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EQUIVALENT NORMS IN HILBERT SPACES WITH UNCONDITIONAL BASES OF REPRODUCING KERNELS

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In a Hilbert space H with an unconditional basis of reproducing kernels, we study the existence of a weighted integral norm with respect to an absolutely continuous measure, which is equivalent to the original H-norm. If the space H is defined via weighted integrals, the problem can be interpreted as restoring the original structure. Bibliography: 16 *titles.*

1 Introduction

Let H be a Hilbert space of entire functions satisfying the following conditions:

(1) H is functional, i.e., the evaluation functionals $\delta_z : f \to f(z)$ are continuous for every $z \in \mathbb{C}$,

(2) H possesses the division property, i.e., $F \in H$ and $F(z_0) = 0$ imply $F(z)(z - z_0)^{-1} \in H$.

In particular, condition (2) means that the evaluation functionals in H are different from zero and condition (1) means that each functional δ_z is generated by an element $k_z(\lambda) \in H$ in the sense that $\delta_z(f) = (f(\lambda), k_z(\lambda))$. The function $k(\lambda, z) = k_z(\lambda)$ is called a *reproducing kernel* of the space H. We denote $K(z) = k(z, z)$. Then the Bergman function for H is defined by

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$\|\delta_z\|_H = (K(z))^{\frac{1}{2}}$ [1]. A basis $\{e_k\}_{k=1}^{\infty}$ for a Hilbert space is called *unconditional* [2] if there are $c, C > 0$ such that for any $x = \sum\limits_{k=1}^{\infty} x_k e_k \in H$

$$c\sum_{j=1}^{\infty} |c_k|^2\|e_k\|^2 \leqslant \Big\| \sum_{j=1}^{\infty} c_k e_k \Big\|^2 \leqslant C\sum_{j=1}^{\infty} |c_k|^2\|e_k\|^2. \tag{1.1}$$

The study of unconditional bases of reproducing kernels for Hilbert spaces of entire functions is an actual problem of complex analysis. (Here and below, for the sake of brevity we write "basis of reproducing kernels" instead of "basis consisting of the values of reproducing kernels") Apparently, the problem in such a formulation was first considered in [3, 4], where the classical interpolation problem was studied for entire functions. Let $\{k(\lambda, \lambda_i)\}_{i=1}^{\infty}$ be an unconditional basis for a Hilbert space H satisfying conditions (1) and (2). Then the biorthogonal system consists of functions

$$L_k(\lambda) = \frac{L(\lambda)}{L'(\lambda_k)(\lambda - \lambda_k)}, \quad k = 1, 2, \dots ,$$

where $L(\lambda)$ is an entire function, called the *generating function*, with simple zeros λ_k, $k = 1, 2, \dots$. This system forms an unconditional basis. By properties of biorthogonal systems,

$$\|L_k\|^2 \asymp \frac{1}{K(\lambda_k)}, \quad k \in \mathbb{N}.$$

The expansion of a function F into the series with respect to this basis has the form

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k) L_k(\lambda), \quad \lambda \in \mathbb{C},$$

i.e., the Lagrange interpolation series with respect to the function L. The research was essentially developed in the paper [5], where radial Hilbert spaces with unconditional bases of reproducing kernels were introduced; namely, the weighted spaces with the weights $(\ln^+|z|)^{\alpha}$, $\alpha \in (1; 2]$. As proved [6, 7], unconditional bases also exist in the case of for more general weights.

Another approach to unconditional bases is to represent functions in the form of series of exponentials. If X is a Hilbert space, where the system $e^{\lambda z}$, $\lambda \in \mathbb{C}$, is complete, then, using the Fourier–Laplace transform $f \longrightarrow \widehat{f}(\lambda) := \langle e^{\lambda z}, f\rangle_z$, $\lambda \in \mathbb{C}$, we can represent X as a Hilbert space $\widehat{X}$ of entire functions. Moreover, the (unconditional) bases of exponentials for the space X correspond to the (unconditional) bases of reproducing kernels for the space $\widehat{X}$. Thus, the exponential bases in the classical space $L_2(-a, ; a)$ are transformed to the bases of reproducing kernels in the Paley–Wiener space of entire functions of exponential type a that are square integrable over the imaginary axis. Let D be a convex polygon in the plane, and let $E_2(D)$ be the Smirnov space of functions that are analytic in D and square integrable on the boundary. It is proved that this space has an unconditional basis of exponentials [8]. Using the Fourier–Laplace transform, one obtains unconditional bases of reproducing kernels for the space

$$\widehat{E}_2(D) = \left\{ \widehat{f}(\lambda) = \int\limits_{\partial D} \overline{f}(z) e^{z\lambda} ds(z), \quad f \in E_2(D) \right\}.$$

Let D be a convex polygon in the plane, and let $B_2(D)$ be the Bergman spaces of square integrable analytic functions in D. Then the space

$$\widehat{B}_2(D) = \left\{ \widehat{f}(\lambda) = \int\limits_D \overline{f}(z) e^{z\lambda} dm(z), \quad f \in B_2(D) \right\}$$

has an unconditional basis [9]. If $k(\lambda, \lambda_n)$ is an unconditional basis for a Hilbert space, then the condition (1.1) for the biorthogonal system L_n takes the form

$$c \sum_{j=1}^{\infty} |F(\lambda_k)|^2 \frac{1}{K(\lambda_k)} \leqslant \|F\|^2 \leqslant C \sum_{j=1}^{\infty} |F(\lambda_k)|^2 \frac{1}{K(\lambda_k)}.$$

Consider the measure $\mu = \sum\limits_{k=1}^{\infty} \delta_{\lambda_k}$. Then the H-norm is equivalent to the integral norm

$$\|F\|_1^2 = \int\limits_{\mathbb{C}} \frac{|F(\lambda)|^2}{K(\lambda)} d\mu(\lambda).$$

When constructing unconditional bases for the above Paley–Wiener spaces $\widehat{E}_2(D)$ and $\widehat{B}_2(D)$, auxiliary "intermediate" weighted integral norms equivalent to the original norms arise. These intermediate norms contain a valuable information about λ_k.

In this paper, we study the existence of an equivalent weighted integral norm with respect to an absolutely continuous measure in a Hilbert space H under the assumption that H has an unconditional bases of reproducing kernel. If the space H is defined via weighted integrals, then the problem can be interpreted as restoring the original structure. Theorem 1.1 below presents a necessary condition on the location of exponents λ_k for the intermediate norms to be equivalent to the original norm in the space. It turns out that this condition is also sufficient for the main spaces having unconditional bases of reproducing kernels (Theorem 1.2).

Assume that $\varphi_\alpha(\lambda) = \ln^\alpha(1 + |\lambda|)$ and introduce the spaces

$$\mathscr{F}_\alpha = \left\{ F \in H(\mathbb{C}) : \int\limits_{\mathbb{C}} |F(\lambda)|^2 e^{-2\varphi_\alpha(\lambda)} dm(\lambda) < \infty \right\}.$$

As shown in [5], the spaces $\mathscr{F}_\alpha$, $\alpha \in (1; 2]$, have unconditional bases of reproducing kernels. We note that some modifications of these spaces were considered in [6, 7].

If H is a HIlbert space with an unconditional basis $k(\lambda, \lambda_k)$, then the most suitable norm "restoring" the original norm is the integral norm with respect to the measure

$$\mu(\lambda) = \frac{1}{K(\lambda)} \sum_{k=1}^{\infty} \frac{\chi(\lambda, \lambda_k, r_k) dm(\lambda)}{r_k^2},$$

where $\chi(\lambda, \lambda_k, r_k)$ is the characteristic function of a disc $B(\lambda_k, r_k)$ and positive numbers r_k satisfy the condition

$$r_k \leqslant \frac{1}{2} \operatorname{dist}(\lambda_k, \mathscr{N} \setminus \{\lambda_k\}), \quad \mathscr{N} = \{\lambda_i\}_{i=1}^{\infty}.$$

Thus, for the sequence r_k we set

$$\|F\|_0^2 = \int\limits_{\mathbb{C}} |F(\lambda)|^2 d\mu(\lambda), \quad F \in H. \tag{1.2}$$

For positive functions f and g we write $f(x) \prec g(x)$, $x \in X$, if there exists a constant $C > 0$ such that $g(x) \leqslant Cf(x)$ for all $x \in X$. The symbols $\succ$ and $\asymp$ are understood in a similar way.

Theorem 1.1. *Let H be a functional Hilbert space with the unconditional basis $\{k(\lambda, \lambda_k)\}_{k=1}^{\infty}$ of reproducing kernels and generating function $L(\lambda)$. If for some sequence of positive number r_k, $k \in \mathbb{N}$, such that $r_k \leqslant \frac{1}{2}\operatorname{dist}(\lambda_k, \mathscr{N} \setminus \{\lambda_k\})$ the weighted integral norm* (1.2) *is equivalent to the H-norm, then*

$$\sum_{n \neq k} \frac{|L'(\lambda_n)|^2}{K(\lambda_n)} \frac{r_n^2}{|\lambda_n - \lambda_k|^2} \prec \frac{|L'(\lambda_k)|^2}{K(\lambda_k)}, \quad k \in \mathbb{N}. \tag{1.3}$$

Conversely, (1.3) *implies $\|F\| \prec \|F\|_0$ for $F \in H$ and $\|L_k\|_0 \asymp \|L_k\|$ for $k \in \mathbb{N}$.*

Theorem 1.2. (1) *In the Paley–Wiener space P, the norms of the form* (1.2) *are equivalent to the original P-norm if the condition* (1.3) *holds.*

(2) *Let D be a bounded convex polygon in the plane, and let θ_j be directions perpendicular to the polygon sides. If the exponents λ_k of the unconditional basis $\{k(\lambda, \lambda_k)\}_{k=1}^{\infty}$ for the Smirnov space $E_2(D)$ or the Bergman space $B_2(D)$ lie in some half-strips*

$$\{r^{i\varphi} : |\operatorname{Im} re^{(\varphi-\theta_j)i}| \leqslant T_j, \operatorname{Re} re^{(\varphi-\theta_j)i} > 0\}$$

and the condition (1.3) *holds, then the norms of the form* (1.2) *are equivalent to the original norm of the space.*

(3) *Assume that $\alpha \in (1;2]$, $1/\alpha + 1/\beta = 1$, $R_m = e^{\alpha^{1-\beta}(m+1)^{\beta-1}}$ and set $Q(R,q) = \{z : \frac{1}{q}R \leqslant |z| \leqslant qR\}$, $R > 0$, $q > 1$. If the exponents λ_n of the unconditional basis for the space $\mathscr{F}_\alpha$ with some q lie in the union of annuli $\bigcup\limits_{m=1}^{\infty} Q(R_m, q)$ and the condition* (1.3) *holds, then the norms of the form* (1.2) *are equivalent to the original norm of the space.*

Remark 1.1. The condition on the location of exponents in assertion (2) of Theorem 1.2 is necessarily satisfied by the bases for the Paley–Wiener space, which is established in [10]. Apparently, this condition seems is also necessary in the case of the Smirnov and Bergman spaces. The condition on the location of exponents in assertion (3) if Theorem 1.2 for bases in the spaces $\mathscr{F}_\alpha$ is necessarily satisfied in the case $\alpha = 2$.

2 Proof of Theorems

The main properties of the generating function are described in [11, 12].

For a continuous function u in the plane and $p > 0$ we denote by $\tau(u, z, p)$ the largest radius of discs with center z, where the function u deviates from harmonic functions by at most p:

$$\tau(u, z, p) = \sup\{r : \inf_H \{ \sup_{w \in B(z,r)} |u(w) - H(w)|, \ H \text{ is harmonic in } \ B(z,r)\} \leqslant p\}.$$

The function $\ln K(z) = 2\sup_{||F|| \leqslant 1} \ln |F(z)|$ is subharmonic and continuous on the whole plane (in view of the stability of $K(z) > 0$).

Theorem 2.1 (cf. [11, 12]). *Let $L(z)$ be the generating function of an unconditional basis. Then for some $P > 1$*

$$\frac{1}{P}K(z) \leqslant \sum_{i=1}^{\infty} \frac{|L(z)|^2 K(z_i)}{|L'(z_i)|^2 |z - z_i|^2} \leqslant PK(z). \tag{2.1}$$

Denote by $\tau(z)$ the function $\tau(\ln K(w), z, \ln(5P))$, where P is a constant in (2.1). Thus,

$$\inf\Big\{ \sup_{z\in\overline{B}(\lambda,\tau(\lambda))} |\ln K(z) - h(z)|, \quad h \text{ is harmonic in } B(z,\tau(z))\Big\} = \ln(5P).$$

Then the following assertions hold.

(1) *Any disc $B(z, 2\tau(z))$ contains at least one zero z_i of the function L.*

(2) *For any i, j, $i \neq j$,*

$$|z_i - z_j| \geqslant \frac{\max(\tau(z_i), \tau(z_j))}{10P^{\frac{3}{2}}}.$$

(3) *For any i*

$$\frac{1}{5^6 P^8} K(z) \leqslant \frac{K(z_i)|L(z)|^2}{|L'(z_i)|^2 |z - z_i|^2} \leqslant PK(z) \quad \text{in } B\Big(z_i, \frac{\tau(z_i)}{20P^{\frac{3}{2}}}\Big).$$

We denote by d_k the distance from λ_k to the set of other zeros of L: $d_k = \inf_{n\neq k} |\lambda_k - \lambda_n|$.

Proposition 2.1.

$$\frac{d_k}{10P^{\frac{3}{2}}} \leqslant \tau(\lambda_k) \leqslant 10P^{\frac{3}{2}} d_k.$$

Proof. The right inequality follows from Theorem 2.1 (2), and the left inequality is obtained from Theorem 2.1 (3) in view of $K(\lambda) > 0$. □

Proof of Theorem 1.1. We show that the condition (1.3) is necessary. If for a sequence $\{r_k\}_{k=1}^{\infty}$ the norm $\|F\|_0$ is equivalent to the original norm $\|F\|$, i.e., there are $m, M > 0$ such that $m\|F\|^2 \leqslant \|F\|_0^2 \leqslant M\|F\|^2$ for all $F \in H$, then

$$\|L_k\|_0^2 \asymp \frac{1}{K(\lambda_k)}, \quad k \in \mathbb{N}.$$

Since $r_n \leqslant \frac{1}{2}d_n$, for $\lambda \in B(\lambda_n, r_n)$, $n \neq k$, we have

$$|\lambda - \lambda_k| \leqslant |\lambda - \lambda_n| + |\lambda_n - \lambda_k| < 2|\lambda_n - \lambda_k|,$$

$$|\lambda_n - \lambda_k| \leqslant |\lambda_n - \lambda| + |\lambda - \lambda_k| \leqslant \frac{1}{2}d_n + |\lambda - \lambda_k| \leqslant 2|\lambda - \lambda_k|.$$

Consequently,

$$\frac{1}{2}|\lambda_n - \lambda_k| \leqslant |\lambda - \lambda_k| \leqslant 2|\lambda_n - \lambda_k|,$$

$$\frac{4r_n^2}{|\lambda_n - \lambda_k|} \geqslant \frac{1}{\pi r_n^2} \int\limits_{B(\lambda_n, r_n)} \frac{|\lambda - \lambda_n|^2}{|\lambda - \lambda_k|^2} dm(\lambda)$$

$$\geqslant \frac{1}{\pi r_n^2} \int\limits_{B(\lambda_n,r_n)\setminus B(\lambda_n,\frac{r_n}{2})} \frac{|\lambda - \lambda_n|^2}{|\lambda - \lambda_k|^2} dm(\lambda) \geqslant \frac{3}{64}\frac{r_n^2}{|\lambda_n - \lambda_k|}.$$

We assume that

$$r_k \leqslant \frac{\tau(\lambda_k)}{20P^{\frac{3}{2}}} := \widetilde{r}_k.$$

By Theorem 2.1 (3),

$$\frac{1}{\pi r_n^2}\int\limits_{B(\lambda_n,r_n)} |L_k(\lambda)|^2 \frac{1}{K(\lambda)} dm(\lambda) \asymp \frac{1}{|L'(\lambda_k)|^2}\frac{|L'(\lambda_n)|^2}{K(\lambda_n)}\frac{1}{\pi r_n^2}\int\limits_{B(\lambda_n,r_n)} \frac{|\lambda-\lambda_n|^2}{|\lambda-\lambda_k|^2} dm(\lambda),$$

where $n, k \in \mathbb{N}$, $n \neq k$. Thereby

$$\frac{1}{\pi r_n^2}\int\limits_{B(\lambda_n,r_n)} |L_k(\lambda)|^2 \frac{1}{K(\lambda)} dm(\lambda) \asymp \frac{1}{|L'(\lambda_k)|^2}\frac{|L'(\lambda_n)|^2}{K(\lambda_n)}\frac{r_n^2}{|\lambda_n-\lambda_k|},$$

where $n, k \in \mathbb{N}$, $n \neq k$. If $r_k > \widetilde{r}_k$, then from Proposition 2.1 it follows that

$$\frac{1}{\pi r_n^2}\int\limits_{B(\lambda_n,r_n)} |L_k(\lambda)|^2 \frac{1}{K(\lambda)} dm(\lambda) \succ \frac{1}{\pi \widetilde{r}_n^2}\int\limits_{B(\lambda_n,\widetilde{r}_n)} |L_k(\lambda)|^2 \frac{1}{K(\lambda)} dm(\lambda)$$

$$\succ \frac{1}{|L'(\lambda_k)|^2}\frac{|L'(\lambda_n)|^2}{K(\lambda_n)}\frac{r_n^2}{|\lambda_n-\lambda_k|},$$

where $n, k \in \mathbb{N}$, $n \neq k$. Summarizing the obtained inequalities with respect to $n \neq k$, we get

$$\sum_{n\neq k}\frac{|L'(\lambda_n)|^2}{K(\lambda_n)}\frac{r_n^2}{|\lambda_n-\lambda_k|^2} \prec \|L_k\|_0^2 |L'(\lambda_k)|^2 \prec \|L_k\|^2 |L'(\lambda_k)|^2 \prec \frac{|L'(\lambda_k)|^2}{K(\lambda_k)}, \quad k \in \mathbb{N}.$$

Thus, the necessity of the condition (1.3) is proved.

We prove the equivalence of the norms of L_k. The function $L_k(\lambda)$ differs from zero in $B(\lambda_k, r_k)$ and $L_k(\lambda_k) = 1$. Hence for any $F \in H$

$$|F(\lambda_k)|^2 \leqslant \frac{1}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{|L_k(\lambda)|^2} dm(\lambda).$$

By Theorem 2.1 (3), for $r_k \leqslant \widetilde{r}_k$

$$|F(\lambda_k)|^2 \prec \frac{K(\lambda_k)}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda), \quad k \in \mathbb{N}.$$

If $r_k > \widetilde{r}_k$, then

$$|F(\lambda_k)|^2 \prec \frac{K(\lambda_k)}{\pi \widetilde{r}_k^2}\int\limits_{B(\lambda_k,\widetilde{r}_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \prec \frac{K(\lambda_k)}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda), \quad k \in \mathbb{N},$$

in view of Proposition 2.1. Since $\{L_k\}_{k=1}^{\infty}$ is an unconditional basis, we have

$$\|F\|^2 \prec \sum_{k=1}^{\infty}\frac{|F(\lambda_k)|^2}{K(\lambda_k)} \prec \sum_{k=1}^{\infty}\frac{1}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) = \|F\|_0^2, \quad F \in H.$$

For L_k, $k \in \mathbb{N}$, the converse assertion is also valid: If $r_k \leqslant \widetilde{r}_k$, then

$$\|L_k\|_0^2 \geqslant \frac{1}{\pi r_k^2} \int\limits_{B(\lambda_k, r_k)} \frac{|L_k(\lambda)|^2}{K(\lambda)} dm(\lambda) \succ \frac{1}{K(\lambda_k)} \asymp \|L_k\|^2, \quad k \in \mathbb{N}.$$

If $r_k > \widetilde{r}_k$, then we use Proposition 2.1. Theorem 1.1 is proved. □

Proof of Theorem 1.2 (1). *The Paley–Wiener spaces.* By the Pavlov theorem [10], the exponents of an unconditional basis lie in some strip $|\operatorname{Re} z| \leqslant d$. As shown in [12], the function $\tau(z) = \tau(\ln K, z, p)$ is continuous and, in the case under consideration, depends only on $\operatorname{Re} z$. Since it is positive, it is not less than some positive constant in $[-d; d]$. By Theorem 2.1 (2), the sequence λ_k is separable, i.e., $\inf\limits_{k \neq n} |\lambda_k - \lambda_n| := \sigma > 0$. By Theorem 2.1 (1), $\sup\limits_{k \in \mathbb{N}} d_k < \infty$ and the sequence $\{r_k\}_{k=1}^{\infty}$ is bounded in the norm (1.2). The Bergman function of the space $P = \widehat{L}_2$ can be calculated by the formula

$$K(\lambda) = \int\limits_{-1}^{1} e^{2t \operatorname{Re} \lambda} dt.$$

Therefore, $K(\lambda) \asymp 1$ on any $[-q; q]$. Thus, we can write (1.3) as

$$\sum_{n \neq k} \frac{r_k^2 |L'(\lambda_n)|^2}{|\lambda_k - \lambda_n|^2} \prec |L'(\lambda_k)|^2, \quad k \in \mathbb{N}. \tag{2.2}$$

If (2.2) is valid, then from Theorem 1.1 it follows that $\|F\| \prec \|F\|_0$, $F \in P$. As known (cf., for example, [13]), the norm in the space P for any $d > 0$ is equivalent to the norm

$$\|F\|_d^2 = \int\limits_{-\infty}^{\infty} \int\limits_{-d}^{d} |F(x + iy)|^2 dx dy.$$

If $\sup\limits_{k \in \mathbb{N}} r_k := r$, then

$$\|F\|_0^2 = \sum_{k=1}^{\infty} \frac{1}{\pi r_k^2} \int\limits_{B(\lambda_k, r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \prec \sum_{k=1}^{\infty} \frac{4}{\pi d_k^2} \int\limits_{B(\lambda_k, d_k/2)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda)$$

$$\prec \sum_{k=1}^{\infty} \int\limits_{B\left(\lambda_k, \frac{d_k}{2}\right)} |F(\lambda)|^2 dm(\lambda) \prec \|F\|_{q+r}^2 \prec \|F\|^2, \quad F \in P.$$

The generating functions of the first unconditional basis for the Paley–Wiener space are sine type functions [8]. Since $|L'(\lambda_k)| \asymp 1$, $k \in \mathbb{N}$, for such functions, they satisfy (2.2).

2. *The Smirnov and Bergman spaces.* The structure of the spaces $\widehat{E}_2(D)$ and $\widehat{B}_2(D)$, regarded as normed spaces, is described in [14, 15]. These spaces are isomorphic to the spaces equipped with the norms

$$\|F\|^2 := \int\limits_0^{\infty} \int\limits_0^{2\pi} \frac{|F(re^{i\varphi})|^2}{K(re^{i\varphi})} d\Delta(\varphi) dr,$$

where $K(\lambda)$ is the squared norm of $e^{\lambda z}$ in the space $E_2(D)$ and, respectively, in $B_2(D)$. In other words, $K(\lambda)$ is the Bergman function of the spaces $\widehat{E}_2(D)$ and $\widehat{B}_2(D)$; moreover,

$$\Delta(\varphi) = h'(\varphi) + \int\limits_0^{\varphi} h(\theta)d\theta,$$

where $h(\varphi) = \max\limits_{z\in\overline{D}} \operatorname{Re} ze^{-i\varphi}$ is the support function of D. If D is a convex n-gon, then

$$d\Delta(\varphi) = \sum_{j=1}^{n} \frac{1}{b_j}\delta(\theta_j),$$

where $\delta(\theta_j)$ is the evaluation functional for the point θ_j and b_j is the side length. We consider the half-strips

$$\Pi_j(d) = \{\operatorname{Re} ze^{i(\varphi-\theta_j)} > 0,\ |\operatorname{Im} ze^{i(\varphi-\theta_j)}| < d\}.$$

By [8, 9], $\tau(\ln K(\lambda), z, p) \asymp 1$ in these strips. Therefore, in the norms of the form (1.2), it is assumed that $r_k \asymp 1$, $k \in \mathbb{N}$, and the set $\{\lambda_j\}_{j\in\mathbb{N}}$ is separable in view of Theorem 2.1 (2). Furthermore, by [9, Lemma 3],

$$\int\limits_{\Pi_j(d)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \prec \|F\|^2, \quad F \in \widehat{B}_2(D).$$

The corresponding assertion for $\widehat{E}_2(D)$ follows from [8]. Since the discs $B(\lambda_k, r_k)$ are pairwise disjoint and $r_k \leqslant d_k/2$, because of the separability and increase of averages we have

$$\|F\|_0^2 = \sum_{k=1}^{\infty} \frac{1}{\pi r_k^2} \int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \leqslant \sum_{k=1}^{\infty} \frac{4}{\pi r_k^2} \int\limits_{B(\lambda_k,\frac{d_k}{2})} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \prec \|F\|^2.$$

3. *The space* $\mathscr{F}_2$. We prove that the relation $\|F\|_0^2 \prec \|F\|^2$ always holds in $\mathscr{F}_2$. As shown in [5, Lemma 2.7], the Bergman function of the space $\mathscr{F}_2$ satisfies the condition

$$K(\lambda) \asymp \frac{e^{2\ln^2(1+|\lambda|)}}{1+|\lambda|^2}, \quad \lambda \in \mathbb{C}, \tag{2.3}$$

and $\tau(\lambda) \asymp |\lambda|$, $\lambda \in \mathbb{C}$. By Theorem 2.1 (2), $d_k \succ |\lambda_k|$ for $k \in \mathbb{N}$. Since the averages of subharmonic functions are increasing, from Theorem 2.1 (3) for $r_k \leqslant \widetilde{r}_k$ we have

$$\frac{1}{\pi r_k^2} \int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \asymp \frac{1}{\pi r_k^2} \int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{|L_k(\lambda)|^2} dm(\lambda)$$

$$\leqslant \frac{1}{\pi \widetilde{r}_k^2} \int\limits_{B(\lambda_k,\widetilde{r}_k)} \frac{|F(\lambda)|^2}{|L_k(\lambda)|^2} dm(\lambda) \asymp \frac{1}{\pi \widetilde{r}_k^2} \int\limits_{B(\lambda_k,\widetilde{r}_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda), \quad k \in \mathbb{N}.$$

By Proposition 2.1, $\widetilde{r}_k \asymp d_k \asymp |\lambda_k|$, $k \in \mathbb{N}$. Therefore,

$$\frac{1}{\pi r_k^2} \int\limits_{B(\lambda_k,r_k)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda) \prec \frac{1}{\pi |\lambda_k|^2} \int\limits_{B\left(\lambda_k,\frac{d_k}{2}\right)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda), \quad k \in \mathbb{N}.$$

From (2.3) we have

$$\frac{1}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)}\frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda)\prec\int\limits_{B(\lambda_k,\frac{d_k}{2})}\frac{|F(\lambda)|^2}{K(\lambda)(1+|\lambda|^2)}dm(\lambda)\asymp\int\limits_{B(\lambda_k,\frac{d_k}{2})}|F(\lambda)|^2e^{-2\ln^2(1+|\lambda|)}dm(\lambda),$$

where $k\in\mathbb{N}$. If $r_k>\widetilde{r}_k$, then

$$\begin{aligned}\frac{1}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)}\frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda)&\prec\frac{1}{\pi\widetilde{r}_k^2}\int\limits_{B(\lambda_k,\frac{d_k}{2})}\frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda)\\&\asymp\frac{4}{\pi d_k^2}\int\limits_{B(\lambda_k,\frac{d_k}{2})}|F(\lambda)|^2e^{-2\ln^2(1+|\lambda|)}dm(\lambda),\quad k\in\mathbb{N},\end{aligned}$$

in view of Proposition 2.1 and

$$\begin{aligned}\frac{1}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)}\frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda)&\prec\frac{4}{\pi d_k^2}\int\limits_{B(\lambda_k,\frac{d_k}{2})}|F(\lambda)|^2e^{-2\ln^2(1+|\lambda|)}dm(\lambda)\\&\asymp\int\limits_{B(\lambda_k,\frac{d_k}{2})}|F(\lambda)|^2e^{-2\ln^2(1+|\lambda|)}dm(\lambda),\quad k\in\mathbb{N}.\end{aligned}$$

Thus,

$$\|F\|_0^2=\sum_{k=1}^{\infty}\frac{1}{\pi r_k^2}\int\limits_{B(\lambda_k,r_k)}\frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda)\prec\sum_{k=1}^{\infty}\int\limits_{B\left(\lambda_k,\frac{d_k}{2}\right)}|F(\lambda)|^2e^{-2\ln^2(1+|\lambda|)}dm(\lambda)<\|F\|^2.$$

4. *The space* $\mathscr{F}_\alpha$, $\alpha\in(1;2)$. We find the asymptotics of the Bergman function $\mathscr{F}_\alpha$ for $\alpha\in(1;2)$. Since these spaces are radial, the monomials λ^n, $n\in\mathbb{N}\cup\{0\}$, form an orthogonal basis. Consequently,

$$K(\lambda)=\sum_{n=0}^{\infty}c_n|\lambda|^{2n},\tag{2.4}$$

where

$$\frac{1}{c_n}=\|\lambda^n\|^2=\int\limits_0^{\infty}t^{2n+1}e^{-2\ln^\alpha(1+t)}dt,\quad n\in\mathbb{N}.$$

Taking into account that

$$\int\limits_0^{\infty}t^{2n+1}e^{-2\ln^\alpha(1+t)}dt\asymp\int\limits_0^{\infty}e^{2(n+1)y-2y^\alpha}dy,\quad n\in\mathbb{N},$$

we calculate the asymptotics of the integral

$$\int\limits_0^{\infty}e^{2xy-2y^\alpha}dy,\quad x\in\mathbb{R}.$$

By [16, Theorem 2.4],

$$\int_0^\infty e^{2xy-2y^\alpha}dy \asymp e^{2A(\alpha)x^\beta}x^{\frac{\beta}{2}-1}, \quad x \in \mathbb{R},$$

where $\beta = \alpha/(\alpha-1)$ and $A(\alpha) = \alpha^{-\beta}(\alpha-1)$. Furthermore if $y(x)$ is the maximum point of $xy - y^\alpha$ and $I_x = \{y : |y - y(x)| \leqslant y(x)^{1-\alpha/2}\}$, then $y(x) = \alpha^{1-\beta}x^{\beta-1}$ and

$$\int_0^\infty e^{xy-y^\alpha}dy \asymp \int_{I(x)} e^{xy-y^\alpha}dy, \quad x \in \mathbb{R}.$$

Thus,

$$\frac{1}{c_n} \asymp \int_0^\infty t^{2n+1}e^{-2\ln^\alpha(1+t)}dt \asymp \int_0^\infty e^{2(n+1)y-2y^\alpha}dy \asymp e^{2A(\alpha)(n+1)^\beta}(n+1)^{\frac{\beta-2}{2}}, \quad n \in \mathbb{N}.$$

By (2.4), we have

$$K(r) \asymp \sum_{n=0}^\infty r^{2n}e^{-2A(\alpha)(n+1)^\beta}(n+1)^{1-\frac{\beta}{2}}, \quad r > 0,$$

and

$$K(r) \succ r^{2m}e^{-2A(\alpha)(m+1)^\beta}(m+1)^{1-\frac{\beta}{2}}, \quad r > 0,\ m \in \mathbb{N}. \tag{2.5}$$

Since $\tau_k \asymp d_k \asymp |\lambda_k|$, $k \in \mathbb{N}$, the number of points in $Q_m := Q(R_m, q)$ is bounded. Hence for any function $F \in H$

$$\sum_{\lambda_k \in Q_m} \frac{1}{\pi r_k^2} \int_{B(\lambda_k, r_k)} \frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda) \prec \frac{1}{R_m^2} \int_{Q(R_m,q)} \frac{|F(\lambda)|^2}{K(\lambda)}dm(\lambda), \quad m \in \mathbb{N}. \tag{2.6}$$

We apply the inequality (2.6) to the function z^n. By (2.5), we have

$$\begin{aligned} I_m(n) &:= \sum_{\lambda_k \in Q_m} \frac{1}{\pi r_k^2} \int_{B(\lambda_k, r_k)} \frac{|\lambda|^{2n}}{K(\lambda)}dm(\lambda) \prec \frac{1}{R_m^2} \int_{Q(R_m,q)} |\lambda|^{2(n-m)}e^{2A(\alpha)(m+1)^\beta}(m+1)^{\frac{\beta}{2}-1}dm(\lambda) \\ &\prec R_m^{2(n-m)}e^{2A(\alpha)(m+1)^\beta}(m+1)^{\frac{\beta}{2}-1}, \quad n, m \in \mathbb{N}. \end{aligned}$$

An elementary calculation shows that

$$c_n I_m(n) \prec e^{-2A(\alpha)[(n+1)^\beta-(m+1)^\beta-\beta(m+1)^{\beta-1}(n-m)]}\left(\frac{n+1}{m+1}\right)^{\frac{2-\beta}{2}}, \quad n, m \in \mathbb{N}.$$

By the Taylor formula for $x \geqslant y > 0$ and some $x_0 \in (y; x)$

$$\begin{aligned} y^\beta &= x^\beta + \beta x^{\beta-1}(y-x) + \frac{1}{2}\beta(\beta-1)x_0^{\beta-2}(y-x)^2 \\ &\geqslant x^\beta + \beta x^{\beta-1}(y-x) + \frac{1}{2}\beta(\beta-1)y^{\beta-2}(y-x)^2. \end{aligned}$$

Consequently, for $m \geqslant n$

$$(n+1)^\beta - (m+1)^\beta - \beta(m+1)^{\beta-1}(n-m) \geqslant \frac{1}{2}\beta(\beta-1)(n+1)^{\beta-2}(m-n)^2,$$

$$c_n I_m(n) \prec e^{-\alpha^{-\beta+1}(\beta-1)(n+1)^{\beta-2}(m-n)^2}\Big(1+\frac{m-n}{n+1}\Big)^{\frac{\beta-2}{2}}, \quad n \in \mathbb{N},\ m \geqslant n.$$

Thus,

$$c_n \sum_{m=n}^{\infty} I_m(n) \prec \sum_{k=0}^{\infty} e^{-\alpha^{-\beta+1}(\beta-1)k^2} k^{\frac{\beta-2}{2}} := C_1, \quad n \in \mathbb{N}.$$

Similarly, for $m < n$

$$(n+1)^\beta - (m+1)^\beta - \beta(m+1)^{\beta-1}(n-m) \geqslant \frac{1}{2}\beta(\beta-1)(m+1)^{\beta-2}(m-n)^2,$$

$$c_n \sum_{m=1}^{n-1} I_m(n) \prec \sum_{k=1}^{n-1} e^{-\alpha^{-\beta+1}(\beta-1)k^2} := C_2, \quad n \in \mathbb{N}.$$

Finally, we obtain the estimate

$$\|z^n\|_2^2 = \sum_{m=1}^{\infty} I_m(n) \prec \frac{C_1 + C_2}{c_n} = (C_1 + C_2)\|z^n\|^2, \quad n \in \mathbb{N},$$

where

$$\|F\|_2^2 := \frac{1}{R_m^2} \int\limits_{Q(R_m,q)} \frac{|F(\lambda)|^2}{K(\lambda)} dm(\lambda), \quad F \in H.$$

Since the system $\{z^n,\ n = 0, 1, \ldots\}$ is an orthogonal basis, we have

$$\|F\|_2^2 = \sum_{n=0}^{\infty} |f_n|^2 \|z^n\|_2^2 \prec \sum_{n=0}^{\infty} |f_n|^2 \|z^n\|^2 = \|F\|^2, \quad F \in H,$$

where f_n are the Taylor coefficients of the function F. In view of (2.6), $\|F\|_0^2 \prec \|F\|^2$ for $F \in H$. Theorem 1.2 is proved. □

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References

1. N. Aronszajn, “Theory of reproducing kernels”, *Trans. Am. Math. Soc.* **68** No. 3, 337–404 (1950).

2. S. V. Khrushchev, N. K. Nikol'skij, and B. S. Pavlov, "Unconditional bases of exponentials and of reproductional kernels", *Lect. Notes Math.* **864**, 214–335 (1981).

3. K. Seip, "Density theorems for sampling and interpolation in the Bargmann–Fock space. I", *J. Reine Angew. Math.* **429**, 91–106 (1992).

4. K. Seip and R. Wallsten, "Density theorems for sampling and interpolation in the Bargmann–Fock space. II", *J. Reine Angew. Math.* **429**, 107–113 (1992).

5. A. Borichev and Yu. Lyubarskii, "Riesz bases of reproducing kernels in Fock type spaces", *J. Inst. Math. Jussieu* **9**, No. 3, 449–461 (2010).

6. A. Baranov, A. Dumont, A. Hartmann. amd K. Kellay, "Sampling, interpolation and Riesz bases in small Fock spaces", *J. Math. Pures Appl.* **103**, No. 6, 1358–1389 (2015).

7. A. Baranov, Yu. Belov, and A. Borichev, "Fock type spaces with Riesz bases of reproducing kernels and de Branges spaces", *Stud. Math.* **236**, No. 2, 127–142 (2017).

8. B. Ya. Levin and Yu. I. Lyubarskij, "Interpolation by entire functions of special classes and related expansions into series of exponentials" [in Russian], *Izv. AN SSSR, Ser. Mat.* **39**, No. 3, 657–702 (1975).

9. K. P. Isaev, "Riesz bases of exponents in Bergman spaces on convex polygons" [in Russian], *Ufim. Math. Zhurn.* **2** (2010), No. 1, 71–86.

10. B. S. Pavlov, "Basicity of an exponential system and Muchkehoupt's condition," *Sov. Math. Dokl.* **20**, 655–659 (1979).

11. K. P. Isaev and R. S. Yulmukhametov, "On unconditional bases of exponentials in Hilbert spaces" [in Russian], *Ufim. Math. Zhurn.* **3**, No. 1, 3–15 (2011).

12. K. P. Isaev and R. S. Yulmukhametov, "Unconditional bases of reproducing kernels in Hilbert spaces of entire functions" [in Russian], *Ufim. Math. Zhurn.* **5**, No. 3, 67–77 (2013).

13. K. P. Isaev, R. S. Yulmukhametov, "On Hilbert spaces of entire functions with unconditional bases of reproducing kernels", *Lobachevskii J. Math.* **40**, No. 9, 1283–1294 (2019).

14. V. I. Lutsenko and R. S. Yulmukhametov, " A generalization of the Paley–Wiener theorem to functionals on Smirnov spaces," *Proc. Steklov Inst. Math.* **200**, 271–280 (1993).

15. K. P. Isaev and R. S. Yulmukhametov, "Laplace transform of functionals on Bergman spaces" *Izv. Math.* **68**, No. 1, 3–41 (2004).

16. M. V. Fedoryuk, *The Saddle-Point Method* [in Russian], Nauka, Moscow (1977).

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