$

then the quasi-metric space X is left ρ-K-complete.

(2) *If for every $\bar\rho$-lsc function $f\colon X \to \mathbb{R}$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*

$$\forall\, x \in X \quad f(y_\varepsilon) \le f(x) + \varepsilon\rho(x, y_\varepsilon), \tag{2.17}$$

then the quasi-metric space X is left $\bar\rho$-K-complete.

Proof. The proof is similar to that of Proposition 2.4, accounting for the fact that a quasi-metric has weaker continuity properties than a metric (see Proposition 2.18).

To prove (1), suppose that (x_n) is a left ρ-K-Cauchy sequence in X. We show first that, for every $n \in \mathbb{N}$, the sequence $\big(\rho(x, x_n)\big)$ is bounded. Indeed, if $n_1 \in \mathbb{N}$ is such that $\rho(x_{n_1}, x_{n_1+k}) \le 1$ for all $k \in \mathbb{N}$, then

$$\rho(x, x_{n_1+k}) \le \rho(x, x_{n_1}) + \rho(x_{n_1}, x_{n_1+k}) \le \rho(x, x_{n_1} + 1,$$

for all $k \in \mathbb{N}$, proving the boundedness of the sequence $\big(\rho(x, x_n)\big)$. Consequently, the function $f\colon X \to [0, \infty)$ given by

$$f(x) = \limsup_{n\to\infty} \rho(x, x_n), \quad x \in X,$$

is well defined.

For $x, x' \in X$,

$$\rho(x, x_n) \le \rho(x, x') + \rho(x', x_n),$$

for all $n \in \mathbb{N}$. Passing to lim sup in both sides of this inequality, one obtains

$$f(x') \ge f(x) - \rho(x, x').$$

Then for every $\varepsilon > 0$, $\rho(x, x') < \varepsilon$ implies $f(x') > f(x) - \varepsilon$, proving that f is ρ-lsc at every $x \in X$. Similarly,

$$\rho(x', x_n) \le \rho(x', x) + \rho(x, x_n), \quad n \in \mathbb{N},$$

implies

$$f(x') \le f(x) + \rho(x', x),$$

from which it follows that the function f is $\bar\rho$-usc at every x.

We show now that

$$\lim_{n\to\infty} f(x_n) = 0. \tag{2.18}$$

Indeed, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall\, n \ge n_\varepsilon \;\forall\, k \in \mathbb{N} \quad \rho(x_n, x_{n+k}) < \varepsilon,$$

implying

$$\forall\, n \geq n_\varepsilon \quad 0 \leq f(x_n) = \limsup_k \rho(x_n, x_{n+k}) \leq \varepsilon,$$

i.e., $\lim_n f(x_n) = 0$.

Now let $y \in X$ satisfy (2.16) for $\varepsilon = 1/2$. Taking $x = x_n$, it follows that

$$\forall\, n \in \mathbb{N} \quad f(y) \leq f(x_n) + \frac{1}{2}\, \rho(y, x_n).$$

Passing to lim sup and taking into account (2.18) one obtains

$$f(y) = \frac{1}{2}\, f(y),$$

which implies $f(y) = 0$. Since

$$f(y) = 0 \iff \limsup_n \rho(y, x_n) = 0 \iff \lim_n \rho(y, x_n) = 0,$$

it follows that the sequence (x_n) is ρ-convergent to y, proving the left ρ-K-completeness of the quasi-metric space X.

The proof of (2) is similar, working with the function $g\colon X \to [0, \infty)$ given by

$$g(x) = \limsup_n \rho(x_n, x), \quad x \in X,$$

which is $\bar{\rho}$-lsc and ρ-usc. □

Remark 2.28. Note that Proposition 2.27 does not contain a proper converse (in the sense of completeness) of the weak Ekeland Principle. We have in fact a kind of "cross" converse, as can be seen from the following explanations.

From Corollary 2.23.2 it follows that if the quasi-metric space (X, ρ) is right $\bar{\rho}$-K-complete, then for every $\bar{\rho}$-lsc function $f\colon X \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a point $y_\varepsilon \in X$ satisfying (2.16).

On the other hand, the fulfillment of (2.16) for any ρ-lsc function implies the left ρ-K-completeness of the quasi-metric space (X, ρ).

Of course, in the metric case, both of these conditions reduce to the completeness of X.

Taking into account the fact that a sequence (x_n) in X is right $\bar{\rho}$-K-Cauchy if and only if it is left ρ-K-Cauchy, one obtains the following completeness results:

$$(X, \rho) \text{ is right } \bar{\rho}\text{-}K\text{-complete} \iff \forall\, (x_n) \text{ a left } \rho\text{-}K\text{-Cauchy sequence in } X\ \exists\, x \in X \text{ such that } x_n \xrightarrow{\bar{\rho}} x,$$

while

$$(X, \rho) \text{ is left } \rho\text{-}K\text{-complete} \iff \forall\, (x_n) \text{ a left } \rho\text{-}K\text{-Cauchy sequence in } X\ \exists\, x \in X \text{ such that } x_n \xrightarrow{\rho} x.$$

The right converse was given by Karapinar and Romaguera [89]. To do this, they need to slightly modify the notion of lsc function.

Let (X, ρ) be a quasi-metric space. A proper function $f\colon X \to \mathbb{R} \cup \{\infty\}$, is called *nearly* ρ-lsc at $x \in X$ if $f(x) \leq \liminf_n f(x_n)$ for every sequence (x_n) of distinct points in X that is ρ-convergent to x.

It is clear that a ρ-lsc function is nearly ρ-lsc and if the topology τ_ρ is T_1 (equivalent to $\rho(x, y) > 0$ for all distinct points $x, y \in X$), then the converse is also true. The following simple example shows that these notions are different in T_0 quasi-metric spaces.

Example 2.29. Let $X = \{0, 1\}$, $\rho(0, 0) = \rho(0, 1) = \rho(1, 1) = 0$ and $\rho(1, 0) = 1$. Then every function $f\colon X \to \mathbb{R} \cup \{\infty\}$ is nearly ρ-lsc (there are no sequences formed of distinct points), but the function $f(0) = 1$, $f(1) = 0$ is not ρ-lsc at $x = 0$.

Indeed, $x_n = 1$ satisfies $\rho(0, x_n) = 0 \to 0$, $f(x_n) = 0$, and $f(0) = 1 > 0 = \liminf_n f(x_n)$.

Theorem 2.30. *For a quasi-semimetric space (X, ρ) the following conditions are equivalent.*

(1) *(X, ρ) is right K-sequentially complete.*

(2) *For every self mapping T of X and every proper bounded below and nearly ρ-lsc function $\varphi\colon X \to \mathbb{R} \cup \{\infty\}$ satisfying the inequality*

$$\rho(T(x), x) + \varphi\big(T(x)\big) \le \varphi(x), \tag{2.19}$$

for all $x \in X$, there exists $z = z_{T,\varphi} \in X$ such that $\varphi(z) = \varphi\big(T(z)\big)$.

(3) *For every proper bounded below and nearly ρ-lsc function $f\colon X \to \mathbb{R} \cup \{\infty\}$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*
   (i) $f(y_\varepsilon) \le \inf f(X) + \varepsilon$;
   (ii) $f(y_\varepsilon) < f(x) + \varepsilon\rho(x, y_\varepsilon)$ *for all* $x \in X \setminus \overline{\{y_\varepsilon\}}$;
   (iii) $f(y_\varepsilon) \le f(x)$ *for all* $x \in \overline{\{y_\varepsilon\}}$.

Proof. We shall present only the proof of the implication (3) $\Longrightarrow$ (1).

We proceed by contradiction. Suppose that the space (X, ρ) is not right K-complete. Then there exists a right K-Cauchy sequence (x_n) in X that has no limit. This implies that (x_n) has no convergent subsequences, see Remarks 2.19.

We shall distinguish two situations.

Suppose that

$$\exists\, m\ \forall\, k \ge m\ \exists\, n_k > k \quad \rho(x_{n_k}, x_k) > 0. \tag{2.20}$$

Then, for $n_1 = m$ there exists $n_2 > n_1$ such that $\rho(x_{n_2}, x_{n_1}) > 0$. Taking $k = n_2$, there follows the existence of $n_3 > n_2$ such that $\rho(x_{n_3}, x_{n_2}) > 0$. Continuing in this manner we obtain a sequence $n_1 < n_2 < \dots$ such that $\rho(x_{n_{k+1}}, x_{n_k}) > 0$ for all $k \in \mathbb{N}$.

Passing to a further subsequence, if necessary, and relabeling, we can suppose that

$$0 < \rho(x_{n+1}, x_n) < \frac{1}{2^{n+1}}, \tag{2.21}$$

for all $n \in \mathbb{N}$.

If (2.20) does not hold, then

$$\forall\, m\ \exists\, k \ge m \text{ such that } \forall\, n > k \quad \rho(x_n, x_k) = 0. \tag{2.22}$$

For $m = 1$ let $k = n_1 \ge 1$ be such that $\rho(x_n, x_{n_1}) = 0$ for all $n > n_1$. Now, for $m = 1 + n_1$ let $n_2 > n_1$ be such that $\rho(x_n, x_{n_2}) = 0$ for all $n > n_2$. It follows that $\rho(x_{n_2}, x_{n_1}) = 0$.

Continuing in this manner we obtain a sequence $n_1 < n_2 < \dots$ such that $\rho(x_{n_{k+1}}, x_{n_k}) = 0$ for all $k \in \mathbb{N}$.

Relabeling, if necessary, we can suppose that the sequence (x_n) satisfies

$$\rho(x_{n+1}, x_n) = 0, \tag{2.23}$$

for all $n \in \mathbb{N}$.

Put

$$B := \{x_n \colon n \in \mathbb{N}\}$$

and define $f\colon X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/2^{n-1} & \text{if } x = x_n \text{ for some } n \in \mathbb{N}, \\ 2 & \text{for } x \in X \setminus B. \end{cases} \tag{2.24}$$

The function f is nearly ρ-lsc. Indeed, let $x \in X$ and (y_n) be a sequence of distinct points in X converging to x. If the set $\{n \in \mathbb{N}\colon y_n \in B\}$ is infinite, then there exists natural numbers $m_1 < m_2 < \dots$ and $n_1 < n_2 < \dots$ such that $y_{m_k} = x_{n_k}$, $k \in \mathbb{N}$. But, this would imply that (x_n) has a subsequence (x_{n_k}) convergent to x, in contradiction to the hypothesis. Consequently, (y_n) must be eventually in $X \setminus B$, and so $f(x) \le 2 = \lim_n f(y_n)$.

For $\varepsilon = 1$ let $y \in X$ satisfy the conditions (i)–(iii). Since

$$\{x \in X\colon f(x) \le \inf f(X) + 1\} = \{x \in X\colon f(x) \le 1\} = B,$$

it follows that $y = x_m \in B$ for some $m \in \mathbb{N}$.

If (2.21) holds, then

$$f(x_{m+1}) + \rho(x_{m+1}, x_m) < \frac{1}{2^m} + \frac{1}{2^{m+1}} = \frac{3}{2^{m+1}} < \frac{1}{2^{m-1}} = f(x_m), \tag{2.25}$$

showing that condition (ii) from (3) is not satisfied, i.e., (3) does not hold.

If (2.23) holds, then, by the triangle inequality,

$$\rho(x_{m+k}, x_m) \le \sum_{i=1}^{k} \rho(x_{m+i}, x_{m+i-1}) = 0,$$

i.e. $x_n \in \overline{\{x_m\}}$ for all $n \ge m$. By (iii),

$$f(x_m) \le f(x_n) = \frac{1}{2^{n-1}},$$

for all $n \ge m$, implying $f(x_m) = 0$, a value not taken by f. □

Remark 2.31. In the proof of Theorem 2 in [89], the possibility that $\rho(x_{n+1}, x_n) = 0$ for all $n \in \mathbb{N}$ (when one can not use x_{m+1} to obtain the contradiction from (2.25)) is not discussed. So the proof given above fills in this gap.

Smyth completeness. We present now some results on Caristi's FPT and Smyth completeness in quasi-metric spaces obtained by Romaguera and Tirado [152].

Let (X, ρ) be a quasi-metric space, $\varphi\colon X \to [0, \infty)$ and $T\colon X \to X$ such that

$$\rho(x, Tx) \le \varphi(x) - \varphi(Tx), \tag{2.26}$$

for all $x \in X$.

The mapping T is called *$\bar{\rho}$-Caristi* if φ is $\bar{\rho}$-lsc and *ρ^s-Caristi* if φ is ρ^s-lsc.

Theorem 2.32 (Romaguera and Tirado [152])**.** *Let (X, ρ) be a quasi-metric space.*

(1) *If (X, ρ) is right $\bar{\rho}$-K-complete, then every $\bar{\rho}$-Caristi map on X has a fixed point.*
(2) *If (X, ρ) is right ρ-K-complete, then every ρ-Caristi map on X has a fixed point.*
(3) *A quasi-metric space (X, ρ) is right $\bar{\rho}$-Smyth complete if and only if every ρ^s-Caristi map has a fixed point.*

Remark 2.33. Some versions of Ekeland's variational principle in asymmetric locally convex spaces were proved in [40]. Other characterizations of completeness of quasi-metric spaces are given by Romaguera and Valero [153].

Some results of Bao and Soubeyran. Inspired by the results of Dancs, Hegedűs, and Medvegyev [43] (see Theorem 2.12), Bao, Cobzaş, and Soubeyran [25], Bao and Soubeyran [28], and Bao and Théra [29] proved versions of Ekeland's principle in quasi-semimetric spaces and obtained characterizations of completeness. They consider a set-valued mapping attached to a function φ and $\lambda > 0$, as in the following proposition.

Proposition 2.34. *Let (X, ρ) be a quasi-semimetric space, $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper function, and $S_\lambda\colon X \rightrightarrows X$ the set-valued mapping defined by*

$$S_\lambda(x) = \{y \in X\colon \lambda\rho(x, y) \le \varphi(x) - \varphi(y)\}. \tag{2.27}$$

Then S_λ enjoys the following properties:

(i) (*nonemptiness*) *$x \in S_\lambda(x)$ for all $x \in \operatorname{dom}(\varphi)$;*
(ii) (*monotonicity*) *$y \in S_\lambda(x) \Longrightarrow \varphi(y) \le \varphi(x)$ and $S_\lambda(y) \subset S_\lambda(x)$.*

Recall that a generalized Picard sequence corresponding to a set-valued mapping $F\colon X \rightrightarrows X$ is a sequence (x_n) in X such that $x_{n+1} \in F(x_n)$ for all n.

Theorem 2.35. *Let (X, ρ) be a quasi-semimetric space, and let $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be proper. Given $x_0 \in \operatorname{dom}(f)$ and $\lambda > 0$, consider the set-valued mapping $S_\lambda\colon X \rightrightarrows X$ defined by (2.27). Assume that*

(C1) (*boundedness from below*) φ *is bounded from below on* $S_\lambda(x_0)$;

(C2) (*nonempty intersection*) *for every generalized Picard sequence* $(x_n)_{n\in\mathbb{N}_0}$ *of* S_λ (*starting with* x_0) *such that* $\varphi(x_n) > \varphi(x_{n+1})$ *for all* $n \in \mathbb{N}_0$, *and* $\sum\limits_{n=0}^{\infty} \rho(x_n, x_{n+1}) < \infty$, *there exists* $y \in X$ *such that* $S_\lambda(y) \subset S_\lambda(x_n)$ *for all* $n \in \mathbb{N}_0$, *where* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Then, there is a generalized Picard sequence $(x_n)_{n\in\mathbb{N}_0}$ (*i.e.,* $x_{n+1} \in S_\lambda(x_n)$ *for all* $n \in \mathbb{N}_0$) *satisfying* $\sum\limits_{n=0}^{\infty} \rho(x_n, x_{n+1}) < \infty$, $\bar{\rho}$*-convergent to some* $\bar{x} \in X$ *such that for every* $\bar{y} \in S_\lambda(\bar{x})$ *the following conditions hold*:

(i) $\lambda\rho(x_0, \bar{y}) \le \varphi(x_0) - \varphi(\bar{y})$;

(ii) $\varphi(\bar{y}) < \varphi(x) + \lambda\rho(\bar{y}, x)$ *for every* $x \in X \setminus S_\lambda(\bar{y})$;

(iii) $\rho(\bar{x}, \bar{y}) = 0$, $\varphi(\bar{y}) = \varphi(\bar{x})$ *and* $S_\lambda(\bar{y}) \subset \overline{\{\bar{y}\}}^{\bar{\rho}}$.

Remark 2.36. It is clear that Theorem 2.35 implies

(ii′) $\varphi(\bar{y}) < \varphi(x) + \lambda\rho(\bar{y}, x)$ for all $x \in X \setminus \overline{\{\bar{y}\}}^{\bar{\rho}}$;

(iii′) $\varphi(\bar{y}) \le \varphi(x)$ for all $x \in \overline{\{\bar{y}\}}^{\bar{\rho}}$.

To obtain a characterization of completeness, one needs a weaker notion of lower semicontinuity.

Definition 2.37. A function $\varphi\colon X \to \mathbb{R} \cup \{\infty\}$, where (X, ρ) is a quasi-semimetric space, is called *strictly-decreasing-ρ-lsc* if for every ρ-convergent sequence (x_n) in X such that the sequence $\big(\varphi(x_n)\big)$ is strictly decreasing, one has

$$\varphi(y) \le \lim_n \varphi(x_n)$$

for every ρ-limit y of the sequence (x_n). A sequence (x_n) such that the sequence $\big(\varphi(x_n)\big)$ is strictly decreasing is called *strictly φ-decreasing*.

Remark 2.38. In [25], it is shown, by an example, that this notion is strictly weaker than that of ρ-lsc, i.e. there exists a function that is strictly decreasing ρ-lsc but not ρ-lsc. In fact, the everywhere discontinuous function $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, defined on $(\mathbb{R}, |\cdot|)$, is strictly decreasing lsc (because there does not exist a sequence (x_n) in $\mathbb{R}$ such that the sequence $\big(\varphi(x_n)\big)$ is strictly decreasing). The function f is not lsc because $f(x) = 1 > 0 = \liminf\limits_{x'\to x} \varphi(x)$ for every $x \in \mathbb{R} \setminus \mathbb{Q}$. Also, it is not usc at every $x \in \mathbb{Q}$.

Remark 2.39. The notion of a function $\varphi\colon X \to \mathbb{R} \cup \{\infty\}$ such that $\varphi(x) \le \lim\limits_n \varphi(x_n)$ for every sequence (x_n) in X converging to x such that $\varphi(x_{n+1}) \le \varphi(x_n)$ for all $n \in \mathbb{N}$, appears also in [100] called *lower semicontinuity from above*, in connection with Ekeland's variational principle and Caristi's fixed point theorem.

Remark 2.40. In [25], the following sufficient condition for the fulfillment of condition (C2) from Theorem 2.35 was given.

(1) Let (X, ρ) be a quasi-semimetric space and $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If every sequence (x_n) in the space (X, ρ) such that $\varphi(x_{n+1}) < \varphi(x_{n+1})$, $n \in \mathbb{N}$, and $\sum\limits_{n+1}^{\infty} \rho(x_n, x_{n+1}) < \infty$ is $\bar{\rho}$-convergent to some $x \in X$, and the function φ is strictly-decreasing-$\bar{\rho}$-lsc on $\operatorname{dom}\varphi$, then condition (C2) is satisfied.

We mention also the following results relating series completeness and completeness in quasi-metric spaces.

Proposition 2.41 ([25, Proposition 3.10]). *Let (X,ρ) be a quasi-semimetric space.*

(1) *If a sequence (x_n) in X satisfies*
$$\sum_{n=1}^{\infty}\rho(x_n,x_{n+1})<\infty,$$
then it is left ρ-K-Cauchy (or, equivalently, right $\bar{\rho}$-K-Cauchy).

(2) *The space X is left ρ-K-complete if and only if every sequence (x_n) in X satisfying*
$$\sum_{n=1}^{\infty}\rho(x_n,x_{n+1})<\infty$$
is ρ-convergent to some $x\in X$.

(3) *The space X is right $\bar{\rho}$-K-complete if and only if every sequence (x_n) in X satisfying*
$$\sum_{n=1}^{\infty}\rho(x_n,x_{n+1})<\infty$$
is $\bar{\rho}$-convergent to some $x\in X$.

Proof (*sketch*). The assertion (1) follows from the triangle inequality and the Cauchy criterion of convergence applied to the series $\sum\limits_{n=1}^{\infty}\rho(x_n,x_{n+1})$:
$$\rho(x_n,x_{n+k})\le\sum_{i=0}^{k-1}\rho(x_{n+i},x_{n+i+1})<\varepsilon.$$

Let us prove (2) and (3). If (x_n) is left ρ-K-Cauchy, then there exist numbers $n_1<n_2<\dots$ such that $\rho(x_{n_k},x_{n_{k+1}})<1/2^k$, $k\in\mathbb{N}$. Then $\sum\limits_{k=1}^{\infty}\rho(x_{n_k},x_{n_{k+1}})<\infty$ so that, by the hypothesis, there exists $x\in X$ with $\lim\limits_k\rho(x,x_{n_k})=0$ ($\lim\limits_k\bar{\rho}(x,x_{n_k})=0$). By Remarks 2.19(2), $\lim\limits_n\rho(x,x_n)=0$ (respectively, $\lim\limits_n\bar{\rho}(x,x_n)=0$). □

We will present now a characterization of completeness of a class of quasi-semimetric spaces (X,q) such that every left ρ-K-Cauchy sequence $\{x_n\}$ in X satisfying the condition that for some $n_0\in\mathbb{N}$, $q(x_{n_0},x_n)=0$ for all $n\ge n_0$, is $\bar{\rho}$-convergent. We denote this class by $\mathcal{QMS}_0$. A sequence $\{x_n\}$ that satisfies the condition that for some $n_0\in\mathbb{N}$, $q(x_n,x_{n+1})=0$ for all $n\ge n_0$ is left ρ-K-Cauchy and, by the triangle inequality, $q(x_{n_0},x_n)=0$ for all $n\ge n_0$. Any quasimetric space (X,q) such that $q(x,y)>0$ for all $x,y\in X$, $x\ne y$, (i.e., a T_1 quasimetric space) belongs to the class $\mathcal{QMS}_0$. Indeed, in this case $q(x_{n_0},x_n)=0$ implies $x_n=x_{n_0}$ for all $n\ge n_0$, so that $\{x_n\}$ is $\bar{\rho}$-convergent to x_{n_0}. We show by an example that the completeness does not hold in arbitrary quasi-semimetric spaces.

Remark 2.42. Using this notion, the condition imposed to the quasi-semimetric space (X,ρ) in Theorem 2.30 means that the conjugate space $(X,\bar{\rho})$ belongs to the class $\mathcal{QMS}_0$.

We will present now a characterization of completeness of quasi-semimetric spaces in terms of Ekeland's variational principle (Theorem 2.35).

Theorem 2.43 (a characterization of completeness). *For any quasi-semimetric space (X,ρ) the following conditions are equivalent.*

(1) *The space X is right $\bar{\rho}$-K-complete.*

(2) *For every proper, bounded from below, and strictly decreasing-$\bar{\rho}$-lsc function $\varphi\colon X\to\mathbb{R}\cup\{+\infty\}$ and for any $x_0\in\operatorname{dom}(\varphi)$ there is $\bar{x}\in X$ such that for every $\bar{y}\in S_1(\bar{x})$ one has*

(i) $\rho(x_0,\bar{y})\le\varphi(x_0)-\varphi(\bar{y})$;

(ii) $\varphi(\bar{y}) < \varphi(x) + \rho(\bar{y}, x)$ *for all* $x \in X \setminus \overline{\{\bar{y}\}}^{\bar{\rho}}$;

(iii) $\varphi(\bar{y}) \le \varphi(x)$ *for all* $x \in \overline{\{\bar{y}\}}^{\bar{\rho}}$.

Proof. (1) $\Longrightarrow$ (2) This implication follows by Theorem 2.35, and Remarks 2.36 and 2.40.

The proof of the implication (2) $\Longrightarrow$ (1) is similar to that of the implication (3) $\Longrightarrow$ (1) in Theorem 2.30, working with the conjugate metric $\bar{\rho}$ instead of ρ. □

The following example shows that the completeness could not hold if we suppose that only the conditions (i) and (ii) from Theorem 2.43 hold.

Example 2.44. Let $x_n = -n$, $n \in \mathbb{N}_0$, and $X = \{x_n : n \in \mathbb{N}_0\}$ with the metric $q(x_n, x_m) = (x_m - x_n)^+ = (-m+n)^+ = n - m$ if $n > m$ and $= 0$ if $n \le m$ (see Example 2.17). Then $q(x_n, x_{n+1}) = 0$ for all $n \in \mathbb{N}_0$ and the space X is not right $\bar{q}$-K-complete (see Example 2.21). Let $\varphi\colon X \to [0, \infty)$ be an arbitrary function. For $\bar{x} = x_0$,

$$\varphi(\bar{x}) = \varphi(x_0) \le \varphi(x_0) = \varphi(x_0) + q(\bar{x}, x_0).$$

Since $q(x_0, x_n) = 0$ for all $n \in \mathbb{N}_0$, the condition

$$\varphi(x_0) < \varphi(x_n) + q(x_0, x_n) \text{ for all } n \in \mathbb{N}_0 \text{ with } q(x_0, x_n) > 0$$

is trivially satisfied.

Remark 2.45. Bao and Soubeyran use the notions of forward and backward in a quasi-semimetric space (X, ρ), where forward means with respect to $\bar{\rho}$, while backward means with respect to ρ.

For instance, a sequence (x_n) in X is

- forward convergent to x if $\bar{\rho}(x, x_n) = \rho(x_n, x) \to 0$, i.e., it is $\bar{\rho}$-convergent to x;
- forward Cauchy if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\rho(x_n, x_{n+k}) < \varepsilon$, for all $n \ge n_\varepsilon$ and all $k \in \mathbb{N}$, i.e., if it is left ρ-K-Cauchy, or equivalently, right $\bar{\rho}$-K-Cauchy;
- backward convergent to x if $\rho(x, x_n) \to 0$, i.e., it is ρ-convergent to x;
- backward Cauchy if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\rho(x_{n+k}, x_n) < \varepsilon$, for all $n \ge n_\varepsilon$ and all $k \in \mathbb{N}$, i.e., if it is right ρ-K-Cauchy, or equivalently, left $\bar{\rho}$-K-Cauchy.

The space X is called forward-forward complete if every forward Cauchy sequence is forward convergent, i.e., if it is right $\bar{\rho}$-K-complete.

The backward notion of completeness is defined on an analogous way: the quasi-metric space (X, ρ) is backward-backward complete if every backward Cauchy sequence is backward convergent, i.e., if it is right ρ-K-complete. One can define also combined notions of completeness: backward-forward (meaning left $\bar{\rho}$-K-completeness) and forward-backward (meaning left ρ-K-completeness).

Bao, Soubeyran, and Mordukhovich applied systematically variational principles in quasi-metric spaces to various domains of human knowledge, such as psychology in [26], capability of well being, and rationality in [27], and group dynamics in [28].

3. Topology and Order

In this section, we shall discuss some results relating topology and order.

3.1. Partially Ordered Sets. Let X be a nonempty set. A *preorder* on X is a reflexive and transitive relation $\le \subset X^2$. The pair $(X, \le)$ is called a *partially preordered set*. An antisymmetric preorder is called an *order* on X. The pair $(X, \le)$ is called a *partially ordered set*, or a *poset* in short.

Let $(X, \le)$ be a partially preordered set. For a nonempty subset A of a partially preordered set $(X, \le)$ one puts

$$\uparrow A := \{y \in X : \exists\, x \in A\ x \le y\}, \quad \downarrow A := \{y \in X : \exists\, x \in A\ y \le x\}. \tag{3.1}$$

In particular, for $x \in X$,

$$\uparrow x := \uparrow\{x\} \quad \downarrow x = \downarrow\{x\}.$$

The set A is called *upward closed* if $A = \uparrow A$ and *downward closed* if $A = \downarrow A$.

A nonempty subset A of X is called *totally ordered* (or a *chain*) if any two elements of A are comparable.

A nonempty subset D of X is called *directed* if for any two elements of $x, y \in A$ there exists $z \in A$ such that $x \le z$ and $y \le z$.

Let $A \subset X$. Then

- an *upper bound* for A is an element $z \in X$ such that $x \le z$ for all $x \in A$;
- if z is an upper bound for A and $z \in A$, then z is the *greatest element* of A;
- the set A is called *upper-bounded* if it has at least one upper bound;
- the least upper bound of A is called the *supremum* of A, denoted by $\sup A$ (or by $\bigvee A$);
- in the case of two elements $x, y \in X$, one uses the notation $x \vee y := \bigvee\{x, y\}$;
- the greatest element of X is called *unity* and is denoted by $\top$ (or by 1); the least element of X is called *zero* and is denoted by $\perp$ (or by 0).

One defines dually lower bounds, infima, etc. (we have used yet the notion of least element). The infimum is denoted by $\inf A$ (or by $\bigwedge A$). Also $x \wedge y := \bigwedge\{x, y\}$.

Definition 3.1. A poset $(X, \le)$ is called

- an *upper semi-lattice* if every two elements $x, y \in X$ have a sup, $x \vee y$;
- a *lower semi-lattice* if every two elements $x, y \in X$ have an inf, $x \wedge y$.
- a *lattice* if $(X, \le)$ is an upper semi-lattice and a lower semi-lattice, i.e., for every $x, y \in X$ there exist $x \wedge y$ and $x \vee y$.

A lattice $(X, \le)$ is called *complete* if for every subset A of X there exist $\sup A$ and $\inf A$.

A poset $(X, \le)$ is called *directed* (*chain*, *boundedly*) *complete* if every directed (totally ordered, upper bounded) subset of X has supremum. A directed complete partially ordered set is denoted in short by **dcpo**.

Remarks 3.2. Let $(X, \le)$ be a poset.

(1) If X has a greatest element $\top$, then $\top = \sup X$. If X has a least element $\perp$, then $\perp = \inf \varnothing$.
(2) In the definition of a complete lattice $(X, \le)$, it suffices to demand that every subset of X has a supremum, because X has a least element $\perp = \sup \varnothing$ and the infimum of a subset A of X is the supremum of the set L_A of all lower bounds of A (this set is nonempty because $\perp \in A$).

Mappings between partially preordered sets. Let $(X, \le)$ and $(Y, \preceq)$ be two partially preordered sets. A mapping $f : X \to Y$ is called

- *increasing* if $\forall\, x, y \in X \;\; x \le y \iff f(x) \preceq f(y)$;
- *decreasing* if $\forall\, x, y \in X \;\; x \le y \iff f(y) \preceq f(x)$;
- *monotonic* if it is increasing or decreasing.

One says that f *preserves*

- *suprema* if and only if for every $A \subset X$, the existence of $\sup A$ implies the existence of $\sup f(A)$ and the equality $\sup f(A) = f(\sup A)$;
- *finite* (*directed*, *chain*) *suprema* if and only if the above condition holds for every finite (respectively, directed, totally ordered) subset A of X.

Similar definitions can be given for infima.

Remark 3.3. A mapping preserving finite suprema is increasing.

Indeed, if $x \le y$ in X, then $y = \sup\{x, y\}$, and so $f(y) = \sup\{f(x), f(y)\}$, implying $f(x) \preceq f(y)$.

3.2. Order Relations in Topological Spaces. The *specialization order* of a topological space (X, τ) is the partial order defined by

$$x \le_\tau y \iff x \in \overline{\{y\}}, \tag{3.2}$$

i.e., y belongs to every open set containing x.

Proposition 3.4. *Let (X, τ) be a topological space. The relation defined by (3.2) is a preorder. It is an order if and only if X is T_0. If X is T_1, then $\leq_\tau$ is the equality relation in X.*

Proof. Since $x \in \overline{\{x\}}$, it follows that $x \leq_\tau x$.

The transitivity follows from the following implication:

$$x \in \overline{\{y\}} \text{ and } \{y\} \subset \overline{\{z\}} \implies x \in \overline{\{y\}} \subset \overline{\overline{\{z\}}} = \overline{\{z\}},$$

i.e.,

$$x \leq_\tau y \text{ and } y \leq_\tau z \implies x \leq_\tau z.$$

Suppose that X is T_0 and x, y are two distinct points in X. Then there exists an open set V that contains exactly one of this points. If $x \in V$ and $y \notin V$, then $x \notin \overline{\{y\}}$, i.e., the relation $x \leq_\tau y$ does not hold. If $y \in V$ and $x \notin V$, then $y \notin \overline{\{x\}}$, i.e., the relation $y \leq_\tau x$ does not hold. This means that we cannot have simultaneously $x \leq_\tau y$ and $y \leq_\tau x$ for a pair of distinct elements in X.

Similar reasonings show that X is T_0 if $\leq_\tau$ is a partial order (i.e., it is antisymmetric).

The topological space X is T_1 if and only if $\overline{\{x\}} = \{x\}$ for every $x \in X$. Consequently,

$$x \leq_\tau y \iff x \in \overline{\{y\}} = \{y\} \iff x = y.$$

One shows also that if the order relation $\leq_\tau$ is equality, then X is T_1. □

In the following results, the order notions are considered with respect to the order $\leq_\tau$.

Proposition 3.5. *Let (X, τ) be a topological space and $A \subset X$.*

(1) *If the set A is open, then it is upward closed, i.e., $\uparrow A = A$.*
(2) *If the set A is closed, then it is downward closed, i.e., $\downarrow A = A$.*

Proof. (1) It is a direct consequence of definitions. Let $x \in A$ and $y \in X$, $x \leq_\tau y$. Since A is open, this inequality implies $y \in A$.

(2) Let $x \in A$ and $y \in X$, $y \leq_\tau x$. Then for any neighborhood V of y, we have $x \in V$, i.e., $V \cap A \neq \varnothing$, showing that $y \in \bar{A} = A$. □

Let us define the *saturation* of a subset A of X as the intersection of all open subsets of X containing A. The set A is called *saturated* if it equals its saturation.

Proposition 3.6. *Let (X, τ) be a topological space.*

(1) *For every $x \in X$, $\downarrow x = \overline{\{x\}}$.*
(2) *For any subset A of X the saturation of A coincides with $\uparrow A$.*

Proof. (1) This follows from the equivalence

$$y \leq_\tau x \iff y \in \overline{\{x\}}.$$

(2) Since every open set is upward closed, $U \in \tau$ and $U \supset A$ implies $U \supset \uparrow A$, i.e.,

$$\uparrow A \subset \bigcap \{U \in \tau \colon A \subset U\}.$$

If $y \notin \uparrow A$, then for every $x \in A$ there exists $U_x \in \tau$ such that $x \in U_x$ and $y \notin U_x$. It follows that $y \notin V := \bigcup \{U_x \colon x \in A\} \in \tau$ and $A \subset V$, hence $y \notin \bigcap \{U \in \tau \colon A \subset U\}$. □

Compactness. We present following [61, p. 69] a result on compactness.

Proposition 3.7. *Let (X, τ) be a topological space. If a subset K of X is compact, then its saturation $\uparrow K$ is also compact. If $\uparrow K$ is compact, then K is compact too.*

Proof. The equivalence follows from the following remark: since every open subset of X is upward closed with respect to the specialization order, the following equivalence is true:

$$K \subset \bigcup\{U_i \colon i \in I\} \iff \uparrow K \subset \bigcup\{U_i \colon i \in I\},$$

for every family $\{U_i \colon i \in I\} \subset \tau$. □

An order $\preceq$ on a topological space (X, τ) is said to be *closed* if and only if its graph

$$\operatorname{Graph}(\preceq) := \{(x, y) \in X \times X \colon x \preceq y\}$$

is closed in $X \times X$ with respect to the product topology. The existence of a closed order on a topological space forces the topology to be Hausdorff.

Proposition 3.8 ([61, Proposition 3.9.12])**.** *If on a topological space (X, τ) there exists a closed order $\preceq$, then the topology τ is Hausdorff.*

Proof. Let x and y be distinct points in X. Then the relations $x \preceq y$ and $y \preceq x$ cannot both hold. Suppose, without the loss of generality, that $x \preceq y$ does not hold, i.e., $(x, y) \notin \operatorname{Graph}(\preceq)$. Then there exist open neighborhoods U and V of x and y, respectively, such that $(U \times V) \cap \operatorname{Graph}(\preceq) = \varnothing$. The proof will be done if we show that $U \cap V = \varnothing$. Indeed, supposing that there exists $z \in W := U \cap V$, one obtains the contradiction

$$(z, z) \in (W \times W) \cap \operatorname{Graph}(\preceq) \subset (U \times V) \cap \operatorname{Graph}(\preceq) = \varnothing.$$ □

For other properties of topological spaces endowed with a closed order (e.g., compactness), see [61, Sec. 9.1.1].

3.3. Topologies on Ordered Sets: Alexandrov's, the Upper Topology, Scott's, the Interval Topology. Consider a partially preordered set $(X, \leq)$. We are interested in defining a topology τ on X such that the specialization preordering $\leq_\tau$ coincides with $\leq$. The answer is, in general no. For instance, on $\mathbb{R}$, with the usual ordering, we can consider several topologies (the usual, the discrete, etc.), all having as specialization preordering the equality.

Let $(X, \leq)$ be a partially preordered set. We shall consider three topologies on X such that the corresponding specialization preorderings coincide with $\leq$, as well as the interval topology and the Moore–Smith order topology.

The Alexandrov topology, the finest. This is the finest of these topologies.

Proposition 3.9 (Alexandov topology)**.** *Let $(X, \leq)$ be a partially preordered set. Then there exists a finest topology τ_{a} on X, called the Alexandrov topology, such that the specialization preordering $\leq_{\tau_{\mathrm{a}}}$ coincides with $\leq$. This topology is characterized by the condition*

(i) *the open sets are exactly the upward closed sets,*

or, equivalently,

(ii) *the closed sets are exactly the downward closed sets.*

Proof. It is clear that the upward closed subsets of X forms a topology τ_{a}. Denote the specialization order determined by this topology by $\leq_a$. If $x \leq y$ and $Z \in \tau_{\mathrm{a}}$ contains x, then $y \in Z$ because Z is upward closed, showing that $x \leq_a y$. Let $x \leq_a y$. Then $\uparrow x \in \tau_{\mathrm{a}}$ and $x \in \uparrow x$ imply $y \in \uparrow x$, so that $x \leq y$. Consequently, $x \leq_a y$ agrees with $x \leq y$.

If τ is a topology on X such that the specialization order $\leq_\tau$ agrees with $\leq$ then, by Proposition 3.5, the sets in τ are upward closed, showing that $\tau \subset \tau_{\mathrm{a}}$. □

We use the notation X_a for (X, τ_{a}).

Remark 3.10. Since for every upward closed subset Z of X, $Z = \bigcup\{\uparrow z \colon z \in Z\}$, it follows that the Alexandrov topology τ_{a} is generated by the family of sets $\{\uparrow x \colon x \in X\}$.

The upper topology, the coarsest.

Proposition 3.11. *Let $(X, \leq)$ be a partially preordered set. Then there exists a coarsest topology τ_{u} on X such that the specialization preordering $\leq_{\tau_{\mathrm{u}}}$ coincides with $\leq$. A subbase of τ_{u} is formed by the complements of the sets $\downarrow x$, $x \in X$. A basis of τ_{u} is formed by the complements of the sets $\downarrow E$ for $E \subset X$, E finite.*

Proof. It is easy to check that the sets $\downarrow E$ for $E \subset X$, E finite, form a basis of a topology τ_{u} on X. Denote by $\leq_{\mathrm{u}}$ the specialization order determined by this topology.

Let $x \leq_{\mathrm{u}} y$. By definition this is equivalent to $x \in \overline{\{y\}}$. But

$$\begin{aligned} x \in \overline{\{y\}} &\iff \forall z\, [x \in X \setminus \downarrow z \implies y \in X \setminus \downarrow z] \\ &\iff \forall z\, [y \in \downarrow z \implies x \in \downarrow z] \\ &\underset{(z=y)}{\implies} x \in \downarrow y \iff x \leq y. \end{aligned}$$

Conversely, suppose $x \leq y$. If for some $z \in X$, $y \in \downarrow z$, then $x \leq y$ and $y \leq z$ would imply $x \leq z$, i.e., $x \in \downarrow z$. Consequently,

$$x \in X \setminus \downarrow z \implies y \in X \setminus \downarrow z,$$

for all $z \in X$, which is equivalent to $x \leq_{\mathrm{u}} y$. □

The Scott topology. This is a topology between τ_{u} and τ_{a}. It is defined in the following way.

Let $(X, \leq)$ be a partially ordered set. A subset U of X is *Scott open* if and only if the following two conditions hold:

(i) U is upward closed,

(ii) for every nonempty directed subset D of X such that $\sup D$ exists (in X) and belongs to U, there exists $d \in D$ such that $d \in U$.

Proposition 3.12. *Let $(X, \leq)$ be a partially ordered set.*

(1) *The family of Scott open subsets of X forms a topology denoted by τ_σ.*

(2) *A subset F of X is Scott closed if and only if the following two conditions hold:*

 (i) *F is downward closed,*

 (ii) *for every nonempty directed subset D of F if $\sup D$ exists (in X), then $\sup D \in F$.*

(3) *The specialization order corresponding to τ_σ agrees with $\leq$ and*

$$\tau_{\mathrm{u}} \leq \tau_\sigma \leq \tau.$$

(4) *Let (X, τ) be a topological space, $\leq_\tau$ the specialization order corresponding to τ, and $\sigma = \sigma(\leq_\tau)$ the Scott topology corresponding to $\leq_\tau$. Then the set $\overline{\{x\}}^\tau$ is Scott closed (i.e., σ-closed) for every $x \in X$.*

Proof. (1) Let U_i, $i = 1, \ldots, n$, be Scott open sets. Then $U := \bigcap\{U_i \colon 1 \leq i \leq n\}$ is upward closed. Suppose that D is a directed set in X such that $\sup D$ exists and belongs to U. Then $\sup D \in U_i$ implies the existence of $x_i \in D \cap U_i$, $i = 1, \ldots, n$. Since D is directed, there exists $x \in D$ with $x_i \leq x$, $i = 1, \ldots, n$. Since each U_i is upward closed, $x \in U_i$, $i = 1, \ldots, n$, i.e., $x \in U$, showing that U is Scott open too. It easy to show that the union of an arbitrary family of Scott open sets is again Scott open.

The proof of (2) follows from the equality $F = X \setminus U$ relating open sets U and closed sets F.

(3) Denote by $\leq_\sigma$ the specialization order corresponding to τ_σ and let $x \leq y$. If U is Scott open and $x \in U$, then $y \in U$, as U is upward closed. Consequently, $x \leq_\sigma y$.

Now suppose $x \leq_\sigma y$. The set $\downarrow y$ is Scott closed. If x belongs to $X \setminus \downarrow y$, then $y \in X \setminus \downarrow y$, a contradiction, so x must belong to $\downarrow y$, i.e., $x \leq y$.

(4) The proof is based on (2). The proof is based on (2). We shall use the notation $\bar{Z}$ instead of $\bar{Z}^\tau$. The symbol $\leq$ stands for $\leq_\tau = \leq_\sigma$.

Show first that the set $\overline{\{x\}}$ is downward closed.

Indeed, let $y \in \overline{\{x\}}$ and $y' \le y$. Then $y \le x$ and $y' \le y$ imply $y' \le x$, i.e., $y' \in \overline{\{x\}}$.

Let us verify condition (ii) from (2).

If $\{x_i : i \in I\}$ is a directed set contained in $\overline{\{x\}}$, then $x_i \le x$ for all $i \in I$, and so $\sup_i x_i \le x$, or equivalently, $\sup_i x_i \in \overline{\{x\}}$. □

In the following proposition, we characterize the continuity with respect to the Scott topology. We use the notation X_σ for (X, τ_σ).

Proposition 3.13. *Let X and Y be a partially ordered sets and $f : X \to Y$. The following are equivalent.*

(1) *The function f is continuous with respect to the Scott topologies on X and Y.*

(2) *The function f satisfies the following conditions:*

 (i) *f is increasing,*

 (ii) *f preserves the suprema of directed sets.*

Proof. All closures that appear in this proof are considered with respect to the Scott topology.

(2) $\Longrightarrow$ (1). The continuity of f is equivalent to each of the following conditions:

(1) $f^{-1}(Z)$ is closed for every closed subset Z of Y,

(2) $f(\bar{A}) \subset \overline{f(A)}$ for every subset A of X.

Let $Z \subset Y$ be Scott closed. We shall use Proposition 3.12(2) to show that $f^{-1}(Z)$ is Scott closed.

If $x \in f^{-1}(Z)$ and $x' \le x$, then $f(x) \in Z$ and $f(x') \le f(x)$. Since Z is downward closed, it follows that $f(x') \in Z$, which is equivalent to $x' \in f^{-1}(Z)$. Consequently, $f^{-1}(Z)$ is downward closed.

Now let $(x_i)_{i \in I}$ be a directed set contained in $f^{-1}(Z)$ such that $x = \sup_i x_i$ exists. Then $\big(f(x_i)\big)_{i\in I}$ is a directed set in Y contained in Z. By the hypothesis, $f(x) = \sup_i f(x_i)$ and, since Z is Scott closed, $f(x) \in Z$, which is equivalent to $x \in f^{-1}(Z)$.

(1) $\Longrightarrow$ (2). Suppose that f is continuous with respect to the Scott topologies on X and Y.

Let $x' \le x$ in X. Taking into account the continuity of f we have

$$x' \le x \iff x' \in \overline{\{x\}} \implies f(x') \in f\big(\overline{\{x\}}\big) \subset \overline{f(x)},$$

which shows that $f(x') \le f(x)$ in Y.

Now let $(x_i)_{i\in I}$ be a directed set in X such that $x = \sup_i x_i$ exists. Since f is increasing, it follows that $f(x_i) \le f(x)$ for all $i \in I$. Let $y \in Y$ be such that $f(x_i) \le y$ for all $i \in I$. Then

$$\forall i \; f(x_i) \le y \iff \forall i \; f(x_i) \in \overline{\{y\}} \iff \forall i \; x_i \in f^{-1}\big(\overline{\{y\}}\big).$$

Since $f^{-1}\big(\overline{\{y\}}\big)$ is Scott closed, it follows $x = \sup_i x_i \in f^{-1}\big(\overline{\{y\}}\big)$, which implies $f(x) \in \overline{\{y\}}$, i.e., $f(x) \le y$. Consequently, $f(x)$ is the least upper bound of $\big(f(x_i)\big)_{i\in I}$. □

Remark 3.14. A mapping satisfying condition (i) and (ii) from Proposition 3.13 is called *Scott continuous*. In fact, by Remark 3.3, it suffices to suppose that f satisfies only the condition (ii).

Example 3.15 ([61])**.** A subset of $\mathbb{R}$ is compact and saturated with respect to the Scott topology if and only if it is the empty set or of the form $[\alpha, \infty)$ for some $\alpha \in \mathbb{R}$.

The interval topology and the Moore–Smith order topology. These topologies were defined by Frink [55]. By a closed interval in a poset $(X, \le)$ one understands a set of the form

$$\uparrow a = \{x \in X : a \le x\}, \quad \downarrow b = \{y \in X : y \le b\}, \quad \text{or} \quad [a, b] = \{x \in X : a \le x \le b\} = \uparrow a \cap \downarrow b, \tag{3.3}$$

for $a, b \in X$. By definition, a subset Y of X is closed with respect to the interval topology if it can be written as the intersection of finite unions of sets of the form (3.3). It is shown in [55] that the family $\mathcal{F}_\le$ of closed sets defined above satisfies the axioms of closed sets:

(i) $\varnothing, X \in \mathcal{F}_\le$,

(ii) $F_i \in \mathcal{F}_{\leq}$, $i \in I$, implies $\bigcap_{i\in I} F_i \in \mathcal{F}_{\leq}$,

(iii) $F_1, F_2 \in \mathcal{F}_{\leq}$ implies $F_1 \cup F_2 \in \mathcal{F}_{\leq}$.

If the set X is totally ordered (i.e., it is a chain), then the interval topology defined above coincides with the *intrinsic topology*, which is the topology having as basis of open sets the intervals

$$(a, b) := \{x \in X : a < x < b\},$$

for $a, b \in X$ (see [55, Theorem 3]). (Recall that we write $x < y$ for $x \leq y$ and $x \neq y$.)

Remark 3.16. By analogy with the upper topology one can define the *lower topology* τ_{l} as that generated by the basis formed of the complements of the sets $\uparrow E$ for $E \subset X$, E finite. The interval topology $\tau_{\leq}$ is the supremum of these two topologies: $\tau_{\leq} = \tau_{\mathrm{u}} \vee \tau_{\mathrm{l}}$.

We mention also the following result.

Theorem 3.17 ([55, Theorem 9])**.** *Every complete lattice is compact in its interval topology.*

Proof. We include the simple proof of this result. Let $(X, \leq)$ be a complete lattice with 0 the least and 1 the greatest element. Then $\uparrow a = [a, 1]$ and $\downarrow b = [0, b]$, so that the intervals $[a, b]$, $a, b \in X$, form a subbasis of the interval topology. By the Alexander subbasis theorem [94, p. 139], it is sufficient to show that every family $[a_i, b_i]$, $i \in I$, of intervals in X having the finite intersection property has nonempty intersection. Since $[a_i, b_i] \cap [a_j, b_j] \neq \varnothing$ if and only if $a_i \vee a_j \leq b_i \wedge b_j$, it follows that $a_i \leq b_j$ for all $i, j \in I$. Hence

$$a := \sup_{i\in I} a_i \leq \inf_{j\in I} b_j =: b,$$

and $\varnothing \neq [a, b] \subset \bigcap_{i\in I} [a_i, b_i]$. □

Frink [55] considered also the Moore–Smith order topology defined in the following way. A net $(x_i : i \in I)$ in a poset $(X, \leq)$ is said to converge to $x \in X$ if there exist an increasing net $(u_i : i \in I)$ and a decreasing one $(v_i : i \in I)$ such that $u_i \leq x_i \leq v_i$ for all $i \in I$ and $\sup_i u_i = x = \inf_i v_i$. By definition, an element $x \in X$ belongs to the closure $\bar{Y}$ of a subset Y of X if and only if there exists a net in Y that converges to x. This closure operation satisfies the conditions

(a) $\bar{\varnothing} = \varnothing$,

(b) $Y \subset \bar{Y}$,

(c) $\overline{Y_1 \cup Y_2} = \bar{Y}_1 \cup \bar{Y}_2$,

for all $Y, Y_1, Y_2 \subset X$, but not the condition $\bar{\bar{Y}} = \bar{Y}$, so it does not generate a topology (see [94, p. 43]). In spite of this fact, we call it the *Moore–Smith order topology.* If $(X, \leq)$ is totally ordered, then it agrees with the interval topology [55, Theorem 3]. If $(X, \leq)$ is a distributive lattice, then the lattice operations $\vee$ and $\wedge$ are continuous with respect to the Moore–Smith order topology [55, Theorem 2].

Remark 3.18. Motivated by applications to computer science, mainly to denotational semantics of functional programming languages, topological and categorical methods applied to partially ordered sets were developed. A branch of this is known under the name of continuous lattices, whose study was initiated by Dana Scott [162] in 1971. Roughly speaking, these are complete lattices $(X, \leq)$ with Scott continuous meet and join operations, which means that

$$x \wedge \sup D = \sup\{x \wedge d : d \in D\} \quad \text{and} \quad x \vee \inf D = \inf\{x \vee d : d \in D\},$$

for every nonempty directed subset D of X.

Another one is the so called domain theory. Essentially it is concerned with the study of lattices or of directed complete partially ordered sets (known as **dcpo**, see Definition 3.1) equipped with a T_0 topology compatible with the order. A good introduction to this area is given in [61] (which we have partially followed in our presentation) and in [2]. For a comprehensive presentation we recommend [59], see also [168]. Notice also that a functional analysis within the context of T_0 topology was recently developed; see for instance [92, 93]. It turned out that a lot of results from Hausdorff functional analysis (Hausdorff topological vector, Hausdorff locally convex spaces, and Banach spaces) have their analogues in some algebraic structures (vector spaces, cones, universal algebras, etc.) equipped with a compatible T_0 topology.

4. Fixed Points in Partially Ordered Sets

In this section, we shall present some fixed point results in partially ordered sets and their impact on the completeness of the underlying ordered set.

4.1. Fixed Point Theorems. These fixed point theorems bear different names in different publications. The explanation is that many mathematicians contribute to their final shape, and the authors choose one or several of them.

Recall that "poset" is a short-hand for "partially ordered set."

Theorem 4.1 (Zermelo)**.** *Let $(X, \leq)$ be a chain-complete poset and $f\colon X \to X$ a mapping such that $x \leq f(x)$ for all $x \in X$. Then f has a fixed point. More precisely, for every $x \in X$, f has a fixed point y above x (i.e., $f(y) = y$ and $x \leq y$). If, further, f is increasing, then, for every $x \in X$, f has a least fixed point above x.*

A mapping $f\colon X \to X$ satisfying $x \leq f(x)$ for all $x \in X$ is called *progressive* in [79], *inflationary* in [61], extensive in [105].

This theorem is attributed to Bourbaki–Witt in [61] (with reference to Bourbaki [35] and Witt [197]), to Bourbaki–Kneser in [201]. As follows from the discussion about this matter in the survey paper by Jachymski [79], who proposed the name Zermelo FPT, this fixed point theorem appears only implicitly in Zermelo's papers on well-ordering (from 1904 and 1908), and it was put in evidence later. Accepting this principle (equivalent to the axiom of choice (AC)), the proof is immediate, but there are proofs independent of AC (see [79]). A brief historical survey is given also in Blanqui [33]. We shall not enter into this delicate question of whether a specific result depends or not on the AC. An exhaustive treatment is given in [69, 154]. Concerning its relevance for fixed points, we recommend the papers by Tasković [188–190] and Mańka [119–121]. Among other things, Mańka has found a proof of Caristi's fixed point theorem independent of the AC.

Remark 4.2. In [35], Zermelo FPT is formulated for a poset in which every well-ordered subset has a supremum, an apparently stronger form. But as was shown by Markowski [123], these conditions are equivalent: a poset X is chain complete if and only if every well-ordered subset of X has a supremum. In fact, according to the comments before Lemma 1.4 in [166], this result can be considered as a part of the folklore; the essential part of the proof (that every chain contains a well-ordered cofinal subset) appears as exercises in Halmos' *naive set theory*, and in Birkhoff's *lattice theory*.

Another important result is the following one.

Theorem 4.3 (Knaster–Tarski)**.** *Let $(X, \leq)$ be a poset and $f\colon X \to X$ an increasing function. If*

(i) *there exists $z \in X$ such that $z \leq f(z)$,*
(ii) *every chain in $\uparrow z$ has a supremum,*

then f has a fixed point above z. Furthermore, there exists a maximal fixed point of f.

In complete lattices, the above theorem takes the following form.

Theorem 4.4 (Birkhoff–Tarski)**.** *Let $(X, \leq)$ be a complete lattice and $f\colon X \to X$ an increasing mapping. Then there exist a smallest fixed point $\underline{x}$ and a greatest fixed point $\bar{x}$ for f, given by $\underline{x} = \inf\{f^n(\top)\colon n \in \mathbb{N}\}$ and by $\bar{x} = \sup\{f^n(\bot)\colon n \in \mathbb{N}\}$, where $\bot$ denotes the least element of X and $\top$ the greatest one. Furthermore, the set of fixed points of the mapping f is a complete lattice.*

Proof. Since $\underline{x} \le \top$ it follows that $f(\underline{x}) \le f(\top)$. Also $\underline{x} \le f^n(\top)$ implies $f(\underline{x}) \le f^{n+1}(\top)$ for all $n \in \mathbb{N}$. Consequently, $f(\underline{x}) \le \underline{x}$. By the definition of $\underline{x}$, $\underline{x} \le f(\top)$, so that $f(\underline{x}) = \underline{x}$.

The case of $\bar{x}$ can be treated similarly. □

In the following theorem, one asks a kind of Scott continuity for the mapping f.

Theorem 4.5 (Tarski–Kantorovich)**.** *Let $(X, \le)$ be a poset such that every countable chain in X has a supremum, and $f\colon X \to X$ be a mapping that preserves the suprema of countable chains. If there exists $z \in X$ such that $z \le f(z)$, then f has a fixed point. Moreover, $z_0 := \sup\{f^n(z)\colon n \in \mathbb{N}\}$ is the least fixed point of f in $\uparrow z$.*

Proof. We include the simple proof of this result following [63]. Since f preserves suprema of countable chains, it follows that it is increasing. From $z \le f(z)$ follows $f(z) \le f^2(z)$ and, by induction, $f^{n-1}(z) \le f^n(z)$ for all $n \in \mathbb{N}$, showing that $\{f^n(z)\colon n \in \mathbb{N}\}$ is a chain in $\uparrow z$. If $x_0 := \sup\{f^n(z)\colon n \in \mathbb{N}\}$, then, by the hypothesis, $f(x_0) = \sup\{f^{n+1}(z)\colon n \in \mathbb{N}\} = x_0$.

Let $x_1 \ge z$ be a fixed point of f. Then $f(z) \le f(x_1) = x_1$ and, by induction $f^n(z) \le x_1$ for all $n \in \mathbb{N}$, that is, x_1 is an upper bound for $\{f^n(z)\colon n \in \mathbb{N}\}$ and so $x_0 \le x_1$. □

Remark 4.6. In Theorem 4.5, it is sufficient to suppose that every countable chain in $\uparrow z$ has a supremum and that f preserves these suprema.

4.2. Converse Results. Apparently, the first converse result in this area was obtained by Davis [44].

Theorem 4.7. *A lattice $(X, \le)$ is complete if and only if every increasing mapping $f\colon X \to X$ has a fixed point.*

By a result of Frink [55] (see Theorem 3.17), a lattice $(X, \le)$ is complete if and only if it is compact with respect to the interval topology. Consequently, Theorem 4.7 admits the following reformulation.

Theorem 4.8. *A lattice $(X, \le)$ is compact in its interval topology if and only if every increasing mapping $f\colon X \to X$ has a fixed point.*

Extensions to lower semi-lattices of this result as well as of the Birkhoff–Tarski fixed point theorem, Theorem 4.4, were given by Ward [195]. Recall that a lower semi-lattice (semi-lattice in short) is a poset $(X, \le)$ such that $x \wedge y$ exists for every $x, y \in X$. It is called complete if every nonempty subset of X has an infimum.

Theorem 4.9 ([195])**.**

1. *A semi-lattice $(X, \le)$ is complete if and only if for every $x \in X$, $\downarrow x$ is compact with respect to the interval topology.*
2. *A semi-lattice $(X, \le)$ is compact with respect to the interval topology if and only if every increasing mapping $f\colon X \to X$ has a fixed point.*

Smithson [167] extended Davis' results to the case of set-valued mappings. Wolk [198] obtained also characterizations of directed completeness of posets (called by him Dedekind completeness) in terms of fixed points of monotonic maps acting on them.

We mention also the following result of Jachymski [80], connecting several properties equivalent to FPP. A *periodic point* for a mapping $f\colon X \to X$ is an element $x_0 \in X$ such that $f^k(x_0) = x_0$, for some $k \in \mathbb{N}$. The set of periodic points is denoted by $\mathrm{Per}(f)$ while the set of fixed points is denoted by $\mathrm{Fix}(f)$. It is obvious that a fixed point is a periodic point with $k = 1$.

Theorem 4.10. *Let X be a nonempty abstract set and f be a self-map of X. The following statements are equivalent.*

1. $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \varnothing$.
2. *There exists a partial ordering $\preceq$ such that every chain in $(X, \preceq)$ has a supremum and f is progressive with respect to $\preceq$ (i.e., $x \preceq f(x)$, $x \in X$) (Zermelo).*

(3) *There exists a complete metric d and a lower semicontinuous function $\varphi\colon X \to \mathbb{R}$ such that f satisfies condition* (2.5) (*Caristi*).
(4) *There exists a complete metric d and a d-Lipschitzian function $\varphi\colon X \to \mathbb{R}$ such that f satisfies condition* (2.5) *and f is nonexpansive with respect to d; i.e.,*

$$d\big(f(x), f(y)\big) \le d(x, y) \quad \text{for all} \quad x, y \in X.$$

(5) *For each $\alpha \in (0, 1)$, there exists a complete metric d such that f is nonexpansive with respect to d and*

$$d\big(f(x), f^2(x)\big) \le \alpha\, d\big(x, f(x)\big) \quad \text{for all} \quad x \in X$$

(*Hicks and Rhoades*).
(6) *There exists a complete metric d such that f is continuous with respect to d and for each $x \in X$ the sequence $\big(f^n(x)\big)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*

For two nonempty sets A and B denote by B^A the family of all mappings from A to B,

$$B^A := \{f\colon f\colon A \to B\}.$$

Let $(X, (\rho_i)_{i\in I})$ be a uniform space, where $\{\rho_i\colon i \in I\}$ is a family of semi-metrics generating the uniformity of X. Define a partial order $\preceq$ on $X \times \mathbb{R}_+^I$ by

$$(x, \varphi) \le (y, \psi) \iff \forall\, i \in I \;\; \rho_i(x, y) \le \varphi(i) - \psi(i), \tag{4.1}$$

for $x, y \in X$ and $\varphi, \psi \in \mathbb{R}_+^I$.

If (X, ρ) is a metric space (i.e., I is a singleton and $\rho_1 = \rho$ is a metric), then the relation order (4.1) becomes

$$(x, \alpha) \le (y, \beta) \iff \rho(x, y) \le \alpha - \beta, \tag{4.2}$$

for $x, y \in X$ and $\alpha, \beta \in \mathbb{R}_+$, an order considered by Ekeland in connection with his variational principle.

Jachymski [77] proved the following results concerning these orders.

Theorem 4.11. *Let $(X, (\rho_i)_{i\in I})$ be a uniform space and $\preceq$ the order on $X \times \mathbb{R}_+^I$ defined by* (4.1). *Then the following are equivalent.*
(1) *Every sequence (x_n) in X such that $\sum\limits_{n=1}^{\infty} \rho_i(x_n, x_{n+1}) < \infty$, for all $i \in I$, is convergent.*
(2) *Every countable chain in $(X \times \mathbb{R}_+^I, \preceq)$ has a supremum.*
(3) *Every increasing sequence in $(X \times \mathbb{R}_+^I, \preceq)$ has a supremum.*

In particular, if the space X is sequentially complete, then each of the above conditions holds.

In the case of a metric space (X, ρ), one obtains a characterization of completeness.

Theorem 4.12. *Let (X, ρ) be a metric space and $\preceq$ the order on $X \times \mathbb{R}_+$ defined by* (4.2). *Then the following are equivalent.*
(1) *The metric space X is complete.*
(2) *Every chain in $(X \times \mathbb{R}_+, \preceq)$ has a supremum.*
(3) *Every countable chain in $(X \times \mathbb{R}_+, \preceq)$ has a supremum.*
(4) *Every increasing sequence in $(X \times \mathbb{R}_+, \preceq)$ has a supremum.*

Jachymski applied these results to obtain proofs of fixed point results for mappings on partially ordered sets. In turn, these order fixed point results were applied to obtain simpler proofs and extensions to various fixed point results in metric and in uniform spaces (see, for instance, the papers by Jachymski [75–77, 79], and the references cited therein).

Klimeš [105] has found a common extension to Theorems 4.1 and 4.3. Let $(X, \le)$ be partially ordered. A mapping $f\colon X \to X$ is called *partially isotone* if for all $x, y \in X$

$$(x \le y \wedge x \le f(y) \wedge f(x) \le y) \implies f(x) \le f(y). \tag{4.3}$$

It is obvious that increasing mappings, "progressive" mappings (satisfying $x \le f(x)$), and "regressive" mappings (satisfying $f(x) \le x$) are partially isotone.

The mapping f is called *comparable* if x is comparable with $f(x)$ for every $x \in X$. The partially ordered set X is called inductive if every chain in X has an upper bound, and *semiuniform* if for every chain C in X the set of upper bounds of C is downward directed.

Klimeš [105] proved that:

- every relatively isotone self-mapping on a complete lattice has a fixed point (Theorem 1.2);
- if the partially ordered set X is chain complete (i.e., every chain in X, including the empty chain, has a supremum) then every relatively isotone self-mapping on X has a fixed point;
- a lattice X is complete if and only if every comparable self-mapping on X has a fixed point;
- a semiuniformly partially ordered set X is chain complete if and only if every relatively isotone self-mapping on X has a fixed point.

In [106], he considered *ascending* maps $f\colon X \to X$, meaning that $f(x) \le y$ implies $f(x) \le f(y)$ for all $x, y \in X$, and proved that the partially ordered set X is inductive if and only if every ascending self-mapping on X has a fixed point. For other related results, see [103, 104]. For instance, in [104] one considers mappings $f\colon X \to X$, X a partially ordered set, such that $x \le y$ and $x \le f(x)$ implies $f(x) \le f(y)$, called by the author extensively isotone.

4.3. Fixed Points in Ordered Metric Spaces. The title of this subsection could be a little confusing: in contrast to ordered Banach spaces, or Banach lattices, it concerns a metric space (X, ρ) equipped with an order relation $\le$ that does not have any connection with the metric structure. Fixed points are proved for mappings $f\colon X \to X$ that are monotonic (increasing or decreasing) with respect to the order and contractive with respect to the metric, but in a restricted manner in the following sense: there exists $0 \le \alpha < 1$ such that

$$\rho\big(f(x), f(y)\big) \le \alpha\rho(x, y) \text{ if } x, y \in X \text{ are comparable (i.e., } x \le y \text{ or } y \le x). \tag{4.4}$$

Theorem 4.13. *Let (X, ρ) be a complete metric space equipped with a partial order $\le$ and $f\colon X \to X$ be a mapping satisfying* (4.4). *Then the following results hold.*

(1) *If the mapping f is increasing and continuous and there exists $x_0 \in X$ such that $x_0 \le f(x_0)$, then f has a fixed point* [134].

(2) *Suppose that for every increasing sequence (x_n) in X converging to some $x \in X$, $x_n \le x$ holds for all $n \in \mathbb{N}$. If f is increasing and there exists $x_0 \in X$ such that $x_0 \le f(x_0)$, then f has a fixed point* [134].

(3) *Suppose that every pair x, y of elements in X has an upper bound or a lower bound. If f is continuous and monotone (i.e., either increasing or decreasing) and there exists $x_0 \in X$ such that $x_0 \le f(x_0)$ or $f(x_0) \le x_0$, then f has a unique fixed point $\bar{x}$ and for every $x \in X$ the sequence $\big(f^n(x)\big)_{n\in\mathbb{N}}$ converges to $\bar{x}$* [150].

(4) *Assume that the ordered set $(X, \le)$ admits a smallest element x_0. Then the conclusions from* (3) *hold for every continuous increasing function $f\colon X \to X$ satisfying* (4.4) [96].

Proof. (1) The proof is simple. Since f is increasing,

$$x_0 \le f(x_0) \implies f(x_0) \le f^2(x_0) \implies f^2(x_0) \le f^3(x_0) \implies \dots,$$

showing that the sequence $\big(f^n(x_0)\big)$ is increasing. By (4.4)

$$\rho\big(f^n(x_0), f^{n+1}(x_0)\big) \le \alpha\rho\big(f^{n-1}(x_0), f^n(x_0)\big) \le \dots \le \alpha^n\rho\big(x_0, f(x_0)\big)$$

for all $n \in \mathbb{N}$. But then, by the triangle inequality,

$$\rho\big(f^n(x_0), f^{n+k}(x_0)\big) \le (\alpha^n + \alpha^{n+1} + \dots + \alpha^{n+k-1})d\big(x_0, f(x_0)\big) \to 0$$

as $n \to \infty$, uniformly with respect to $k \in \mathbb{N}$, which shows that $\big(f^n(x_0)\big)$ is a Cauchy sequence and so, by the completeness of the metric space X, it converges to some $\bar{x} \in X$. By the continuity of f,

$$f(\bar{x}) = f\Big(\lim_n f^n(x_0)\Big) = \lim_n f^{n+1}(x_0) = \bar{x}.$$

(2) As in the proof of (1), the sequence $\big(f^n(x_0)\big)$ is increasing and convergent to some $\bar{x} \in X$. By the hypothesis, it follows that $f^n(x_0) \le \bar{x}$ for all $n \in \mathbb{N}$, so that

$$\rho\big(f^{n+1}(x_0), f(\bar{x})\big) \le \alpha\rho(f^n(x_0), \bar{x}) \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that $\rho\big(\bar{x}, f(\bar{x})\big) = 0$, i.e., $f(\bar{x}) = \bar{x}$.

(3) Suppose that f is increasing and that there exists $x_0 \in X$ such that $x_0 \le f(x_0)$. Then $\big(f^n(x_0)\big)$ is an increasing sequence, convergent to some $\bar{x} \in X$ which is a fixed point for f. The proof will be done if we show that, for every $x \in X$, the sequence $\big(f^n(x)\big)$ is convergent to $\bar{x}$.

Let $x \in X$. If $x \le x_0$, then $f^n(x) \le f^n(x_0)$ so that, by (4.4),

$$\rho\big(f^n(x), f^n(x_0)\big) \le \alpha\rho\big(f^{n-1}(x), f^{n-1}(x_0)\big) \le \cdots \le \alpha^n\rho(x, x_0) \to 0.$$

It follows that

$$\lim_n f^n(x) = \lim_n f^n(x_0) = \bar{x}.$$

The situation is the same if $x \ge x_0$.

If $x \in X$ is not comparable to x_0, then, by the hypothesis, x and x_0 have a lower bound or an upper bound in $(X, \le)$.

If they have a lower bound x_1, then $x_1 \le x_0$ and $x_1 \le x$, so that by the first part of the proof

$$\bar{x} = \lim_n f^n(x_0) = \lim_n f^n(x_1) = \lim_n f^n(x).$$

The situation is the same if x and x_0 have an upper bound x_2 in X.

(4) In this case, $x_0 \le f(x_0)$ and, for every $x \in X$, $x_0 \le x$, so we can proceed as in the proof of (3). □

Remark 4.14. Usually results as those from Theorem 4.13 are called fixed point of Ran–Reurings type [150].

Refinements of the above results were given in [81, 133, 135, 148].

5. Partial Metric Spaces

These spaces were introduced by Matthews [124–127] in connection with his research on computer science. They are only T_0 topological spaces, a feature that fits the needs of denotational semantics of dataflow networks. In this section, we shall first present the basic notions and results following [37, 124, 126, 127] (see also [98, 160]). Although all the included results on partial metric spaces can be found in the papers of Matthews or in other ones dealing with fixed point results in such spaces, we include full proofs of the results for the reader's convenience. At the same time, different approaches concerning convergence of sequences and completeness notions in partial metric spaces, used by various authors, are put in a proper light.

5.1. Definition and Topological Properties.

Let X be a nonempty set.

Definition 5.1. A mapping $p\colon X \times X \to \mathbb{R}_+$ satisfying the following conditions

$$\begin{aligned}
&\text{(PM1)} \quad x = y \iff p(x,x) = p(y,y) = p(x,y),\\
&\text{(PM2)} \quad 0 \le p(x,x) \le p(x,y),\\
&\text{(PM3)} \quad p(x,y) = p(y,x),\\
&\text{(PM4)} \quad p(x,z) \le p(x,y) + p(y,z) - p(y,y)
\end{aligned}$$

for all $x, y, z \in Z$, is called a *partial metric* on X. The pair (X, p) is called a *partial metric space.*

This means that, in contrast to the metric case, one admits the possibility that $d(x,x) > 0$ for some points $x \in X$.

A point $x \in X$ is called *complete* if $p(x,x) = 0$ and *partial* if $p(x,x) > 0$, giving an explanation for the term "partial" coined by Matthews.

The following property follows from (PM2) and (PM1):

$$p(x,y) = 0 \implies x = y. \tag{5.1}$$

The following characterization of partial metric spaces is given by M. & V. Anisiu [19].

Theorem 5.2. *A function $p\colon X \times X \to [0,\infty)$ is a partial metric on X if and only if there exist a metric d and a nonexpansive with respect to d function $\varphi\colon X \to [0,\infty)$, such that*

$$p(x,y) = d(x,y) + \varphi(x) + \varphi(y) \quad \text{for all} \quad x,y \in X.$$

Furthermore, d and φ are uniquely determined by p.

The following two examples of partial metric spaces are related to some questions in theoretical computer science.

Example 5.3. Let $X = 2^{\mathbb{N}}$. The function $p\colon X \times X \to [0,\infty)$ defined by

$$p(x,y) = 1 - \sum_{n \in x \cap y} 2^{-n},$$

with the convention that the sum over the empty set is 0, is a partial metric on X.

Example 5.4. For a nonempty set S let $X = S^* \cup S^{\mathbb{N}}$ be the set of all finite (belonging to S^*) or infinite sequences (belonging to $S^{\mathbb{N}}$). The length $\ell(x)$ of a finite sequence $x = (x_1, x_2, \ldots, x_n)$ is n and the length of an infinite sequence $x\colon \mathbb{N} \to S$ is ∞. If

$$i(x,y) = \sup\{n \in \mathbb{N} \colon n \le \ell(x) \wedge \ell(y),\ x_j = y_j \text{ for all } j \le n\},$$

then

$$p(x,y) = 2^{-i(x,y)}, \quad x,y \in X,$$

with the convention $2^{-\infty} = 0$, is a partial metric on X.

The function p is a metric on $S^{\mathbb{N}}$ called the Baire metric and a partial metric on $S^* \cup S^{\mathbb{N}}$, because $p(x,x) = 2^{-n} > 0$ for $x = (x_1, \ldots, x_n) \in S^*$.

We define the open balls as in the metric case:

$$B_p(x,\varepsilon) := \{y \in X \colon p(x,y) < \varepsilon\}, \tag{5.2}$$

for $x \in X$ and $\varepsilon > 0$.

In this case, the possibility that $B_p(x,\varepsilon) = \varnothing$ is not excluded.

Remark 5.5. If $p(x,x) > 0$, then $B_p(x,\varepsilon) = \varnothing$ for every $0 < \varepsilon \le p(x,x)$.

If $B_p(x,\varepsilon) \ne \varnothing$, then $x \in B_p(x,\varepsilon)$.

Indeed, by (PM2), $p(x,y) \ge p(x,x) \ge \varepsilon$ for every $y \in X$ implies $B_p(x,\varepsilon) = \varnothing$. Also, if $y \in B_p(x,\varepsilon)$, then, again by (PM2), $p(x,x) \le p(x,y) < \varepsilon$, i.e., $x \in B_p(x,\varepsilon)$.

Consider also the balls

$$B'_p(x,\varepsilon) := \{y \in X \colon p(x,y) < \varepsilon + p(x,x)\}, \tag{5.3}$$

for $x \in X$ and $\varepsilon > 0$.

The following proposition contains some properties of these two kinds of balls.

Proposition 5.6. *Let (X,p) be a partial metric space.*

(1) *If $y \in B_p(x,\varepsilon)$, then*

$$y \in B_p(y,\delta) \subset B_p(x,\varepsilon),$$

where $\delta := \varepsilon - p(x,y) + p(y,y) > 0$.

(2) *The balls* B_p *and* B'_p *are related by the following equalities*:

$$B'_p(x,\varepsilon) = B_p\big(x, \varepsilon + p(x,x)\big), \tag{5.4}$$

and

$$B_p(x,\varepsilon) = \begin{cases} B'_p\big(x, \varepsilon - p(x,x)\big) & \text{if } \varepsilon > p(x,x), \\ \varnothing & \text{if } 0 < \varepsilon \le p(x,x). \end{cases}$$

Proof. (1) Let $\delta := \varepsilon - p(x,y) + p(y,y)$. Then $\delta > 0$ (because $p(x,y) < \varepsilon$) and $p(y,y) < \delta$, so that $y \in B_p(y,\delta)$.

If $z \in B_p(y,\delta)$, then the inequalities

$$p(y,z) < \varepsilon - p(x,y) + p(y,y) \quad \text{and} \quad p(x,z) \le p(x,y) + p(y,z) - p(y,y)$$

yield by addition $p(x,z) < \varepsilon$, i.e., $z \in B_p(x,\varepsilon)$ and so $B_p(y,\delta) \subset B_p(x,\varepsilon)$.

The equalities from (2) are obvious by the definitions of the corresponding balls (see also Remark 5.5). □

Now we introduce the topology of a partial metric space and present some of its properties.

Theorem 5.7. *Let* (X,p) *be a partial metric space.*

(1) *The family of open balls*

$$\mathcal{B} := \{B_p(x,\varepsilon),\ x \in X,\ \varepsilon > 0\} \tag{5.5}$$

is a basis of a topology on X*, denoted by* τ_p *(sometimes by* $\tau(p)$*).*

(2) *The family* $\mathcal{B}'$ *of sets*

$$B'_p(x,\varepsilon) := \{y \in X : p(x,y) < \varepsilon + p(x,x)\}, \quad x \in X, \quad \varepsilon > 0, \tag{5.6}$$

is also a basis for the topology τ_p.

(3) *Any ball* $B_p(x,\varepsilon)$ *is open and for every* $x \in X$ *the family* $\mathcal{V}_p(x)$ *of neighborhoods of* x *is given by*

$$\mathcal{V}_p(x) = \{V \subset X : \exists\, \delta > 0,\ x \in B_p(x,\delta) \subset V\}. \tag{5.7}$$

(4) *The topology* τ_p *is* T_0.

Proof. Since $x \in B_p\big(x, 1 + p(x,x)\big)$, it follows that

$$X = \bigcup \{B_p\big(x, 1 + p(x,x)\big) : x \in X\}.$$

Also, by Proposition 5.6, $B_p(z,\eta_z) \subset B_p(x,\varepsilon) \cap B_p(y,\delta)$, for any $z \in B_p(x,\varepsilon) \cap B_p(y,\delta)$, where

$$\eta_z := p(z,z) + \min\{\varepsilon - p(x,z), \varepsilon - p(y,z)\},$$

and so

$$B_p(x,\varepsilon) \cap B_p(y,\delta) = \bigcup \{B_p(z,\eta_z) : z \in B_p(x,\varepsilon) \cap B_p(y,\delta)\}.$$

These two properties show that the family (5.5) forms a basis of a topology τ_p on X, i.e., every set in τ_p can be written as a union of open balls of the form $B_p(x,\varepsilon)$.

(2) The fact that $\mathcal{B}'$ is also a basis for τ_p follows from the equalities from Proposition 5.6(2).

(3) By Proposition 5.6, every ball in (X,p) can be written as

$$B_p(x,\varepsilon) = \bigcup \{B_p(y,\delta_y) : y \in B_p(x,\varepsilon)\} \in \tau_p,$$

where $\delta_y = \varepsilon - p(x,y) + p(y,y)$, $y \in B_p(x,\varepsilon)$.

Since the open balls form a basis of the topology τ_p, $V \in \mathcal{V}_p(x)$ if and only if there exists $y \in X$ and $\varepsilon > 0$ such that $x \in B_p(y,\varepsilon) \subset V$. Appealing again to Proposition 5.6, it follows that $x \in B_p(x,\delta) \subset B_p(y,\varepsilon) \subset V$, where $\delta = \varepsilon - p(x,y) + p(x,x)$.

(4) We have to show that for any pair x, y of distinct points in X there exists a τ_p-open set containing exactly one of them.

Let $x \ne y$ be two points in X. Then by (PM1) and (PM2) either $p(x,x) < p(x,y)$ or $p(y,y) < p(x,y)$.

Suppose $p(x,x) < p(x,y)$ and let $\varepsilon := \big(p(x,x) + p(x,y)\big)/2$. Then

$$2p(x,x) < p(x,x) + p(x,y) = 2\varepsilon \implies p(x,x) < \varepsilon \iff x \in B_p(x,\varepsilon).$$

On the other hand,

$$p(x,y) > p(x,x) = 2\varepsilon - p(x,y) \implies p(x,y) > \varepsilon \implies y \notin B_p(x,\varepsilon).$$

The case $p(y,y) < p(x,y)$ can be treated similarly. □

Remark 5.8. We adopt the convention that

$$\bigcup\{A_i : i \in \varnothing\} = \varnothing$$

(implying, by de Morgan rules, $\bigcap\{A_i : i \in \varnothing\} = X$), and so $\varnothing$ belongs to the family of arbitrary unions of sets in $\mathcal{B}$. If one considers only unions over nonempty index sets, then we must say that the family $\mathcal{B}$ plus the empty set generates the topology τ_p.

5.2. Convergent Sequences, Completeness and the Contraction Principle. The convergence of sequences with respect to τ_p can be characterized in the following way.

Proposition 5.9. *Let (X,p) be a partial metric space. A sequence (x_n) in X is τ_p-convergent to $x \in X$ if and only if*

$$\lim_{n\to\infty} p(x,x_n) = p(x,x). \tag{5.8}$$

Proof. Suppose that $x_n \xrightarrow{\tau_p} x$. Given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that, for all $n \geq n_0$,

$$x_n \in B_p\big(x, \varepsilon + p(x,x)\big) \iff p(x,x_n) < \varepsilon + p(x,x).$$

Taking into account (PM2), it follows that

$$0 \leq p(x,x_n) - p(x,x) < \varepsilon,$$

for all $n \geq n_0$, showing that (5.8) holds.

Conversely, suppose that (5.8) holds and let $V \in \mathcal{V}_p(x)$. Since, by Theorem 5.7(2), $\mathcal{B}'$ is also a basis for the topology τ_p, there exists $\varepsilon > 0$ such that $B'_p(x,\varepsilon) \subset V$. Let $n_0 \in \mathbb{N}$ be such that $0 \leq p(x,x_n) - p(x,x) < \varepsilon$ for all $n \geq n_0$. Then

$$0 \leq p(x,x_n) - p(x,x) < \varepsilon \iff p(x,x_n) < \varepsilon + p(x,x) \iff x_n \in B'_p(x,\varepsilon) \subset V,$$

for all $n \geq n_0$, proving that $x_n \xrightarrow{\tau_p} x$. □

Remark 5.10. Since the topology τ_p of a partial metric space is only T_0, a convergent sequence can have many limits. In fact, if $x_n \xrightarrow{\tau_p} x$, then $x_n \xrightarrow{\tau_p} y$ for any $y \in X$ such that $p(x,y) = p(y,y)$.

Indeed,

$$0 \leq p(y,x_n) - p(y,y) \leq p(y,x) + p(x,x_n) - p(x,x) - p(y,y) = p(x,x_n) - p(x,x) \to 0.$$

To obtain uniqueness and to define a reasonable notion of completeness, a stronger notion of convergence is needed.

Definition 5.11. One says that a sequence (x_n) in a partial metric space *converges properly* to $x \in X$ if and only if

$$\lim_{n\to\infty} p(x,x_n) = p(x,x) = \lim_{n\to\infty} p(x_n,x_n). \tag{5.9}$$

In other words, (x_n) converges properly to x if and only if (x_n) converges to x with respect to τ_p and further

$$\lim_{n\to\infty} p(x_n,x_n) = p(x,x). \tag{5.10}$$

Proposition 5.12. *Let (X,p) be a partial metric space and (x_n) be a sequence in X that converges properly to $x \in X$. Then*

(i) *the limit is unique,*

(ii) $\lim_{m,n\to\infty} p(x_m, x_n) = p(x, x)$.

Proof. Suppose that $x, y \in X$ are such that (x_n) converges properly to both x and y. Then

$$p(x,y) \le p(x,x_n) + p(x_n,y) - p(x_n,x_n) \to p(y,y) \quad \text{as} \quad n \to \infty,$$

implying $p(x,y) \le p(y,y)$. But, by (PM2), $p(y,y) \le p(x,y)$, so that

$$p(x,y) = p(y,y) = p(x,x), \tag{5.11}$$

which by (PM1) yields $x = y$.

To prove (ii) observe that

$$p(x_m,x_n) \le p(x_m,x) + p(x,x_n) - p(x,x),$$

so that

$$p(x_m,x_n) - p(x,x) \le p(x_m,x) - p(x,x_n) - 2p(x,x) \to 0 \quad \text{as} \quad m,n \to \infty.$$

Also

$$p(x,x) \le p(x,x_m) + p(x_m,x) - p(x_m,x_m) \le p(x,x_m) + p(x_m,x_n) + p(x_n,x) - p(x_n,x_n) - p(x_m,x_m)$$

implies

$$p(x,x) - p(x_m,x_n) \le p(x,x_m) + p(x_n,x) - p(x_n,x_n) - p(x_m,x_m) \to 0 \quad \text{as} \quad m,n \to \infty.$$

Consequently, $\lim_{m,n\to\infty} p(x_m,x_n) = p(x,x)$. □

Remark 5.13. Some authors take the condition (ii) from Proposition 5.12 in the definition of a properly convergent sequence. As was shown, this is equivalent to the condition from Definition 5.11

The definition of Cauchy sequences in partial metric spaces takes the following form.

Definition 5.14. A sequence (x_n) in a partial metric space (X,p) is called a *Cauchy sequence* if there exists $a \ge 0$ in $\mathbb{R}$ such that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ with

$$|p(x_n,x_m) - a| < \varepsilon,$$

for all $m,n \ge n_\varepsilon$, written also as $\lim_{m,n\to\infty} p(x_n,x_m) = a$.

The partial metric space (X,p) is called *complete* if every Cauchy sequence is properly convergent to some $x \in X$.

A mapping f on a partial metric space (X,p) is called a *contraction* if there exists $0 \le \alpha < 1$ such that

$$p\big(f(x), f(y)\big) \le \alpha\, p(x,y), \tag{5.12}$$

for all $x, y \in X$.

The analogue of the Banach contraction principle holds in partial metric spaces too.

Theorem 5.15 ([124, 127])**.** *Let (X,p) be a complete partial metric spaces. Then every contraction $f\colon X \to X$ has a fixed point x_0 such that $p(x_0,x_0) = 0$.*

Proof (sketch). Let f be an α-contraction on X with $0 \le \alpha < 1$.

First, one shows that for every $z \in X$ the sequence of iterates $\big(f^n(z)\big)$ satisfies the condition

$$\lim_{m,n\to\infty} p\big(f^n(z), f^m(z)\big) = 0,$$

i.e., it is Cauchy. By the completeness of (X,p) there exists $x_0 \in X$ such that

$$0 = \lim_{n\to\infty} p\big(f^n(z), f^n(z)\big) = p(x_0,x_0) = \lim_{n\to\infty} p\big(x_0, f^n(z)\big).$$

But

$$0 \le p\big(x_0, f(x_0)\big) \le p\big(x_0, f^n(x_0)\big) + p\big(f^n(x_0), f(x_0)\big) - p\big(f^n(x_0), f^n(x_0)\big) \le$$
$$\le p\big(x_0, f^n(x_0)\big) + \alpha p(f^{n-1}(x_0), x_0) - p\big(f^n(x_0), f^n(x_0)\big) \to 0 \text{ as } n \to \infty.$$

It follows that $p\big(x_0, f(x_0)\big) = 0 = p(x_0, x_0)$. The relations $0 \le p\big(f(x_0), f(x_0)\big) \le \alpha p(x_0, x_0) = 0$ imply $p\big(f(x_0), f(x_0)\big) = 0$, so that, by (PM1), $f(x_0) = x_0$. □

Remark 5.16. O'Neill [136] considered partial metrics that take values in $\mathbb{R}$ (not in $\mathbb{R}_+$ as in the case of Matthews' partial metric) and related them to domain theory. These kind of spaces are called by some authors *dualistic partial metric space*. The extension of the Banach fixed point theorem to this setting was given by Oltra and Valero [137] (see also [192]). In this case, the contraction condition is given by

$$\exists\, 0 \le \alpha < 1 \text{ such that } \big|p\big(f(x), f(y)\big)\big| \le \alpha |p(x, y)| \text{ for all } x, y \in X.$$

Extensions of various fixed point results from metric spaces to partial metric spaces were given by O. Valero in cooperation with other mathematicians, see [6, 7, 163, 164, 193] (see also [159]).

5.3. Topology and Order on Partial Metric Spaces. In this section, we shall examine the behavior of the specialization order (3.2) with respect to the topology $\tau(p)$ generated by a partial metric p.

Proposition 5.17. *Let (X, p) be a partial metric space and $\le_p$ the specialization order on X.*

(1) *The specialization order can be characterized by the following condition*:
$$x \le_p y \iff p(x, x) = p(x, y). \tag{5.13}$$
(2) *Every open ball $B_p(x, \varepsilon)$ is upward closed. Consequently, every τ_p-open sets is upward closed.*
(3) *The Alexandrov topology $\tau_{\mathrm{a}}(\le_p)$ generated by $\le_p$ (see Proposition 3.9) is finer than $\tau(p)$. The equality $\tau(p) = \tau_{\mathrm{a}}(\le_p)$ holds if and only if*
$$\forall\, x \in X\ \exists \varepsilon_x > 0\ \ B_p(x, \varepsilon_x) = \uparrow x. \tag{5.14}$$

Proof. (1) Suppose $x \le_p y$. By definition

$$x \le_p y \iff x \in \overline{\{y\}},$$

so that

$$\forall\, \varepsilon > 0\ \ \{y\} \cap B'_p(x, \varepsilon) \ne \varnothing \iff \forall\, \varepsilon > 0\ \ p(x, y) < \varepsilon + p(x, x) \implies p(x, y) \le p(x, x).$$

But, by (PM2), $p(x, x) \le p(x, y)$, and so $p(x, x) = p(x, y)$.

Conversely, if $p(x, x) = p(x, y)$, then $p(x, y) < \varepsilon + p(x, x)$ for all $\varepsilon > 0$, showing that $x \in \overline{\{y\}}$, i.e., $x \le_p y$.

(2) Let $y \in B_p(x, \varepsilon)$ and

$$y \le_p z \iff p(y, z) = p(y, y).$$

Then

$$p(x, z) \le p(x, y) + p(y, z) - p(y, y) = p(x, y) < \varepsilon,$$

i.e., $z \in B_p(x, \varepsilon)$.

Let $U \subset X$ be τ_p-open. Then for every $x \in U$ there exists $\varepsilon_x > 0$ such that $B_p(x, \varepsilon_x) \subset U$. If $x \in U$ and $x \le_p y$, then, since $B_p(x, \varepsilon_x)$ is upward closed, $y \in B_p(x, \varepsilon_x) \subset U$. Consequently, U is upward closed.

(3) Since the Alexandrov topology is the finest such that the induced order specialization agrees with $\le_p$ (Proposition 3.9), it follows that $\tau(p) \subset \tau(\le_p)$.

Now suppose that the condition (5.14) holds and let $Z \in \tau(\le_p)$. Since open sets are upward closed, it follows that

$$Z = \bigcup \{\uparrow x \colon x \in Z\} = \bigcup \{B_p(x, \varepsilon_x) \colon x \in Z\} \in \tau(p).$$

Consequently, $\tau(\le_p) \subset \tau(p)$, so that, taking into account the first statement in (2), $\tau(\le_p) = \tau(p)$.

Conversely, suppose that $\tau(\le_p) = \tau(p)$. Then for every $x \in X$, $\uparrow x \in \tau(p)$, implying the existence of $\varepsilon_x > 0$ such that $x \in B_p(x, \varepsilon_x) \subset \uparrow x$.

If $y \in \uparrow x$, then $p(x, y) = p(x, x) < \varepsilon_x$, i.e., $y \in B_p(x, \varepsilon_x)$, showing that $B_p(x, \varepsilon_x) = \uparrow x$. □

Remark 5.18. In terms of the specialization order $\le_p$ of a partial metric space (X, p), Remark 5.10 says in fact that if a sequence (x_n) in X converges to $x \in X$, then it converges to every y with $y \le_p x$. Also the equalities (5.11) say that if (x_n) converges properly to x and y, then $x \le_p y$ and $y \le_p x$, and so $x = y$.

5.4. The Specialization Order in Quasi-Metric Spaces. In this section, we shall describe the specialization order in a quasi-metric space.

Proposition 5.19. *Let (X, q) be a quasi-metric space.*

(1) *The specialization order $\le_q$ corresponding to q is given by*

$$x \le_q y \iff q(x, y) = 0. \tag{5.15}$$

(2) *Every open set is upward closed.*

Proof. (1) For $x, y \in X$,

$$x \le_q y \iff x \in \overline{\{y\}} \iff \forall\, \varepsilon > 0 \;\; y \in B_q(x, \varepsilon) \iff \forall\, \varepsilon > 0 \;\; q(x, y) < \varepsilon \iff q(x, y) = 0.$$

(2) Let us show first that an open ball $B_q(x, \varepsilon)$ is upward closed. Indeed, $y \in B_q(x, \varepsilon)$ and $y \le_q z$ imply

$$q(x, z) \le q(x, y) + q(y, z) = q(x, y) < \varepsilon.$$

Now if $U \subset X$ is τ_q-open, then for every $x \in U$ there exists $\varepsilon_x > 0$ such that $B_q(x, \varepsilon_x) \subset U$. If $x \le_q y$, then $y \in B_q(x, \varepsilon_x) \subset U$. □

Remark 5.20. If q is only a quasi-semimetric (see Definition 2.16), then (5.15) defines only a preorder $\le_q$, which is an order if and only if q is a quasi-metric.

Indeed,

$$(x \le_q y \wedge y \le_q x) \iff (q(x, y) = 0 \wedge q(y, x) = 0) \iff x = y.$$

A contraction principle holds in this case too. A mapping f on a quasi-metric space (X, q) is called a contraction if there exists $\alpha \in [0, 1)$ such that

$$q\big(f(x), f(y)\big) \le \alpha\, q(x, y), \tag{5.16}$$

for all $x, y \in X$.

Theorem 5.21 (contraction principle in quasi-metric spaces [126])**.** *Let (X, q) be a quasi-metric space such that the associated metric space (X, q^s) is complete. Then every contraction on (X, q) has a fixed point.*

5.5. Partial Metrics and Quasi-Metrics. In this section, we put in evidence some relations between partial metrics and quasi-metrics.

Proposition 5.22. *Let (X, p) be a partial metric space. Then the mapping $q\colon X^2 \to \mathbb{R}_+$ given by*

$$q(x, y) = p(x, y) - p(x, x), \quad x, y \in X, \tag{5.17}$$

is a quasi-metric on X. The topology $\tau(p)$ generated by p agrees with the topology $\tau(q)$ generated by q and the corresponding specialization orders $\le_p$ and $\le_q$ coincide as well.

Proof. It is a routine verification to show that the mapping q defined by (5.17) is a quasi-metric on X.

For $0 < \varepsilon \le p(x, x)$, $B_p(x, \varepsilon) = \varnothing \in \tau(q)$. If $\varepsilon > p(x, x)$, then $B_p(x, \varepsilon) = B_q\big(x, \varepsilon - p(x, x)\big) \in \tau(q)$, relations that imply $\tau(p) \subset \tau(q)$.

Since, for every $\varepsilon > 0$, $B_q(x, \varepsilon) = B_p\big(x, \varepsilon + p(x, x)\big) \in \tau(p)$, it follows that $\tau(q) \subset \tau(p)$.

Taking into account (5.13), we have

$$x \le_p y \iff p(x,y)=p(x,x) \iff q(x,y)=0 \iff \forall\, \varepsilon>0\ \ y\in B_q(x,\varepsilon)$$
$$\iff x\in \overline{\{y\}}^q \iff x\le_q y. \qquad \square$$

Remark 5.23. It follows that

$$q^s(x,y)=q(x,y)+q(y,x)=2q(x,y)-p(x,x)-p(y,y), \quad x,y\in X, \tag{5.18}$$

is a metric on X, called the associate metric to the partial metric p.

The next result shows that the completeness of the partial metric space (X,p) is equivalent to the completeness of the associate metric space (X,q^s).

Proposition 5.24. *Let (X,p) be a partial metric space, and q^s the associated metric to p given by* (5.18).

(1) *The convergence and completeness properties of the spaces (X,p) and (X,q^s) are related in the following way*:
  (i) *a sequence (x_n) in X is properly convergent to $x\in X$ if and only if $x_n \xrightarrow{q^s} x$;*
  (ii) *a sequence (x_n) in X is p-Cauchy if and only if it is q^s-Cauchy;*
  (iii) *the partial metric space (X,p) is complete if and only if the associated metric space (X,q^s) is complete.*

(2) *For any fixed $x\in X$ the mapping $p(x,\cdot)$ is q^s-lsc on X. The mapping $\beta\colon X\to[0,\infty)$, given by $\beta(x)=p(x,x)$, $x\in X$, is q^s-continuous* [151].

Proof. (1)(i) By definition

$$x_n \xrightarrow{q^s} x \iff p(x_n,x)-p(x,x)+p(x_n,x)-p(x_n,x_n)\to 0.$$

Since $p(x_n,x)-p(x,x)\ge 0$ and $p(x_n,x)-p(x_n,x_n)\ge 0$, the last condition from above is equivalent to

$$\begin{cases} p(x_n,x)\to p(x,x),\\ p(x_n,x)-p(x_n,x_n)\to 0\end{cases} \iff \begin{cases} p(x_n,x)\to p(x,x),\\ p(x_n,x_n)\to p(x,x),\end{cases}$$

i.e., to the fact that (x_n) converges properly to x.

(1)(ii) I. *Any p-Cauchy sequence is q^s-Cauchy.*

Let (x_n) be a p-Cauchy sequence in X, i.e.,

$$\lim_{m,n\to\infty} p(x_m,x_n)=a,$$

for some $a\in\mathbb{R}_+$. Then $\lim_{k\to\infty} p(x_k,x_k)=a$, so that

$$q^s(x_m,x_n)=2\,p(x_m,x_n)-p(x_n,x_n)-p(x_m,x_m)\to 0 \ \text{ as } \ m,n\to\infty,$$

which shows that the sequence (x_n) is q^s-Cauchy.

II. *Any q^s-Cauchy sequence is p-Cauchy.*

Let (x_n) be a q^s-Cauchy sequence in X, i.e.,

$$q^s(x_m,x_m)=p(x_m,x_n)-p(x_n,x_n)+p(x_m,x_n)-p(x_m,x_m)\to 0 \ \text{ as } \ m,n\to\infty,$$

which is equivalent to

$$0\le p(x_m,x_n)-p(x_n,x_n)\to 0 \ \text{ and } \ 0\le p(x_m,x_n)-p(x_m,x_m)\to 0 \tag{5.19}$$

as $m,n\to\infty$. By subtraction one obtains

$$p(x_m,x_m)-p(x_n,x_n)\to 0 \ \text{ as } \ m,n\to\infty. \tag{5.20}$$

Now we show that the net $\big(p(x_m,x_n)\big)_{(m,n)\in\mathbb{N}^2}$ is Cauchy in $\mathbb{R}_+$.

Let $\varepsilon > 0$. By (5.19) and (5.20) there exists $k_0 \in \mathbb{N}$ such that

$$0 \le p(x_m, x_n) - p(x_n, x_n) < \varepsilon,$$
$$0 \le p(x_{m'}, x_{n'}) - p(x_{n'}, x_{n'}) < \varepsilon,$$
$$|p(x_n, x_n) - p(x_{n'}, x_{n'})| < \varepsilon,$$

for all $m, n, m', n' \ge k_0$. Then

$$|p(x_m, x_n) - p(x_{m'}, x_{n'})| \le |p(x_m, x_n) - p(x_n, x_n)| + |p(x_n, x_n) - p(x_{n'}, x_{n'})| + |p(x_{n'}, x_{n'}) - p(x_{m'}, x_{n'})| < 3\varepsilon,$$

for all $m, n, m', n' \ge k_0$. It follows that the net $\big(p(x_m, x_n)\big)_{(m,n)\in\mathbb{N}^2}$ is Cauchy in $\mathbb{R}_+$, so it converges to some $a \in \mathbb{R}_+$, which means that the sequence (x_n) is p-Cauchy.

(1)(iii) This follows from the definition of the completeness of the partial metric space (X, p) and from (i) and (ii).

(2) Let $x, y \in X$ be fixed. If (y_n) is a sequence in X that is q^s-convergent to y, then by (1)(i), $\lim_n p(y_n, y) = p(y, y) = \lim_n p(y_n, y_n)$, so that $\lim_n [p(y_n, y) - p(y_n, y_n)] = 0$.

Passing to $\liminf$ in the inequality

$$p(x, y) \le p(x, y_n) + p(y_n, y) - p(y_n, y_n),$$

one obtains $p(x, y) \le \liminf_n p(x, y_n)$, which shows that $p(x, \cdot)$ is q^s-lsc at y.

Now let $x \in X$ be fixed and (x_n) be a sequence in X that is q^s-convergent to x. By the first assertion of the proposition, this is equivalent to the fact that (x_n) converges properly to x, which, by Definition 5.11, implies $\beta(x_n) = p(x_n, x_n) \to p(x, x) = \beta(x)$. □

Remark 5.25. Definition 5.14 of a Cauchy sequence in a partial metric space is taken from [127] (see also [37]). In [124], the following equivalent definition is proposed: a sequence (x_n) in a partial metric space (X, p) is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$0 \le p(x_n, x_m) - p(x_m, x_m) < \varepsilon,$$

for all $m, n \ge n_\varepsilon$.

Indeed, the relations (5.19) show that this is equivalent to the fact that (x_n) is q^s-Cauchy, which in turn is equivalent to the fact that (x_n) is p-Cauchy.

Remark 5.26. Another metric on a partial metric space (X, p) is given by $d(x, y) = 0$ if $x = y$ and $d(x, y) = p(x, y)$ for $x \ne y$. In this case, $\tau_{q^s} \subset \tau_d$, and the metric space (X, d) is complete if and only if the partial metric space (X, p) is complete. This result can be used to show that some fixed points results in partial metric spaces can be obtained directly from their analogues (in the metric) case, see [64]. A similar situation occurs in the case of the so called cone-metric spaces (see, for instance, the survey paper [83]).

5.6. The Existence of Suprema in Partial Metric Spaces. In this section, we shall prove that every increasing sequence in a partial metric space has a supremum and it is properly convergent to its supremum. We agree to call a mapping $f\colon (X_1, p_1) \to (X_2, p_2)$ *properly continuous* if $\big(f(x_n)\big)$ properly converges to $f(x)$ for every sequence (x_n) in X_1 properly convergent to x.

Proposition 5.27. *Let (X, p) be a partial metric space and $\le_p$ the specialization order corresponding to p.*

1. *If (X, p) is complete, then every increasing sequence $x_1 \le_p x_2 \le_p \dots$ in X has a supremum x and the sequence (x_n) converges properly to x.*
2. *Let (X_1, p_1) and (X_2, p_2) be complete partial metric spaces with the specialization orders $\le_1$ and $\le_2$, respectively, and $f\colon (X_1, p_1) \to (X_2, p_2)$ be a mapping. If f is properly continuous and monotonic, then f preserves suprema of increasing sequences, i.e., $\sup_n f(x_n) = f(x)$ for every increasing sequence $x_1 \le_1 x_2 \le_1 \dots$ in X_1 with $\sup_n x_n = x$*

Proof. (1) We show first that the sequence (x_n) is Cauchy. Indeed,

$$x_n \leq_p x_{n+k} \iff p(x_n, x_{n+k}) - p(x_n, x_n) = 0,$$

so that, taking into account Remark 5.25, it follows that (x_n) is Cauchy. The completeness hypothesis implies the existence of $x \in X$ such that the sequence (x_n) is properly convergent to x, i.e.,

$$\lim_n p(x, x_n) = p(x, x) = \lim_n p(x_n, x_n). \tag{5.21}$$

We show that $x = \sup_n x_n$, i.e.,

$$\begin{aligned} &x_n \leq x \text{ for all } n \in \mathbb{N};\\ &\text{if } x_n \leq y \text{ for all } n \in \mathbb{N}, \text{ then } x \leq y. \end{aligned} \tag{5.22}$$

We have for all $n, k \in \mathbb{N}$

$$p(x_n, x) \leq p(x_n, x_{n+k}) + p(x_{n+k}, x) - p(x_{n+k}, x_{n+k}) = p(x_n, x_n) + p(x_{n+k}, x) - p(x_{n+k}, x_{n+k}).$$

Letting $k \to \infty$ and taking into account (5.21), one obtains $p(x_n, x) \leq p(x_n, x_n)$, so that, by (PM2) from Definition 5.1, $p(x_n, x) = p(x_n, x_n)$, i.e., $x_n \leq_p x$.

Now suppose that $x_n \leq_p y$ for all $n \in \mathbb{N}$. Then

$$p(x, y) \leq p(x, x_n) + p(x_n, y) - p(x_n, x_n) = p(x_n, y) = p(x_n, x_n),$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ one obtains (by (5.21), $p(x, y) \leq p(x, x)$. It follows that $p(x, y) = p(x, x)$, i.e., $x \leq_p y$. Consequently, both conditions (i) and (ii) from (5.22) hold.

(2) Let $x_1 \leq_1 x_2 \leq_1 \dots$ be an increasing sequence in X with $\sup_n x_n = x$. Then (x_n) is p_1-properly convergent to x. Then the sequence $\big(f(x_n)\big)$ is $\leq_2$-increasing and properly convergent to $f(x)$. By (1), this implies that $\sup_n f(x_n) = f(x)$. □

Remark 5.28. It is possible that the property from the first statement of Proposition 5.27 characterizes the completeness of the partial metric space (X, p) (like in Theorem 4.12). Concerning the second statement, I do not know whether the Scott continuity is equivalent to the continuity of the mapping f.

5.7. Caristi's Fixed Point Theorem and Completeness in Partial Metric Spaces. In this section, we shall present, following [151], the equivalence of Caristi's fixed point theorem to the completeness of the underlying partial metric space.

Let (X, p) be a partial metric space. Recall the Caristi condition for a mapping $f\colon X \to X$:

$$p\big(x, f(x)\big) \leq \varphi(x) - \varphi\big(f(x)\big), \tag{Car$_\varphi$}$$

for all $x \in X$. Here φ is a function $\varphi\colon X \to \mathbb{R}$. According to the continuity properties of the function φ we distinguish two kinds of Caristi conditions. One says that the mapping f is

- p-Caristi if (Car$_\varphi$) holds for some p-lsc bounded from below function $\varphi\colon X \to \mathbb{R}$,
- q^s-Caristi if (Car$_\varphi$) holds for some q^s-lsc bounded from below function $\varphi\colon X \to \mathbb{R}$,

where q^s is the metric associated to p by (5.18).

As was shown in [151], the completeness of a partial metric space (X, p) cannot be characterized by the existence of fixed points of p-Caristi mappings.

Example 5.29. Consider the set $\mathbb{N}$ with the partial metric $p(m, n) = \max\{m^{-1}, n^{-1}\}$. The associated metric q^s is given by $q^s(m, n) = |m^{-1} - n^{-1}|$, $m, n \in \mathbb{N}$. If $0 < \varepsilon < [n(n+1)]^{-1}$, then $B_{q^s}(n, \varepsilon) = \{n\}$, i.e., the topology $\tau(q^s)$ is the discrete metric on $\mathbb{N}$, and so the only convergent sequences are the ultimately constant ones. The space $(\mathbb{N}, q^s)$ is not complete because the sequence $x_n = n$, $n \in \mathbb{N}$, is q^s-Cauchy and not q^s-convergent. On the other hand, there are no p-Caristi maps on $\mathbb{N}$.

To obtain a characterization of this kind, another notion is needed.

Definition 5.30. Let (X,p) be a partial metric space. A sequence (x_n) in X is called 0-*Cauchy* if and only if $\lim_{m,n\to\infty} p(x_m,x_n) = 0$. The partial metric space (X,p) is called 0-*complete* if every 0-Cauchy sequence (x_n) is convergent with respect to τ_p to some $x \in X$ such that $p(x,x) = 0$.

Remark 5.31. The above definition is given in [151]. Taking into account Proposition 5.24, the following assertions hold:

$$\begin{cases} \lim_{m,n\to\infty} p(x_n,x_m) = 0, \\ \lim_{n\to\infty} p(x,x_n) = p(x,x), \\ p(x,x) = 0 \end{cases} \iff \begin{cases} \lim_{m,n\to\infty} p(x_n,x_m) = 0, \\ \lim_{n\to\infty} p(x,x_n) = p(x,x), \\ p(x,x) = 0 = \lim_{n\to\infty} p(x_n,x_n) \end{cases} \implies \begin{cases} (x_n) \text{ is } q^s\text{-Cauchy}, \\ x_n \xrightarrow{q^s} x. \end{cases}$$

Consequently, a partial metric space (X,p) is 0-complete if and only if every 0-Cauchy sequence is properly convergent and if and only if every 0-Cauchy sequence is q^s-convergent.

Remark 5.32. It is obvious that a complete partial metric space is 0-complete, but the converse is not true (see [151]).

Notice also the following property.

Remark 5.33 ([1])**.** Let (X,p) be a partial metric space, (x_n) be a sequence in X and $x \in X$. If $\lim_{n\to\infty} p(x_n,x) = 0$, then $\lim_{n\to\infty} p(x_n,y) = p(x,y)$ for every $y \in Y$.

Indeed,

$$p(x_n,y) \le p(x_n,x) + p(x,y) - p(x,x) \le p(x_n,x) + p(x,y)$$

implies $p(x_n,y) - p(x,y) \le p(x_n,x)$, while

$$p(x,y) \le p(x,x_n) + p(x_n,y) - p(x_n,x_n) \le p(x,x_n) + p(x_n,y)$$

implies $p(x,y) - p(x_n,y) \le p(x,x_n)$.

Consequently,

$$|p(x,y) - p(x_n,y)| \le p(x,x_n) \to 0.$$

The characterization result is the following one.

Theorem 5.34 ([98,151])**.** *Let (X,p) be a partial metric space. Then (X,p) is 0-complete if and only if every q^s-Caristi mapping on X has a fixed point.*

Proof. Suppose that (X,p) is 0-complete and let $f\colon X \to X$ be a q^s-Caristi mapping for some q^s-lsc bounded from below function $\varphi\colon X \to \mathbb{R}$. For $x \in X$, let

$$A_x := \{y \in X : p(x,y) + \varphi(y) \le \varphi(x)\}.$$

Then, by (Car_φ), $f(x) \in A_x$ and A_x is q^s-closed because, by Proposition 5.24, the mapping $p(x,\cdot) + \varphi(\cdot)$ is q^s-lsc.

Starting with an arbitrary $x_0 \in X$, we shall construct inductively a sequence of q^s-closed sets A_{x_n} such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &x_k \in A_{x_{k-1}} \quad\text{and}\quad A_{x_k} \subset A_{x_{k-1}}, \\ &p(x_k,x) < \frac{1}{2^k} \quad\text{for all}\quad x \in A_{x_k}. \end{aligned} \tag{5.23}$$

Suppose that x_k and A_{x_k}, $k = 0,1,\ldots,n$, satisfy the conditions (5.23). Choose $x_{n+1} \in A_{x_n}$ such that

$$\varphi(x_{n+1}) < \inf \varphi\big(A_{x_n}\big) + \frac{1}{2^{n+1}}.$$

If $y \in A_{x_{n+1}}$, then

$$\begin{aligned} p(x_n,y) &\le p(x_n,x_{n+1}) + p(x_{n+1},y) - p(x_{n+1},x_{n+1}) \\ &\le \varphi(x_n) - \varphi(x_{n+1}) + \varphi(x_{n+1}) - \varphi(y) - p(x_{n+1},x_{n+1}) \le \varphi(x_n) - \varphi(y), \end{aligned}$$

which shows that $y \in A_{x_n}$, and so $A_{x_{n+1}} \subset A_{x_n}$.

For $x \in A_{x_{n+1}} \subset A_{x_n}$,

$$p(x_{n+1}, x) \le \varphi(x_{n+1}) - \varphi(x) \le \inf \varphi\big(A_{x_n}\big) + \frac{1}{2^{n+1}} - \varphi(x) \le \varphi(x) + \frac{1}{2^{n+1}} - \varphi(x) = \frac{1}{2^{n+1}}.$$

For $m > n$, $x_m \in A_{x_{m-1}} \subset A_{x_n}$, so that $p(x_n, x_m) < 1/2^n$, showing that the sequence (x_n) is 0-Cauchy. It follows that there exists $z \in X$ with $p(z,z) = 0$ such that

$$\lim_n p(x_n, z) = 0.$$

By Remark 5.31, $x_n \xrightarrow{q^s} z$. Since each set A_{x_n} is q^s-closed and $x_{n+k} \in A_{x_{n+k-1}} \subset A_{x_n}$ for all $k \in \mathbb{N}$, it follows that $z \in A_{x_n}$, for all $n \in \mathbb{N}$.

Also, the inequalities

$$p\big(x_n, f(z)\big) \le p(x_n, z) + p\big(z, f(z)\big) \le \varphi(x_n) - \varphi(z) + \varphi(z) - \varphi\big(f(z)\big) \le \varphi(x_n) - \varphi\big(f(z)\big)$$

show that

$$f(z) \in \bigcap_{n=1}^{\infty} A_{x_n}.$$

Consequently, $p\big(x_n, f(z)\big) < 1/2^n$ and, by the q^s-lsc of $p\big(\cdot, f(z)\big)$,

$$0 \le p\big(z, f(z)\big) \le \liminf_n p\big(x_n, f(z)\big) \le \lim_n \frac{1}{2^n} = 0,$$

so that $p\big(z, f(z)\big) = 0$. From

$$p\big(f(z), f(z)\big) \le p(f(z), z) + p\big(z, f(z)\big) - p(z,z) = 0$$

follows

$$p\big(z, f(z)\big) = p(z,z) = p\big(f(z), f(z)\big) = 0,$$

which implies $f(z) = z$.

To prove the converse, suppose that the partial metric space (X,p) is not 0-complete. Then there exists a 0-Cauchy sequence $(x_n)_{n=0}^{\infty}$ that is not properly convergent in (X,p). Passing, if necessary, to a subsequence, we can suppose further that the points x_n are pairwise distinct and

$$p(x_n, x_{n+1}) < \frac{1}{2^{n+1}} \quad \text{for all} \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{5.24}$$

Let

$$A := \{x_n \colon n \in \mathbb{N}_0\}.$$

By Proposition 5.24 the sequence (x_n) is q^s-Cauchy and not q^s-convergent, so it has no limit points, implying that the set A is q^s-closed.

Consider the functions $f \colon X \to X$ and $\varphi \colon X \to [0, \infty)$ given by

$$f(x) = \begin{cases} x_0 & \text{for } x \in X \setminus A, \\ x_{n+1} & \text{for } x = x_n,\ n \in \mathbb{N}_0, \end{cases} \quad \text{and} \quad \varphi(x) = \begin{cases} p(x_0, x) + 1 & \text{for } x \in X \setminus A, \\ 1/2^n & \text{for } x = x_n,\ n \in \mathbb{N}_0. \end{cases} \tag{5.25}$$

It is obvious that f has no fixed points.

I. *The function φ is q^s-lsc.*

Let (y_n) be a sequence in X q^s-convergent to some $y \in X$.

If $y \in X \setminus A$, then there exists $n_0 \in \mathbb{N}$ such that $y_n \in X \setminus A$ for all $n \ge n_0$. Since $p(x_0, \cdot)$ is q^s-lsc (Proposition 5.24), it follows that $\varphi(y) \le \liminf_n \varphi(y_n)$.

Now suppose that $y = x_k$ for some $k \in \mathbb{N}_0$ and denote by (y_{m_j}), $m_1 < m_2 < \ldots$, the terms of the sequence (y_n) that belong to A and by (y_{n_i}), $n_1 < n_2 < \ldots$, those in $X \setminus A$. If the set $\{m_j \colon j \in \mathbb{N}\}$ is infinite, then we must have $y_{m_j} = x_k$, $j \ge j_0$, for some $j_0 \in \mathbb{N}$, so that $\varphi(x_k) = 2^{-k} = \lim_j \varphi(y_{m_j})$.

If the set $\{n_i \colon i \in \mathbb{N}\}$ is infinite, then

$$\varphi(x_k) = \frac{1}{2^k} \leq 1 \leq \liminf_i [p(x_0, y_{n_i}) + 1] = \liminf_i \varphi(y_{n_i}). \tag{5.26}$$

Consequently, $\varphi(y) \leq \liminf_n \varphi(y_n)$ in all cases.

II. *f is a Caristi mapping with respect to φ.*

Indeed, if $x \in X \setminus A$, then $f(x) = x_0$ and

$$p\big(x, f(x)\big) = p(x, x_0) = \varphi(x) - 1 = \varphi(x) - \varphi\big(f(x)\big).$$

If $x = x_k$ for some $k \in \mathbb{N}_0$, then $f(x_k) = x_{k+1}$ and, by (5.24),

$$p\big(x_k, f(x_k)\big) = p(x_k, x_{k+1}) < \frac{1}{2^{k+1}} = \frac{1}{2^k} - \frac{1}{2^{k+1}} = \varphi(x_k) - \varphi\big(f(x_k)\big).$$

Consequently, f is a q^s-Caristi mapping without fixed points. □

Remark 5.35. Caristi-type fixed point theorems in complete partial metric spaces were also proved by Karapinar *et al.* in [21, 88]. Since a complete partial metric space is 0-complete, but the converse is not true (see [151]), these results follow from those proved by Romaguera [151]

Another definition of Caristi condition in partial metric spaces was given by Acar, Altun, and Romaguera [3]. A mapping $f \colon X \to X$ is called *AR-Caristi* if

$$p\big(x, f(x)\big) \leq p(x, x) + \varphi(x) - \varphi\big(f(x)\big), \tag{AR-Car$_\varphi$}$$

for some q^s-lsc bounded from below function $\varphi \colon X \to \mathbb{R}$.

Theorem 5.36 (Acar, Altun, and Romaguera [3])**.** *A partial metric space (X, p) is complete if and only if every AR-Caristi mapping on X has a fixed point.*

Proof. Suppose that (X, p) is complete. Let $f \colon X \to X$ be a mapping satisfying the condition (AR-Car$_\varphi$) for some q^s-lsc bounded from below function $\varphi \colon X \to \mathbb{R}$. By Proposition 5.24 the function $\beta \colon X \to [0, \infty)$ given by $\beta(x) = p(x, x)$, $x \in X$, is q^s-continuous, so that the function $\psi := \beta + 2\varphi$ is q^s-lsc and bounded from below (by $2 \inf \varphi(X)$).

Putting $\varphi = 2^{-1}(\psi - \beta)$ in (AR-Car$_\varphi$) and taking into account the definition (5.18) of the metric q^s associated to the partial metric p, one obtains

$$q^s\big(x, f(x)\big) \leq \psi(x) - \psi\big(f(x)\big). \tag{5.27}$$

Since, by Proposition 5.24 the metric space (X, q^s) is complete, we can apply Caristi's fixed point theorem (Theorem 2.6) to the mapping f and the q^s-lsc function ψ to conclude that f has a fixed point.

The proof of the converse follows the same line as that of the corresponding implication in Theorem 5.34.

Suppose that $(x_n)_{n \in \mathbb{N}_0}$ ($\mathbb{N}_0 = \{0, 1, 2, \dots\}$) is a Cauchy sequence in (X, p) that is not convergent. Passing to a subsequence, if necessary, we can suppose further that

$$p(x_n, x_{n+1}) - p(x_n, x_n) < \frac{1}{2^{n+1}}, \tag{5.28}$$

for all $n \in \mathbb{N}_0$ (see Remark 5.25). It follows that the set

$$A := \{x_n \colon n \in \mathbb{N}_0\}$$

is q^s-closed in (X, q^s).

Define the mappings $f \colon X \to X$ and $\varphi \colon X \to [0, \infty)$ by the formulas (5.25). Then φ is q^s-lsc. It is obvious that the mapping f has no fixed points, so it remains to show that it satisfies the condition (AR-Car$_\varphi$).

For $x \in X \setminus A$,

$$p\big(x, f(x)\big) = p(x, x_0) = \varphi(x) - \varphi\big(f(x)\big) \leq p(x, x) + \varphi(x) - \varphi\big(f(x)\big),$$

while for $x = x_n \in A$,

$$p\big(x_n, f(x_n)\big) = p(x_n, x_{n+1}) < p(x_n, x_{n+1}) + \frac{1}{2^{n+1}} = p(x_n, x_n) + \varphi(x_n) - \varphi\big(f(x_n)\big). \qquad \square$$

Remark 5.37. One can think to use the relations

$$\psi = \beta + 2\varphi \iff \varphi = \frac{1}{2}(\psi - \beta),$$

in the proof of the converse. Indeed, if (X, p) is not complete, then (X, q^s) is not complete (see Proposition 5.24), so, by Corollary 2.8, there exists a mapping $f\colon X \to X$ without fixed points that satisfies (5.27) for some q^s-lsc bounded from below function $\psi\colon X \to \mathbb{R}$. The function $\varphi = (1/2)(\psi - \beta)$ is q^s-lsc (because β is q^s-continuous) and replacing ψ by $\beta + 2\varphi$ in (5.27), one obtains (AR-Car$_\varphi$).

Unfortunately, we are not sure that the function $\varphi = (1/2)(\psi - \beta)$ is bounded form below in order to obtain a contradiction.

Remark 5.38. Caristi's fixed point theorem for set-valued mappings on partial metric spaces is discussed in a recent paper by Alsiary and Latif [9].

5.8. Ekeland's Variational Principle in Partial Metric Spaces. In this section, we shall show that in partial metric spaces Caristi's FPT is also equivalent to the weak Ekeland principle.

Theorem 5.39 (Ekeland's variational principle — weak form)**.** *Let (X, p) be a 0-complete partial metric space and $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a q^s-lsc bounded below proper function. Then for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that*

$$\forall\, x \in X \setminus \{x_\varepsilon\} \quad \varphi(x_\varepsilon) < \varphi(x) + \varepsilon p(x, x_\varepsilon). \tag{5.29}$$

Proof. Suppose on the contrary that there exists $\varepsilon > 0$ such that

$$\forall\, x \in X \ \exists\, y_x \in X \setminus \{x\} \ \text{ with } \ \varphi(x) \geq \varphi(y_x) + \varepsilon p(x, y_x). \tag{5.30}$$

Consider a point $x_0 \in X$ such that $\varphi(x_0) \leq \inf \varphi(X) + \varepsilon$, and let

$$Y := \{x \in X \colon \varphi(x) + \varepsilon p(x_0, x) \leq \varphi(x_0) + \varepsilon\, p(x_0, x_0)\}. \tag{5.31}$$

Since the function $\varphi(\cdot) + \varepsilon\, p(x_0, \cdot)$ is q^s-lsc (see Proposition 5.24), the set Y is p^s-closed, and so 0-complete. Indeed, if (x_n) is a 0-Cauchy sequence in Y, then it has a τ_p-limit $x \in X$ such that $p(x, x) = 0$. But this implies $x_n \xrightarrow{q^s} x$ (see Remark 5.31) and so $x \in Y$. Also $Y \neq \varnothing$ because $x_0 \in Y$ and φ is finite on Y (i.e., $\varphi(x) \in \mathbb{R}$ for all $x \in Y$).

Observe that the element y_x given by (5.30) belongs to Y for every $x \in Y$. Indeed, if $x \in Y$, then

$$\begin{aligned}\varphi(y_x) + \varepsilon p(x_0, y_x) &\leq \varphi(x) - \varepsilon p(x, y_x) + \varepsilon p(x_0, y_x)\\ &\leq \varphi(x_0) + \varepsilon\, p(x_0, x_0) + \varepsilon[p(x_0, y_x) - p(x_0, x) - p(x, y_x)] \leq \varphi(x_0) + \varepsilon\, p(x_0, x_0),\end{aligned}$$

because $p(x_0, y_x) - p(x_0, x) - p(x, y_x) \leq 0$. This last inequality follows from

$$p(x_0, y_x) \leq p(x_0, x) + p(x, y_x) - p(x, x) \leq p(x_0, x) + p(x, y_x).$$

Now put $\tilde{\varphi} := \varepsilon^{-1} \varphi|_Y \colon Y \to \mathbb{R}$, and let $f\colon Y \to Y$ be defined by $f(x) = y_x$, where, for $x \in Y$, $y_x \neq x$ is the element of Y satisfying (5.30).

Then the inequality (5.30) is equivalent to

$$p\big(x, f(x)\big) \leq \tilde{\varphi}(x) - \tilde{\varphi}\big(f(x)\big), \quad x \in Y,$$

which shows that f is a Caristi mapping with respect to $\tilde{\varphi}$. Since f has no fixed points, this is in contradiction to Caristi's fixed point theorem (Theorem 5.34). $\square$

We show now that the converse implication also holds.

Proposition 5.40. *Ekeland's variational principle in its weak form* (*Theorem* 5.39) *implies Caristi's fixed point theorem* (*Theorem* 5.34).

Proof. Let (X,p) be a 0-complete partial metric space, $\varphi\colon X \to \mathbb{R}$ be a q^s-lsc bounded from below function and $f\colon X \to X$ be a Caristi mapping with respect to φ. By Theorem 5.39 applied to φ for $\varepsilon = 1$ there exists a point $x_1 \in X$ such that

$$\varphi(x_1) < \varphi(x) + p(x_1, x),$$

for all $x \in X \setminus \{x_1\}$. Supposing $f(x_1) \neq x_1$, we can take $x = f(x_1)$ in the above inequality to obtain

$$p\big(x_1, f(x_1)\big) > \varphi(x_1) - \varphi\big(f(x_1)\big),$$

in contradiction to the inequality (Car_φ) satisfied by f.

Consequently, $f(x_1) = x_1$, i.e., x_1 is a fixed point of f. □

Remark 5.41. It follows that the validity of Ekeland's variational principle in its weak form, as given in Theorem 5.39, is also equivalent to the 0-completeness of the partial metric space (X,p).

We shall present now the version of Ekeland's variational principle that can be obtained from Theorem 5.36.

Theorem 5.42 (Ekeland's variational principle 2 — weak form)**.** *Let (X,p) be a complete partial metric space and $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper q^s-lsc bounded below proper function. Then for every $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that*

$$\forall\, x \in X \setminus \{x_\varepsilon\} \quad \varphi(x_\varepsilon) + \varepsilon p(x_\varepsilon, x_\varepsilon) < \varphi(x) + \varepsilon p(x, x_\varepsilon). \tag{5.32}$$

Proof. Suppose, by contradiction, that there exists an $\varepsilon > 0$ such that

$$\forall\, x \in X\ \exists\, y_x \in X \setminus \{x\} \ \text{ with } \ \varphi(x) + \varepsilon p(x,x) \geq \varphi(y_x) + \varepsilon p(x, y_x), \tag{5.33}$$

and let $x_0 \in X$ be such that $\varphi(x_0) \leq \varepsilon + \inf \varphi(X)$.

To get rid of the points where φ takes the value $+\infty$, consider again the set Y given by (5.31). Then Y is nonempty ($x_0 \in Y$) and q^s-closed and so complete with respect to the partial metric p. Indeed, if (x_n) is a Cauchy sequence in (X,p) then, by the definition of the completeness, it converges properly to some $x \in X$. By Proposition 5.24, (x_n) is q^s-convergent to x and so $x \in Y$.

Observe that $x \in Y$ implies that the element y_x given by (5.33) also belongs to Y. Indeed, if $x \in Y$, then

$$\begin{aligned}\varphi(y_x) + \varepsilon p(x_0, y_x) &\leq \varphi(x) + \varepsilon[p(x_0, y_x) - p(x, y_x) + p(x,x)]\\ &\leq \varphi(x_0) + \varepsilon[p(x_0,x_0) - p(x_0,x) + p(x_0,y_x) - p(x,y_x) + p(x,x)] \leq \varphi(x_0) + \varepsilon\, p(x_0,x_0),\end{aligned}$$

because

$$p(x_0, y_x) - p(x_0, x) - p(x, y_x) + p(x,x) \leq 0 \iff p(x_0,y_x) + p(x,x) \leq p(x_0,x) + p(x,y_x),$$

and the last inequality is true, by the triangle inequality (PM4) from Definition 5.1.

Taking again $\tilde{\varphi} = \varepsilon^{-1}\varphi|_Y$ and $f\colon Y \to Y$ defined by $f(x) = y_x$, where for $x \in Y$ the element $y_x \in Y$ is given by (5.33), the function $\tilde{\varphi}$ is q^s-lsc and f is a mapping on Y without fixed points, satisfying (AR-Car_φ) for $\varphi = \tilde{\varphi}$. □

The converse implication holds in this case too. The proof is similar to that of Proposition 5.40.

Proposition 5.43. *Ekeland's variational principle in its weak form, as given in Theorem* 5.42*, implies Caristi's fixed point theorem, as given in Theorem* 5.36*.*

Proof. Let (X,p) be a complete partial metric space, $\varphi\colon X \to \mathbb{R}$ be a q^s-lsc bounded from below function and $f\colon X \to X$ be a mapping satisfying (AR-Car_φ). Applying Theorem 5.39 to φ for $\varepsilon = 1$ there follows the existence of a point $x_1 \in X$ such that

$$\varphi(x_1) + p(x_1, x_1) < \varphi(x) + p(x_1, x),$$

for all $x \in X \setminus \{x_1\}$. Supposing $f(x_1) \neq x_1$, we can take $x = f(x_1)$ in the above inequality to obtain

$$p\big(x_1, f(x_1)\big) > p(x_1, x_1) + \varphi(x_1) - \varphi\big(f(x_1)\big),$$

in contradiction to the inequality (AR-Car$_\varphi$) satisfied by f.

Consequently, $f(x_1) = x_1$, i.e., x_1 is a fixed point of f. □

Remark 5.44. It follows that the validity of Ekeland's variational principle in its weak form, as given in Theorem 5.42, is equivalent to the completeness of the partial metric space (X, p).

Remark 5.45. A version of Ekeland's variational principle in partial metric spaces was proved by Aydi, Karapinar, and Vetro [22].

5.9. Dislocated Metric Spaces. This class of spaces was considered by Hitzler and Seda [65] in connection with some problems in logic programming. A *dislocated metric* on a set X is a function $\rho\colon X \times X \to \mathbb{R}_+$ satisfying the conditions

$$\begin{aligned} &\text{(DM1)} \quad \rho(x, y) = 0 \implies x = y, \\ &\text{(DM2)} \quad \rho(x, y) = \rho(y, x), \\ &\text{(DM3)} \quad \rho(x, y) \le \rho(x, z) + \rho(z, y), \end{aligned}$$

for all $x, y, z \in X$. If ρ satisfies only (DM1) and (DM3), then it is called a *dislocated quasi-metric*. The pair (X, ρ) is called a *dislocated metric* (respectively, a *dislocated quasi-metric*) *space*.

These spaces are close to partial metric spaces (in this case it is also possible that $\rho(x, x) > 0$ for some $x \in X$), with the exception that a dislocated metric satisfies the usual triangle inequality (DM3) instead of the inequality (PM4) from Definition 5.1. In fact, any partial metric is a dislocated metric.

For $x \in X$ and $r > 0$ the open ball $B(x, r)$ is defined by $B(x, r) = \{y \in X : \rho(x, y) < r\}$.

Hitzler and Seda [65] defined a kind of topology on a dislocated metric space (X, ρ) in the following way. They defined first a relation $\prec$ in $X \times 2^X$ as a substitute for the membership relation $\in$. For $(x, A) \in X \times 2^X$ put

$$x \prec A \iff \exists\, \varepsilon > 0 \ \text{ such that } \ B(x, \varepsilon) \subset A.$$

One defines the d-*neighborhood system* $\mathcal{V}(x)$ of a point $x \in X$ by the condition

$$V \in \mathcal{V}(x) \iff V \subset X \ \text{ and } \ x \prec V.$$

(Here "d" denotes "dislocated.")

The neighborhood axioms are satisfied with the relation $\prec$ instead of $\in$:

$$\begin{aligned} &\text{(V1)} \quad V \in \mathcal{V}(x) \implies x \prec V, \\ &\text{(V2)} \quad V \in \mathcal{V}(x) \text{ and } V \subset U \implies U \in \mathcal{V}(x), \\ &\text{(V3)} \quad U, V \in \mathcal{V}(x) \implies U \cap V \in \mathcal{V}(x), \\ &\text{(V4)} \quad V \in \mathcal{V}(x) \implies \exists\, W \in \mathcal{V}(x),\ W \subset V \text{ such that } V \in \mathcal{V}(y) \text{ for all } y \prec W. \end{aligned}$$

It is easy to check the validity of these properties. As a sample, let us check (V4). For $V \in \mathcal{V}(x)$ let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subset V$. If $y \prec B(x, \varepsilon)$, then there exists $\varepsilon' > 0$ such that $B(y, \varepsilon') \subset B(x, \varepsilon) \subset V$, so that $V \in \mathcal{V}(y)$. It follows that we can take $W = B(x, \varepsilon)$.

The so defined "neighborhood system" is not a proper neighborhood system (i.e., with respect to the relation $\in$), because the relation $x \in V$ is not always satisfied: we are not sure that $x \in B(x, \varepsilon)$ and further, the ball $B(x, \varepsilon)$ could be empty for some ε.

Example 5.46. Let X be a set of cardinality at least 2. Define $\rho(x, x) = 1$ and $\rho(x, y) = 2$ if $x \neq y$, for all $x, y \in X$. Then $B(x, \varepsilon) = \varnothing$ for $0 < \varepsilon \le 1$, implying that every subset of X (including the empty set) is a d-neighborhood of x.

In fact, the following properties hold.

Proposition 5.47 ([65, Proposition 3.2]). *Let (X,ρ) be a dislocated metric space.*

(1) *The following conditions are equivalent*:
 (i) *ρ is a metric,*
 (ii) *$\rho(x,x)=0$ for all $x\in X$.*
 (iii) *$B(x,\varepsilon)\neq\varnothing$ for all $x\in X$ and $\varepsilon>0$.*

(2) *The subset* $\ker\rho := \{x\in X\colon \rho(x,x)\}$ *is a metric space with respect to ρ.*

A sequence (x_n) in X is called d-*convergent* to $x\in X$ if

$$\forall\, V\in\mathcal{V}(x)\ \exists\, n_0\in\mathbb{N}\ \text{ such that }\ x_n\in V\ \text{ for all }\ n\geq n_0.$$

The sequence (x_n) in X is called *ρ-convergent* to x if $\lim\limits_{n\to\infty}\rho(x,x_n)=0$.

Remark 5.48. Note again that this type of convergence is not a proper convergence. For instance, if $\rho(x,x)>0$, then the constant sequence $x_n=x$, $n\in\mathbb{N}$, is not ρ-convergent to x.

The following property holds.

Proposition 5.49 ([65, Proposition 3.9]). *Let (X,ρ) be a dislocated metric space. A sequence (x_n) in X is ρ-convergent to $x\in X$ if and only if it is* d-*convergent to x.*

Proof. Suppose that (x_n) is d-convergent to x. For $\varepsilon>0$, $B(x,\varepsilon)$ is a d-neighborhood of x, so there exists $n_0\in\mathbb{N}$ such that $x_n\in B(x,\varepsilon)$ if and only if $\rho(x,x_n)<\varepsilon$, for all $n\geq n_0$, showing that $\rho(x,x_n)\to 0$.

Now suppose that $\rho(x,x_n)\to 0$. For $V\in\mathcal{V}(x)$, let $\varepsilon>0$ be such that $B(x,\varepsilon)\subset V$. By the hypothesis, there exists $n_0\in\mathbb{N}$ such that $\rho(x,x_n)<\varepsilon$ for all $n\geq n_0$. It follows that $x_n\in B(x,\varepsilon)\subset V$ for all $n\geq n_0$. □

A sequence (x_n) in X is called *ρ-Cauchy* if for every $\varepsilon>0$ there exists exists $n_0\in\mathbb{N}$ such that $\rho(x_n,x_m)<\varepsilon$ for all $m,n\geq n_0$. The dislocated metric space (X,ρ) is called *complete* if every Cauchy sequence is ρ-convergent. Hitzler and Seda [65, Theorem 2.7] proved that Banach's contraction principle holds in complete dislocated metric spaces.

Pasicki [143] defined a topology τ_ρ on a dislocated metric space (X,ρ) in the following way. The family of subsets $\{B(x,r)\colon x\in X,\ r>0\}$ satisfies

$$X=\bigcup\{B(x,r)\colon x\in X,\ r>0\},$$

so it is a subbase for a topology τ_ρ on X (see [94, Theorem 12, p. 47]).

It follows that a subset U of X is a neighborhood of $x\in X$ if and only if

$$\exists\, n\in\mathbb{N}\ \exists\, y_1,\dots,y_n\in X\ \exists\, r_1,\dots,r_n>0\ \text{ such that }\ x\in B(y_1,r_1)\cap\cdots\cap B(y_n,r_n)\subset U. \tag{5.34}$$

Denote by $\mathcal{U}_\rho(x)$ the neighborhood system of a point $x\in X$ with respect to τ_ρ.

Remark 5.50. Let (x_n) be a sequence in a dislocated metric space (X,ρ) and $x\in X$. If $\lim\limits_{n}\rho(x,x_n)=0$, then the sequence (x_n) is τ_ρ-convergent to $x\in X$.

Indeed, for any τ_ρ-neighborhood U of x there exists $y\in X$ and $\varepsilon>0$ such that $x\in B(y,\varepsilon)\subset U$. Then $\varepsilon-\rho(x,y)>0$ so that, by the hypothesis, there exists $n_0\in\mathbb{N}$ such that $\rho(x,x_n)<\varepsilon-\rho(x,y)$ for all $n\geq n_0$. It follows that

$$\rho(y,x_n)\leq\rho(y,x)+\rho(x,x_n)<\rho(x,y)+\varepsilon-\rho(x,y)=\varepsilon,$$

i.e., $x_n\in B(y,\varepsilon)\subset U$ for all $n\geq n_0$, showing that (x_n) is τ_ρ-convergent to x.

Remark 5.51. I do not know of a characterization of the τ_ρ-convergence in terms of the sequence $\big(\rho(x,x_n)\big)_{n\in\mathbb{N}}$.

Apparently unaware of Hitzler and Seda paper [65], Amini-Harandi [14] defined dislocated metric spaces calling them metric-like spaces. He defined the balls by analogy with partial metric spaces:

$$\tilde{B}(x,\varepsilon) = \{y \in X : |\rho(x,y) - \rho(x,x)| < \varepsilon\}.$$

The family of balls $\tilde{B}(x,r)$, $x \in X$, $\varepsilon > 0$, forms the base of a topology $\tilde{\tau}_\rho$ on the dislocated metric space (X,ρ).

A sequence (x_n) in X is $\tilde{\tau}_\rho$-convergent to $x \in X$ if and only if $\lim_{n\to\infty} \rho(x,x_n) = \rho(x,x)$.

A sequence (x_n) in X is called *Cauchy* if there exists the limit $\lim_{m,n\to\infty} \rho(x_n,x_m) \in \mathbb{R}$. The space (X,ρ) is called complete if for every Cauchy sequence (x_n) in X there exists $x \in X$ such that

$$\lim_{n\to\infty} \rho(x,x_n) = \rho(x,x) = \lim_{m,n\to\infty} \rho(x_m,x_n)$$

(compare with Sec. 5.2, Definition 5.11).

The paper [65] contains some fixed point theorems in complete dislocated metric spaces. The same approach is adopted in the paper [90] (and possibly in other papers).

Remark 5.52. In fact, in a preliminary version of the paper [143], Pasicki called these spaces near metric spaces. After the reviewer drew his attention to Hitzler and Seda paper, he changed them to dislocated metric spaces. There are a lot of papers dealing with fixed point results in dislocated metric spaces (or in metric like spaces), as can be seen by a simple search on MathSciNet, ZbMATH, or Google Scholar. I do not know if there are any converse results, i.e., completeness implied by the validity of some fixed point results.

5.10. Other Generalized Metric Spaces. In this section, we shall present some completeness results in other classes of generalized metric spaces: dislocated metric spaces, w-spaces and τ-spaces. Good surveys of various generalizations of metric spaces are given in the papers by Ansari [20], Berinde and Choban [31], and in the books [46], [98], and [160].

w-distances. This notion was introduced by Kada *et al.* [87]. Let (X,ρ) be a metric space. A mapping $p\colon X \times X \to \mathbb{R}_+$ is called a *w-distance* if, for all $x,y,z \in X$,

(w1) $p(x,y) \le p(x,z) + p(z,y)$,
(w2) $p(x,\cdot$ is ρ-lsc,
(w3) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $p(x,y) < \delta$ and $p(x,z) < \delta$ implies $p(y,z) < \varepsilon$.

τ-distances. A more involved notion was introduced by Suzuki [172]. Let (X,ρ) be a metric space and $\eta\colon X \times \mathbb{R}_+ \to \mathbb{R}_+$. A mapping $p\colon X \times X \to \mathbb{R}_+$ is called a *τ-distance* if

(τ1) $p(x,y) \le p(x,z) + p(z,y)$ for all $x,y,z \in X$,
(τ2) for every $x \in X$ the function $\eta(x,\cdot)$ is concave and continuous, $\eta(x,0) = 0$ and $\eta(x,t) \ge t$ for all $(x,t) \in X \times \mathbb{R}_+$,
(τ3) $\lim_n x_n = x$ and $\lim_n \Big(\sup_{m\ge n} \eta\big(z_n, p(z_n,x_m)\big)\Big) = 0$ imply $p(w,x) \le \liminf_n p(w,x_n)$ for all $w \in X$,
(τ4) $\lim_n \Big(\sup_{m\ge n} p(x_n,y_m)\Big) = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$,
(τ5) $\lim_n \eta\big(z_n, p(z_n,x_n)\big) = 0$ and $\lim_n \eta\big(z_n, p(z_n,y_n)\big) = 0$ imply $\lim_n \rho(x_n,y_n) = 0$.

Remark 5.53. It was shown in [172] that (τ2) can be replaced by

($\tau 2'$) for every $x \in X$ the function $\eta(x,\cdot)$ is increasing and $\inf_{t>0} \eta(x,t) = 0$.

Lin and Du [113, 115] propose a slightly simplified version of a τ-function.

Let (X,ρ) be a metric space. A mapping $p\colon X \times X \to \mathbb{R}_+$ is called a (LDτ)-*distance* if

(LD-τ1) $p(x,y) \le p(x,z) + p(z,y)$ for all $x,y,z \in X$,
(LD-τ2) for every $x \in X$ and every sequence (y_n) in X converging to some $y \in X$ if for some $M > 0$, $p(x,y_n) \le M$, for all n, then $p(x,y) \le M$,

(LD-τ3) if (x_n) and (y_n) are sequences in X such that $\lim_n\Big(\sup_{m\ge n} p(x_n,x_m)\Big)=0$ and $\lim_n p(x_n,y_n)=0$, then $\lim_n \rho(x_n,y_n)=0$,

(LD-τ4) for all $x,y,z\in X$, $p(x,y)=p(x,z)=0$ implies $y=z$.

Remark 5.54.

(1) If, for every $x\in X$, $p(x,\cdot)$ is lsc, then condition (LD-τ2) is satisfied.
(2) If p satisfies (LD-τ3), then every sequence (x_n) in X satisfying

$$\lim_n\Big(\sup_{m\ge n} p(x_n,x_m)\Big)=0$$

is a Cauchy sequence.

Lin and Du proved in [113,115] variational principles of Ekeland type for this kind of function, and for the w-distance in [114].

Tataru distance. This was defined by Tataru [191] in the following way. Let X be a subset of a Banach space E. A family $\{T(t)\colon t\in\mathbb{R}_+\}$ of mappings on X is called a *strongly continuous semigroup of nonexpansive mappings* on X if

(Sg1) for every $t\in\mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on X,
(Sg2) $T(0)x=x$ for all $x\in X$,
(Sg3) $T(s+t)=T(s)T(t)$ for all $s,t\in\mathbb{R}_+$,
(Sg4) for each $x\in X$ the mapping $T(\cdot)x\colon\mathbb{R}_+\to X$ is continuous.

The *Tataru distance* corresponding to a strongly continuous semigroup $\{T(t)\colon t\in\mathbb{R}_+\}$ of nonexpansive mappings on X is defined for $x,y\in X$ by

$$p(x,y)=\inf\{t+\|T(t)x-y\|\colon t\in\mathbb{R}_+\}. \tag{5.35}$$

It was shown by Suzuki [172,178] that any w-distance is a τ-distance, but the converse does not hold; for instance, the Tataru distance is a w-distance but not a τ-distance. The paper [178] contains many examples of w-distances and τ-distances, other τ-distances that are not w-distances, and conditions under which the Tataru distance is a τ-distance.

Various fixed point results, Ekeland-type principles, and completeness for τ-distances were proved by Suzuki in [97,172–176,179,180,182].

Fixed points for contractions and completeness results in quasi-metric spaces endowed with a w-distance were proved by Alegre *et al.* [5], for single-valued maps, and by Marín *et al.* [122], for set-valued ones. Similar results in the case of partial metric spaces were obtained by Altun and Romaguera [10].

A mapping f on a metric space (X,ρ) for which there exist a w-distance p on X and a number $\alpha\in[0,1)$ such that

$$p\big(f(x),f(x')\big)\le\alpha p(x,y)\quad\text{for all}\quad x,x'\in X, \tag{5.36}$$

is called *w-contractive.* In the case of a set-valued mapping $F\colon X\rightrightarrows X$, the condition (5.36) is replaced by

$$\forall\, x,x'\in X\ \exists\, y\in F(x)\ y'\in F(x')\text{ such that } p(y,y')\le\alpha p(x,y). \tag{5.37}$$

Direct and converse fixed point results involving completeness for w-contractive mappings and for other types of mappings (e.g., Kannan maps) on metric spaces endowed with a w-distance were proved in [38,72,87,165,183,185,187] (see also [98,186]). For instance, in [183] it is proved that a metric space X is complete if and only if every weakly contractive mapping on X has a fixed point. Also, the result of Borwein [34] (see Corollary 1.20), on the completeness of convex subsets of normed spaces on which every contraction has a fixed point, is rediscovered.

Branciari's distance — generalized metric spaces. Branciari [36] (see [161] for some corrections) introduced a new class of spaces, called *generalized metric spaces*, in the following way. A function $d\colon X \times X \to \mathbb{R}_+$, where X is a nonempty set, is called a *generalized metric* if the following conditions hold:

(GM1) $d(x, y) = 0 \iff x = y$,
(GM2) $d(x, y) = d(y, x)$,
(GM3) $d(x, y) \le d(x, u) + d(u, v) + d(v, y)$,

for all $x, y, u, v \in X$. The generalized triangle inequality (GM3) causes several troubles concerning the topology of these spaces (it is not always Hausdorff and the distance function $d(\cdot, \cdot)$ is continuous only under a supplementary condition, see [98, Chap. 13]) and the completeness. Branciari loc. cit. proved a Banach contraction principle within this context (some flaws in the original proof are corrected in [98, Chap. 13]).

Ghosh and Deb Ray [58] considered Suzuki's generalized contractions for these spaces and proved direct fixed point results as well as converse completeness results.

Probabilistic metric spaces. Completeness as well as relations between completeness and fixed point results in probabilistic metric spaces are explored in [4, 8, 62, 68]. We do not enter into the details of this matter.

Appendix. A Pessimistic Conclusion

In conclusion, we quote from the review of the paper [142].

MR835839 (87m:54125) Park, Sehie; Rhoades, B. E. *Comments on characterizations for metric completeness. Math. Japon.* 31 (1986), No. 1, 95–97.

> There are many papers in which the completeness of a metric space is characterized by using a fixed point theorem. In the present paper, the authors prove two very simple and general theorems that "**encompass some previous as well as future theorems of this type**."

(Reviewed by J. Matkowski)

Under these circumstances, it seems that the best we can hope to do in this domain is to prove some particular cases of these very general results.

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