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ON SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS WITH DRIFT AND ANOMALOUS DIFFUSION

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We study the solvability of certain linear nonhomogeneous elliptic equations and establish that, under some technical assumptions, the L^2-convergence of the right-hand sides yields the existence and convergence of solutions in an appropriate Sobolev space. The problems involve differential operators with or without Fredholm property, in particular, the one-dimensional negative Laplacian in a fractional power, on the whole real line or on a finite interval with periodic boundary conditions. We prove that the presence of the transport term in these equations provides regularization of the solutions. Bibliography: 23 titles.

1 Introduction

We consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$, $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant, and $V(x)$ converges to 0 at infinity. For $a \geqslant 0$ the essential spectrum of the operator $A : E \to F$ corresponding to the left-hand side of (1.1) contains the origin. Consequently, the operator does not possess the Fredholm property. For such operators the image is not closed and for $d > 1$ the kernel dimension and the image codimension are not finite. In the present paper, we study some properties of such operators. We note that elliptic equations with non-Fredholm operators were treated extensively in recent years (cf. [1]–[6]) along with their potential applications to the theory of reaction-diffusion problems (cf. [7, 8]). In the particular case $a = 0$, the operator A satisfies the Fredholm property in some properly chosen weighted spaces (cf. [6] and [9]–[12]). However, the case $a \neq 0$ is significantly different and the approach developed in the cited works cannot be applied.

One of the important issues about equations with non-Fredholm operators concerns their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of

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functions from $H^2(\mathbb{R}^d)$ such that $Au_n = f_n$, $n \in \mathbb{N}$. Since the operator A fails to satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n such that $Au_n \to f$ a *solution in the sense of sequences* of the problem $Au = f$ (cf. [13]). If this sequence converges to a function u_0 in the E-norm, then u_0 is a solution to this problem. A solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators, the convergence may not hold or it can occur in some weaker sense. Then the solution in the sense of sequences may not imply the existence of the usual solution. In the this paper, we obtain sufficient conditions for the equivalence of solutions in the sense of sequences and the usual solutions. In other words, we obtain conditions on the sequences f_n guaranteeing the strong convergence of the corresponding sequence u_n.

In the first part of the paper, we study the problem with the transport term

$$\Big(-\frac{d^2}{dx^2}\Big)^s u - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R}, \quad 0 < s < 1, \tag{1.2}$$

where $a \geqslant 0$ and $b \in \mathbb{R}$, $b \neq 0$, are constants and the right-hand side belongs to $L^2(\mathbb{R})$. The operator $\Big(-\frac{d^2}{dx^2}\Big)^s$ can be defined by means of the spectral calculus and is extensively used, for example, in the study of anomalous diffusion and related problems (cf. [14] and the references therein). Anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at infinity of the probability density function determines the value of the power of the Laplace operator (cf. [15]). The form boundedness criterion for the relativistic Schrödinger operator was proved in [16]. The article [17] deals with establishing the embedding theorems and the studies of the spectrum of a certain pseudodifferential operator. The equation with drift in the context of the Darcy law describing the fluid motion in a porous medium was treated in [4]. The transport term is significant when studying the emergence and propagation of patterns arising in the theory of speciation (cf. [18]). Nonlinear propagation phenomena for reaction-diffusion type equations including the drift term was studied in [19]. Weak solutions of the Dirichlet and Neumann problems with drift were considered in [20]. Apparently, the operator involved on the left-hand side of (1.2)

$$L_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a: \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad 0 < s \leqslant \frac{1}{2}, \tag{1.3}$$

$$L_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a: \quad H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}), \quad \frac{1}{2} < s < 1, \tag{1.4}$$

is nonselfadjoint. Using the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx, \quad p \in \mathbb{R}, \tag{1.5}$$

it can be easily derived that the essential spectrum of the operator $L_{a,\ b,\ s}$ is given by

$$\lambda_{a,\ b,\ s}(p) := |p|^{2s} - a - ibp, \quad p \in \mathbb{R}.$$

It is evident that, in the case $a > 0$, the operator $L_{a,\ b,\ s}$ is Fredholm since the origin does not belong to its essential spectrum. But if a vanishes, the operator $L_{0,\ b,\ s}$ does not satisfy the Fredholm property because its essential spectrum contains the origin.

Note that, in the absense of the transport term, we deal with the selfadjoint operator

$$\Big(-\frac{d^2}{dx^2}\Big)^s - a: \quad H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}), \quad a > 0,$$

which fails to satisfy the Fredholm property (cf. [21]). Let us write down the corresponding sequence of approximate equations with $m \in \mathbb{N}$ as

$$\Big(-\frac{d^2}{dx^2}\Big)^s u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \quad 0 < s < 1, \tag{1.6}$$

where the right-hand sides tend to the right-hand side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$. The inner product of two functions is defined by

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx, \tag{1.7}$$

with a slight abuse of notation when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, then the integral on the right-hand side of (1.7) makes sense like, for example, in the case of functions involved in the orthogonality conditions (1.10) and (1.11) of Theorems 1.1 and 1.2 below. For the problems on the finite interval $I := [0, 2\pi]$ with periodic boundary conditions, we use the inner product analogous to (1.7), replacing the real line with I. In the first part of the present work, we consider the spaces $H^1(\mathbb{R})$ and $H^{2s}(\mathbb{R})$, $0 < s < 1$, equipped with the norms

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \Big\|\frac{du}{dx}\Big\|^2_{L^2(\mathbb{R})}, \tag{1.8}$$

$$\|u\|^2_{H^{2s}(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \Big\|\Big(-\frac{d^2}{dx^2}\Big)^s u\Big\|^2_{L^2(\mathbb{R})} \tag{1.9}$$

respectively. Using the norms in $H^1(I)$ and $H^{2s}(I)$, $0 < s < 1$, in the second part of the article, we replace $\mathbb{R}$ with I in formulas (1.8) and (1.9) respectively. Our first main proposition is as follows.

Theorem 1.1. *Let $f(x) : \mathbb{R} \to \mathbb{R}$, $f(x) \in L^2(\mathbb{R})$.*

(a) *If $a > 0$ and $0 < s \leqslant 1/2$, then Equation* (1.2) *has a unique solution $u(x) \in H^1(\mathbb{R})$.*

(b) *If $a > 0$ and $1/2 < s < 1$, then Equation* (1.2) *has a unique solution $u(x) \in H^{2s}(\mathbb{R})$.*

(c) *If $a = 0$, $0 < s < 1/4$, and, in additon, $f(x) \in L^1(\mathbb{R})$, then Equation* (1.2) *has a unique solution $u(x) \in H^1(\mathbb{R})$.*

(d) *If $a = 0$, $1/4 \leqslant s \leqslant 1/2$, and, in addition, $xf(x) \in L^1(\mathbb{R})$, then Equation* (1.2) *has a unique solution $u(x) \in H^1(\mathbb{R})$ if and only if*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0. \tag{1.10}$$

(e) *If $a = 0$, $1/2 < s < 1$, and, in addition, $xf(x) \in L^1(\mathbb{R})$, then Equation* (1.2) *has a unique solution $u(x) \in H^{2s}(\mathbb{R})$ if and only if the orthogonality relation* (1.10) *holds.*

It is evident that the expression on the left-hand side of (1.10) is well defined by simple arguments analogous to the proof of Fact 1 in [2]. We formulate the result on the solvability in the sense of sequences on the whole real line.

Theorem 1.2. *Assume that $m \in \mathbb{N}$, $f_m(x) : \mathbb{R} \to \mathbb{R}$, and $f_m(x) \in L^2(\mathbb{R})$. Let $f_m(x) \to f(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$.*

(a) *If $a > 0$ and $0 < s \leqslant 1/2$, then Equations (1.2) and (1.6) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$ such that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$.*

(b) *If $a > 0$ and $1/2 < s < 1$, then Equations (1.2) and (1.6) have unique solutions $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R})$ such that $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$.*

(c) *if $a = 0$, $0 < s < 1/4$, and, in addition, $f_m(x) \in L^1(\mathbb{R})$ and $f_m(x) \to f(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$, then Equations (1.2) and (1.6) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$ such that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$.*

(d) *Assume that $a = 0$, $1/4 \leqslant s \leqslant 1/2$, $xf_m(x) \in L^1(\mathbb{R})$, and $xf_m(x) \to xf(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$. Let*
$$(f_m(x), 1)_{L^2(\mathbb{R})} = 0, \quad m \in \mathbb{N}. \tag{1.11}$$
Then Equations (1.2) and (1.6) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$ such that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$.

(e) *Assume that $a = 0$, $1/2 < s < 1$, and, in addition, $xf_m(x) \in L^1(\mathbb{R})$ and $xf_m(x) \to xf(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$. Let the orthogonality relations (1.11) hold. Then Equations (1.2) and (1.6) have unique solutions $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R})$ such that $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$.*

Note that the orthogonality conditions are not used in assertions (a) and (b) of Theorems 1.1 and 1.2, unlike assertion (e) of Theorems 1.1 and 1.2 in [21]. Another issue is that we establish the solvability of our equations in $H^1(\mathbb{R})$ for $0 < s \leqslant 1/2$ in Theorems 1.1 and 1.2, whereas in cases (a) and (e) in Theorems 1.1 and 1.2 of [21], the solvability was established without a transport term only in $H^{2s}(\mathbb{R})$. Finally, in Theorem 1.1 (e) and Theorem 1.2(e) above, only one orthogonality condition is requred, unlike assertion (a) in Theorems 1.1 and 1.2 of [21], where the second orthogonality relation is required for $s \in [3/4, 1)$ along with the assumption that $x^2 f(x)$, $x^2 f_m(x) \in L^1(\mathbb{R})$, $m \in \mathbb{N}$. Hence the introduction of the transport term provides the regularization for the solutions to the equations under consideration.

In the second part of the paper, we study the above equation on the finite interval $I := [0,\ 2\pi]$ with periodic boundary conditions
$$\Big(-\frac{d^2}{dx^2} \Big)^s u - b\frac{du}{dx} - au = f(x), \quad x \in I, \tag{1.12}$$
where $a \geqslant 0$ and $b \in \mathbb{R}$, $b \neq 0$, are constants and the right-hand side of (1.12) is bounded and periodic. It is obvious that
$$\|f\|_{L^1(I)} \leqslant 2\pi \|f\|_{L^\infty(I)} < \infty, \quad \|f\|_{L^2(I)} \leqslant \sqrt{2\pi}\|f\|_{L^\infty(I)} < \infty. \tag{1.13}$$
Thus, $f(x) \in L^1(I) \cap L^2(I)$. We use the Fourier transform
$$f_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}, \tag{1.14}$$

such that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

It is clear that the nonselfadjoint operator on the left-hand side of (1.12)

$$\mathscr{L}_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a: \quad H^1(I) \to L^2(I), \quad 0 < s \leqslant \frac{1}{2}, \tag{1.15}$$

$$\mathscr{L}_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a: \quad H^{2s}(I) \to L^2(I), \quad \frac{1}{2} < s < 1, \tag{1.16}$$

is Fredholm. By (1.14), it is easy to verify that the spectrum of $\mathscr{L}_{a,\ b,\ s}$ is given by $\lambda_{a,\ b,\ s}(n) := |n|^{2s} - a - ibn$, $n \in \mathbb{Z}$, and the corresponding eigenfunctions are the Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. The eigenvalues of the operator $\mathscr{L}_{a,\ b,\ s}$ are simple, unlike the situation without the transport term, where the eigenvalues corresponding to $n \neq 0$ are double-degenerate. The appropriate function spaces $H^1(I)$ and $H^{2s}(I)$ are

$$\{u(x): I \to \mathbb{R} \mid u(x),\ u'(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\},$$

$$\Big\{u(x): I \to \mathbb{R} \mid u(x),\ \Big(-\frac{d^2}{dx^2}\Big)^s u(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\Big\}$$

respectively. For the technical purposes we introduce the auxiliary constrained subspaces

$$H_0^1(I) = \{u(x) \in H^1(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{1.17}$$

$$H_0^{2s}(I) = \{u(x) \in H^{2s}(I) \mid (u(x), 1)_{L^2(I)} = 0\} \tag{1.18}$$

which are Hilbert spaces (cf., for example, [22, Chapter 2.1]). It is clear that for $a > 0$ the kernel of the operator $\mathscr{L}_{a,\ b,\ s}$ is trivial. In the case $a = 0$, we consider

$$\mathscr{L}_{0,\ b,\ s}: \quad H_0^1(I) \to L^2(I), \quad 0 < s \leqslant \frac{1}{2},$$

$$\mathscr{L}_{0,\ b,\ s}: \quad H_0^{2s}(I) \to L^2(I), \quad \frac{1}{2} < s < 1.$$

It is evident that such an operator has the trivial kernel. We write the corresponding sequence of the approximate equations with $m \in \mathbb{N}$:

$$\Big(-\frac{d^2}{dx^2}\Big)^s u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in I, \tag{1.19}$$

where the right-hand sides are bounded, periodic and converge to the right-hand side of (1.12) in $L^\infty(I)$ as $m \to \infty$. The purpose of Theorems 1.3 and 1.4 below is to demonstrate the formal similarity of the results on the finite interval with periodic boundary conditions to the ones derived for the whole real line situation in Theorems 1.1 and 1.2.

Theorem 1.3. *Assume that $f(x): I \to \mathbb{R}$, $f(0) = f(2\pi)$, $f(x) \in L^\infty(I)$.*

(a) *If $a > 0$ and $0 < s \leqslant 1/2$, then Equation (1.12) has a unique solution $u(x) \in H^1(I)$.*

(b) *If $a > 0$ and $1/2 < s < 1$, then Equation* (1.12) *has a unique solution $u(x) \in H^{2s}(I)$.*

(c) *If $a = 0$ and $0 < s \leqslant 1/2$, then Equation* (1.12) *has a unique solution $u(x) \in H^1_0(I)$ if and only if*

$$(f(x), 1)_{L^2(I)} = 0. \tag{1.20}$$

(d) *If $a = 0$ and $1/2 < s < 1$, then Equation* (1.12) *has a unique solution $u(x) \in H^{2s}_0(I)$ if and only if the orthogonality relation* (1.20) *holds.*

Our final main statement deals with the solvability in the sense of sequences on the finite interval I.

Theorem 1.4. *Let $m \in \mathbb{N}$ and $f_m(x) : I \to \mathbb{R}$ be such that $f_m(0) = f_m(2\pi)$. Furthermore, $f_m(x) \in L^\infty(I)$ and $f_m(x) \to f(x)$ in $L^\infty(I)$ as $m \to \infty$.*

(a) *If $a > 0$ and $0 < s \leqslant 1/2$, then Equations* (1.12) *and* (1.19) *have unique solutions $u(x) \in H^1(I)$ and $u_m(x) \in H^1(I)$ such that $u_m(x) \to u(x)$ in $H^1(I)$ as $m \to \infty$.*

(b) *If $a > 0$ and $1/2 < s < 1$, then Equations* (1.12) *and* (1.19) *have unique solutions $u(x) \in H^{2s}(I)$ and $u_m(x) \in H^{2s}(I)$ such that $u_m(x) \to u(x)$ in $H^{2s}(I)$ as $m \to \infty$.*

(c) *IF $a = 0$, $0 < s \leqslant 1/2$, and*

$$(f_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}, \tag{1.21}$$

then Equations (1.12) *and* (1.19) *have unique solutions $u(x) \in H^1_0(I)$ and $u_m(x) \in H^1_0(I)$ such that $u_m(x) \to u(x)$ in $H^1_0(I)$ as $m \to \infty$.*

(d) *If $a = 0$, $1/2 < s < 1$, and the orthogonality relations* (1.21) *hold, then EQuations* (1.12) *and* (1.19) *have unique solutions $u(x) \in H^{2s}_0(I)$ and $u_m(x) \in H^{2s}_0(I)$ such that $u_m(x) \to u(x)$ in $H^{2s}_0(I)$ as $m \to \infty$.*

Note that the orthogonality relations are not needed in assertions (a) and (b). If there is no transport term in the problems under consideration, the situation is more singular (cf. formulas (3.2) and (3.8) below with $a = n_0^{2s}$, $n_0 \in \mathbb{N}$).

2 The Whole Real Line Case

Proof of Theorem 1.1. We first show that it suffices to solve the equation in $L^2(\mathbb{R})$. Indeed, if $u(x)$ is a square integrable solution to Equation (1.2), we have

$$\Big(-\frac{d^2}{dx^2}\Big)^s u - b\frac{du}{dx} \in L^2(\mathbb{R}).$$

Using the standard Fourier transform (1.5), we derive $(|p|^{2s} - ibp)\widehat{u}(p) \in L^2(\mathbb{R})$, so that

$$\int_{-\infty}^{\infty} (|p|^{4s} + b^2p^2)|\widehat{u}(p)|^2 dp < \infty. \tag{2.1}$$

Let $0 < s \leqslant 1/2$. From (2.1) we easily deduce that

$$\int_{-\infty}^{\infty} p^2 |\widehat{u}(p)|^2 dp < \infty.$$

Hence $\dfrac{du}{dx} \in L^2(\mathbb{R})$ and $u(x) \in H^1(\mathbb{R})$.

Let $1/2 < s < 1$. Then (2.1) yields

$$\int_{-\infty}^{\infty} |p|^{4s} |\widehat{u}(p)|^2 dp < \infty.$$

Therefore, $\Big(-\dfrac{d^2}{dx^2}\Big)^s u \in L^2(\mathbb{R})$ so that $u(x) \in H^{2s}(\mathbb{R})$.

Let us prove the uniqueness of a solution to Equation (1.2) in the case $0 < s \leqslant 1/2$. For $1/2 < s < 1$ the proof is similar. Assume that $u_1(x),\ u_2(x) \in H^1(\mathbb{R})$ satisfy (1.2). Then their difference $w(x) := u_1(x) - u_2(x) \in H^1(\mathbb{R})$ solves the homogeneous problem

$$\Big(-\frac{d^2}{dx^2}\Big)^s w - b\frac{dw}{dx} - aw = 0.$$

Since the operator $L_{a,\ b,\ s}$ defined in (1.3) does not possess any nontrivial zero modes in $H^1(\mathbb{R})$, we have $w(x) = 0$ identically on $\mathbb{R}$.

Applying the standard Fourier transform (1.5) to both sides of (1.2), we get

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - a - ibp}, \quad p \in \mathbb{R}, \quad 0 < s < 1. \tag{2.2}$$

Thus,

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{(|p|^{2s} - a)^2 + b^2 p^2} dp. \tag{2.3}$$

Let us first consider assertions (a) and (b). From (2.3) it follows that

$$\|u\|^2_{L^2(\mathbb{R})} \leqslant \frac{1}{C} \|f\|^2_{L^2(\mathbb{R})} < \infty$$

due to the above assumptions. Here and below, C denotes a finite positive constant. By the above, if $a > 0$, then Equation (1.2) has a unique solution $u(x) \in H^1(\mathbb{R})$ if $0 < s \leqslant 1/2$ and $u(x) \in H^{2s}(\mathbb{R})$ if $1/2 < s < 1$.

Let us consider the case $a = 0$. Formula (2.2) yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p| \leqslant 1\}} + \frac{\widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p| > 1\}}. \tag{2.4}$$

Throughout the paper, χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}$. It is obvious that the second term on the right-hand side of (2.4) can be estimated from above in the absolute value by

$$\frac{|\widehat{f}(p)|}{\sqrt{1 + b^2}} \in L^2(\mathbb{R})$$

since $f(x)$ is square integrable as assumed. It is clear that

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{\sqrt{2\pi}}\|f(x)\|_{L^1(\mathbb{R})}. \tag{2.5}$$

If $0 < s < 1/4$, then we use (2.5) to derive

$$\left|\frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}\right| \leqslant \frac{|\widehat{f}(p)|}{|p|^{2s}}\chi_{\{|p|\leqslant 1\}} \leqslant \frac{\|f(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|p|^{2s}}\chi_{\{|p|\leqslant 1\}}.$$

This allows us to obtain the upper bound on the norm

$$\left\|\frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}\right\|^2_{L^2(\mathbb{R})} \leqslant \frac{\|f(x)\|^2_{L^1(\mathbb{R})}}{\pi(1-4s)} < \infty$$

since $f(x) \in L^1(\mathbb{R})$ as assumed. By the argument above, Equation (1.2) has a unique solution $u(x) \in H^1(\mathbb{R})$ in assertion (c).

To prove assertions (d) and (e), we express

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds}ds.$$

Hence the first term on the right-hand side of (2.4) can be written as

$$\frac{\widehat{f}(0)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}} + \frac{\int_0^p \frac{d\widehat{f}(s)}{ds}ds}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}. \tag{2.6}$$

By the definition (1.5) of the standard Fourier transform, we easily get

$$\left|\frac{d\widehat{f}(p)}{dp}\right| \leqslant \frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}.$$

This enables us to estimate the second term in (2.6) from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}}\frac{\|xf(x)\|_{L^1(\mathbb{R})}}{|b|}\chi_{\{|p|\leqslant 1\}} \in L^2(\mathbb{R})$$

due to the assumptions. Let us analyze the first term in (2.6), which is given by

$$\frac{\widehat{f}(0)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}. \tag{2.7}$$

It is clear that for $1/4 \leqslant s \leqslant 1/2$, the expression (2.7) can be bounded below in the absolute value by

$$\frac{|\widehat{f}(0)|}{|p|^{2s}\sqrt{1+b^2}}\chi_{\{|p|\leqslant 1\}},$$

which does not belong to $L^2(\mathbb{R})$ unless $\widehat{f}(0)$ vanishes. This gives us orthogonality relation (1.10). In case (d), the square integrability of the solution $u(x)$ to Equation (1.2) is equivalent to $u(x) \in H^1(\mathbb{R})$.

It is evident that for $1/2 < s < 1$ the expression (2.7) can be estimated below in the absolute value by the quantity

$$\frac{|\widehat{f}(0)|}{|p|\sqrt{1+b^2}}\chi_{\{|p|\leqslant 1\}},$$

which is not square integrable unless the orthogonality condition (1.10) holds. In case (e), the square integrability of the solution $u(x)$ to the problem (1.2) is equivalent to $u(x) \in H^{2s}(\mathbb{R})$. $\square$

We proceed by establishing the solvability in the sense of sequences on the whole real line.

Proof of Theorem 1.2. We assume that Equations (1.2) and (1.6) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$, $m \in \mathbb{N}$ if $0 < s \leqslant 1/2$ and, similarly, $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R})$, $m \in \mathbb{N}$, for $1/2 < s < 1$ such that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. Then $u_m(x)$ also converges to $u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ for $0 < s \leqslant 1/2$ and, similarly, $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ if $1/2 < s < 1$. Indeed, from (1.2) and (1.6) it follows that

$$\left\|\left(-\frac{d^2}{dx^2}\right)^s(u_m-u) - b\frac{d(u_m-u)}{dx}\right\|_{L^2(\mathbb{R})} \leqslant \|f_m - f\|_{L^2(\mathbb{R})} + a\|u_m - u\|_{L^2(\mathbb{R})}. \tag{2.8}$$

The right-hand side of the upper bound (2.8) tends to zero as $m \to \infty$ by the above assumptions. Using the standard Fourier transform (1.5), we easily get

$$\int_{-\infty}^{\infty}(|p|^{4s} + b^2p^2)|\widehat{u}_m(p) - \widehat{u}(p)|^2dp \to 0, \quad m \to \infty. \tag{2.9}$$

Let $0 < s \leqslant 1/2$. By (2.9),

$$\int_{-\infty}^{\infty} p^2|\widehat{u}_m(p) - \widehat{u}(p)|^2dp \to 0, \quad m \to \infty,$$

so that

$$\frac{du_m}{dx} \to \frac{du}{dx} \quad \text{in} \quad L^2(\mathbb{R}), \quad m \to \infty.$$

Therefore, if $0 < s \leqslant 1/2$, we have $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$.

Assume that $1/2 < s < 1$. By (2.9),

$$\int_{-\infty}^{\infty} |p|^{4s}|\widehat{u}_m(p) - \widehat{u}(p)|^2dp \to 0, \quad m \to \infty.$$

Then

$$\left(-\frac{d^2}{dx^2}\right)^s u_m \to \left(-\frac{d^2}{dx^2}\right)^s u \quad \text{in} \quad L^2(\mathbb{R}), \quad m \to \infty.$$

Hence for $1/2 < s < 1$, we have $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$.

Applying the standard Fourier transform (1.5) to both sides of (1.6), we get

$$\widehat{u}_m(p) = \frac{\widehat{f}_m(p)}{|p|^{2s} - a - ibp}, \quad m \in \mathbb{N}, \quad p \in \mathbb{R}, \quad 0 < s < 1. \tag{2.10}$$

Let us discuss assertions (a) and (b). By Theorem 1.1, (a), (b), if $a > 0$, then Equations (1.2) and (1.6) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$, $m \in \mathbb{N}$ provided that $0 < s \leqslant 1/2$ and, similarly $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R})$, $m \in \mathbb{N}$, if $1/2 < s < 1$. From (2.10) and (2.2) we get

$$\|u_m - u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}_m(p) - \widehat{f}(p)|^2}{(|p|^{2s} - a)^2 + b^2p^2} dp.$$

Hence

$$\|u_m - u\|_{L^2(\mathbb{R})} \leqslant \frac{1}{C}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty,$$

by the above assumptions. Therefore, if $a > 0$, then $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ for $0 < s \leqslant 1/2$ and $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ for $1/2 < s < 1$ by the above argument.

We complete the proof by considering the case $a = 0$. By [23, Lemma 3.3(a)], under our assumptions,

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.11}$$

in asserations (d) and (e). By Theorem 1.1, (c), (d), (e), Equations (1.2) and (1.6) with $a = 0$ have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$, $m \in \mathbb{N}$, if $0 < s \leqslant 1/2$ and, similarly, $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R})$, $m \in \mathbb{N}$, for $1/2 < s < 1$. From (2.10) and (2.2) it follows that

$$\widehat{u}_m(p) - \widehat{u}(p) = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}} + \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|>1\}}. \tag{2.12}$$

Apparently, the second term on the right-hand side of (2.12) can be bounded from above in the $L^2(\mathbb{R})$-norm by

$$\frac{1}{\sqrt{1+b^2}}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

in view of the above assumptions. Let $0 < s < 1/4$. Using an analog of the inequality (2.5), we derive

$$\left|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}\right| \leqslant \frac{|\widehat{f}_m(p) - \widehat{f}(p)|}{|p|^{2s}}\chi_{\{|p|\leqslant 1\}} \leqslant \frac{\|f_m - f\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|p|^{2s}}\chi_{\{|p|\leqslant 1\}}$$

so that

$$\left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}\right\|_{L^2(\mathbb{R})} \leqslant \frac{\|f_m - f\|_{L^1(\mathbb{R})}}{\sqrt{\pi(1-4s)}} \to 0, \quad m \to \infty,$$

by the above assumptions. Arguing as above, we see that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ in the case $a = 0$ and $0 < s < 1/4$.

To prove assertions (d) and (e), we use the orthogonality conditions (2.11) and (1.11). By the standard Fourier transform (1.5), we have $\widehat{f}(0) = 0$, $\widehat{f}_m(0) = 0$, $m \in \mathbb{N}$. Thus,

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_m(p) = \int_0^p \frac{d\widehat{f}_m(s)}{ds}ds, \quad m \in \mathbb{N}. \tag{2.13}$$

Hence the first term on the right-hand side of (2.12) in assertions (d) and (e) can be written as

$$\frac{\int_0^p \left[\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}\right]ds}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}.$$

Using the standard Fourier transform (1.5), we easily derive

$$\left|\frac{d\widehat{f}_m(p)}{dp} - \frac{d\widehat{f}(p)}{dp}\right| \leqslant \frac{1}{\sqrt{2\pi}}\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}.$$

Hence

$$\left|\frac{\int_0^p \Big[\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}\Big]ds}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}\right| \leqslant \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b|}\chi_{\{|p|\leqslant 1\}}$$

so that

$$\left\|\frac{\int_0^p \Big[\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}\Big]ds}{|p|^{2s} - ibp}\chi_{\{|p|\leqslant 1\}}\right\|_{L^2(\mathbb{R})} \leqslant \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{\pi}|b|} \to 0$$

as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. By the above argument in the case $a = 0$, we have $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ if $1/4 \leqslant s \leqslant 1/2$ and $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ for $1/2 < s < 1$. □

3 The Problem on Finite Interval

Proof of Theorem 1.3. We first show that it suffices to solve the problem in $L^2(I)$. Indeed, if $u(x)$ is a square integrable solution to Equation (1.12), periodic on I along with its first derivative, then

$$\Big(-\frac{d^2}{dx^2}\Big)^s u - b\frac{du}{dx} \in L^2(I).$$

From (1.14) we obtain $(|n|^{2s} - ibn)u_n \in l^2$ so that

$$\sum_{n=-\infty}^{\infty} (|n|^{4s} + b^2n^2)|u_n|^2 < \infty. \tag{3.1}$$

Let $0 < s \leqslant 1/2$. By (3.1), we have $\sum_{n=-\infty}^{\infty} n^2|u_n|^2 < \infty$, which yields $\frac{du}{dx} \in L^2(I)$. Hence $u(x) \in H^1(I)$.

Let $1/2 < s < 1$. By (3.1), we have $\sum_{n=-\infty}^{\infty} |n|^{4s}|u_n|^2 < \infty$, which yields $\Big(-\frac{d^2}{dx^2}\Big)^s u(x) \in L^2(I)$. Hence $u(x) \in H^{2s}(I)$.

To prove the uniqueness of a solution to Equation (1.12), we discuss the situation where $a > 0$ and $0 < s \leqslant 1/2$. If $a > 0$ and $1/2 < s < 1$, similar ideas can be exploited for $H^{2s}(I)$. For $a = 0$, $0 < s \leqslant 1/2$ and $a = 0$, $1/2 < s < 1$, our argument can be generalized by using the above constrained subspaces $H^1_0(I)$ and $H^{2s}_0(I)$ respectively. Assume that $u_1(x),\ u_2(x) \in H^1(I)$ solve (1.12). Then the difference $w(x) := u_1(x) - u_2(x) \in H^1(I)$ satifies the homogeneous equation

$$\Big(-\frac{d^2}{dx^2}\Big)^s w - b\frac{dw}{dx} - aw = 0.$$

Since the operator $\mathscr{L}_{a,\ b,\ s}$ introduced in (1.15) does not have any nontrivial $H^1(I)$ zero modes, $w(x) \equiv 0$ on I.

Applying the Fourier transform (1.14) to both sides of the problem (1.12), we get

$$u_n = \frac{f_n}{|n|^{2s} - a - ibn}, \quad n \in \mathbb{Z}. \tag{3.2}$$

Hence

$$\|u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_n|^2}{(|n|^{2s} - a)^2 + b^2 n^2}. \tag{3.3}$$

We begin with assertions (a) and (b). By (3.3), we have

$$\|u\|^2_{L^2(I)} \leqslant \frac{1}{C}\|f\|^2_{L^2(I)} < \infty$$

in view of the above assumptions and (1.13). Arguing as in the case $a > 0$, we see that Equation (1.12) has a unique solution $u(x) \in H^1(I)$ if $0 < s \leqslant 1/2$ and $u(x) \in H^{2s}(I)$ for $1/2 < s < 1$.

To conclude the proof, we consider the case $a = 0$. In this case, (3.2) yields

$$u_n = \frac{f_n}{|n|^{2s} - ibn}, \quad n \in \mathbb{Z}. \tag{3.4}$$

It is evident that the right-hand side of (3.4) belongs to l^2 if and only if

$$f_0 = 0 \tag{3.5}$$

so that

$$\|u\|^2_{L^2(I)} = \sum_{n\in\mathbb{Z},\quad n\neq 0} \frac{|f_n|^2}{n^{4s} + b^2 n^2} \leqslant \frac{1}{1+b^2}\|f\|^2_{L^2(I)} < \infty,$$

in view of the above assumtpions and (1.13). Arguing in the same way as in the proof of assertions (c) and (d), we conclude that $u(x) \in H^1_0(I)$ and $u(x) \in H^{2s}_0(I)$ respectively. It is obvious that (3.5) is equivalent to the orthogonality condition (1.20). □

We proceed by establishing the solvability in the sense of sequences on the interval I with periodic boundary conditions.

Proof of Theorem 1.4. Under the given assumptions, we get

$$|f(0) - f(2\pi)| \leqslant |f(0) - f_m(0)| + |f_m(2\pi) - f(2\pi)| \leqslant 2\|f_m - f\|_{L^\infty(I)} \to 0, \quad m \to \infty.$$

Hence $f(0) = f(2\pi)$. By (1.13), for $f_m(x)$, $f(x)$ bounded on the interval I we get $f_m(x),\ f(x) \in L^1(I) \cap L^2(I)$, $m \in \mathbb{N}$. An analog of (1.13) implies

$$\|f_m(x) - f(x)\|_{L^1(I)} \leqslant 2\pi\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty. \tag{3.6}$$

Hence $f_m(x) \to f(x)$ in $L^1(I)$ as $m \to \infty$. Similarly, (1.13) yields

$$\|f_m(x) - f(x)\|_{L^2(I)} \leqslant \sqrt{2\pi}\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty. \tag{3.7}$$

Hence $f_m(x) \to f(x)$ in $L^2(I)$ as $m \to \infty$ as well. We apply the Fourier transform (1.14) to both sides of (1.19) and derive

$$u_{m,n} = \frac{f_{m,n}}{|n|^{2s} - a - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.8}$$

We first prove assertions (a) and (b). By Theorem 1.3, (a), (b), in the case $a > 0$, Equations (1.12) and (1.19) have unique solutions $u(x) \in H^1(I)$ and $u_m(x) \in H^1(I)$, $m \in \mathbb{N}$, for $0 < s \leqslant 1/2$ and, similarly, $u(x) \in H^{2s}(I)$ and $u_m(x) \in H^{2s}(I)$, $m \in \mathbb{N}$, for $1/2 < s < 1$. From (3.8) along with (3.2) and (3.7) we have

$$\|u_m - u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_{m,n} - f_n|^2}{(|n|^{2s} - a)^2 + b^2 n^2} \leqslant \frac{1}{C} \|f_m - f\|^2_{L^2(I)} \to 0, \quad m \to \infty.$$

Thus, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. By (1.12) and (1.19),

$$\left\| \left(-\frac{d^2}{dx^2} \right)^s (u_m - u) - b \frac{d(u_m - u)}{dx} \right\|_{L^2(I)} \leqslant \|f_m - f\|_{L^2(I)} + a\|u_m - u\|_{L^2(I)}.$$

The right-hand side of this inequality converges to zero as $m \to \infty$ due to (3.7). The Fourier transform (1.14) gives us

$$\sum_{n=-\infty}^{\infty} (|n|^{4s} + b^2 n^2)|u_{m,n} - u_n|^2 \to 0, \quad m \to \infty. \tag{3.9}$$

Let $0 < s \leqslant 1/2$. Then (3.9) yields $\sum_{n=-\infty}^{\infty} n^2 |u_{m,n} - u_n|^2 \to 0$ as $m \to \infty$. Therefore,

$$\frac{du_m}{dx} \to \frac{du}{dx} \quad \text{in} \ \ L^2(I), \quad m \to \infty,$$

which implies $u_m(x) \to u(x)$ in $H^1(I)$ as $m \to \infty$, as well as in case (a).

Let $1/2 < s < 1$. By (3.9), we have $\sum_{n=-\infty}^{\infty} |n|^{4s} |u_{m,n} - u_n|^2 \to 0$, $m \to \infty$, so that

$$\left(-\frac{d^2}{dx^2} \right)^s u_m \to \left(-\frac{d^2}{dx^2} \right)^s u \quad \text{in} \ \ L^2(I), \quad m \to \infty.$$

Therefore, $u_m(x) \to u(x)$ in $H^{2s}(I)$ as $m \to \infty$ as well as in case (b).

Finally, we consder the case where the constant a vanishes. Then (1.21) and (3.6) imply

$$|(f(x), 1)_{L^2(I)}| = |(f(x) - f_m(x), 1)_{L^2(I)}| \leqslant \|f_m - f\|_{L^1(I)} \to 0, \quad m \to \infty.$$

Hence the limiting orthogonality condition

$$(f(x), 1)_{L^2(I)} = 0 \tag{3.10}$$

holds. By Theorem 1.3, (a), (d), in the case $a = 0$, Equations (1.12) and (1.19) have possess unique solutions $u(x) \in H_0^1(I)$ and $u_m(x) \in H_0^1(I)$, $m \in \mathbb{N}$ respectively for $0 < s \leqslant 1/2$ and, similarly, $u(x) \in H_0^{2s}(I)$ and $u_m(x) \in H_0^{2s}(I)$, $m \in \mathbb{N}$ if $1/2 < s < 1$. By (3.2) and (3.8),

$$u_{m,n} - u_n = \frac{f_{m,n} - f_n}{|n|^{2s} - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.11}$$

The orthogonality relations (3.10) and (1.21) give us $f_0 = 0$, $f_{m,0} = 0$, $m \in \mathbb{N}$. We obtain the upper bound on the norm

$$\|u_m - u\|_{L^2(I)} = \sqrt{\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{|f_{m,n} - f_n|^2}{|n|^{4s} + b^2 n^2}} \leqslant \frac{\|f_m - f\|_{L^2(I)}}{\sqrt{1+b^2}} \to 0, \quad m \to \infty,$$

by using (3.7). Hence $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. Therefore, if a vanishes and $0 < s \leqslant 1/2$, we prove then $u_m(x) \to u(x)$ in $H_0^1(I)$ as $m \to \infty$ as in assertion (a). If $a = 0$ and $1/2 < s < 1$, then $u_m(x) \to u(x)$ in $H_0^{2s}(I)$ as $m \to \infty$ as in assertion (b). □

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