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THE BOUNDARY BEHAVIOR OF A SOLUTION TO THE DIRICHLET PROBLEM FOR THE p-LAPLACIAN WITH WEIGHT UNIFORMLY DEGENERATE ON A PART OF DOMAIN WITH RESPECT TO SMALL PARAMETER

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We consider the Dirichlet problem for the p-Laplacian with weight and continuous boundary function in a domain D divided into two parts by the hyperplane Σ. The weight is equal to 1 in some part of the domain D and coincides with a small parameter ε in the other. We estimate the modulus of continuity for the solution at a boundary point $x_0 \in \partial D \cap \Sigma$ with a constant independent of ε. Bibliography: 22 titles.

Dedicated to the 80th anniversary of Vasilii Vasil'evich Zhikov

1 Introduction

In a bounded domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, we consider the equation

$$Lu = \operatorname{div}\,\bigl(\omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u\bigr) = 0, \quad p = \text{const} > 1. \tag{1.1}$$

We assume that the domain D is divided into the parts $D^{(1)} = D \cap \{x_n > 0\}$ and $D^{(2)} = D \cap \{x_n < 0\}$ by the hyperplane $\Sigma = \{x_n = 0\}$ and

$$\omega_\varepsilon(x) = \omega_\varepsilon(x_n) = \begin{cases} \varepsilon, & x_n > 0, \\ 1, & x_n < 0, \end{cases} \quad \varepsilon \in (0, 1]. \tag{1.2}$$

Below, $W^{1,p}(D)$ denotes the Sobolev space of functions that, together with all generalized first order derivatives, belong to $L^p(D)$ and $W_0^{1,p}(D)$ is the closure of the set $C_0^\infty(D)$ of compactly

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supported and infinitely differentiable functions in D in the $W^{1,p}(D)$-norm. We say that a function $u \in W^{1,p}(D)$ is a *solution* to Equation (1.1) in D if the integral identity

$$\int_D \omega_\varepsilon |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = 0 \tag{1.3}$$

holds for any test function $\varphi \in W_0^{1,p}(D)$. A function $u \in W^{1,p}(D)$ is called a *supersolution* to Equation (1.1) in D if for all nonnegative $\varphi \in W_0^{1,p}(D)$

$$\int_D \omega_\varepsilon |\nabla u|^{p-2} \, \nabla u \cdot \nabla \varphi \, dx \geqslant 0. \tag{1.4}$$

We consider the Dirichlet problem

$$Lu = 0 \ \text{ in } \ D, \quad u \in W^{1,p}(D), \quad h \in W^{1,p}(D), \quad (u - h) \in W_0^{1,p}(D). \tag{1.5}$$

The solution to this problem coincides with the minimizer of the variational problem

$$\min_{\psi - h \in W_0^{1,p}(D)} F(\psi), \quad F(\psi) = \int_D \omega_\varepsilon(x) \frac{|\nabla \psi|^p}{p} \, dx.$$

This paper is devoted to the study of boundary properties of solutions to the Dirichlet problem

$$Lu_f = 0 \ \text{ in } \ D, \quad u_f|_{\partial D} = f, \tag{1.6}$$

where f is continuous on ∂D.

A solution to the problem (1.6) is defined as follows. Using the Tietze–Uryson theorem, we extend the boundary function f by continuity to $\mathbb{R}^n$, preserving the same notation. We consider a sequence of infinitely differentiable functions f_k in $\mathbb{R}^n$, uniformly converging to f in $\overline{D}$. We solve the Dirichlet problem

$$Lu_k = 0 \ \text{ in } \ D, \quad u_k \in W_p^1(D), \ (u_k - f_k) \in \overset{\circ}{W}{}_p^1 (D).$$

By the maximum principle, the sequence u_k converges uniformly in D to a function u that belongs to the space $W^{1,p}(D')$ in an arbitrary subdomain $D' \Subset D$ and satisfies the integral identity (1.3) with test functions $\varphi \in W^{1,p}(D)$ with compact support in D. The limit function is independent of the extension method and approximation of the boundary function f and is called a weak solution to the Dirichlet problem (1.6). We refer, for example, to [1]–[3] for details of this construction.

Definition 1.1. A boundary point $x_0 \in \partial D$ is said to be *regular* if $\lim\limits_{D \ni x \to x_0} u_f(x) = f(x_0)$ for any continuous function f on ∂D.

In what follows, we need the notion of a capacity. The *capacity* of a compact set $K \subset B$ relative to a ball $B \subset \mathbb{R}^n$ is the number

$$C_p(K, B) = \inf \left\{ \int_B |\nabla \varphi|^p \, dx \ : \ \varphi \in C_0^\infty(B), \ \varphi \geqslant 1 \ \text{ on } \ K \right\}.$$

We denote by $B_r^{x_0}$ an open ball in $\mathbb{R}^n$ with center x_0 and radius r and by $\overline{B}_r^{x_0}$ its closure. The criterion for regularity of a boundary point $x_0 \in \partial D$ for the classical p-Laplacian ((1.1) with $\varepsilon = 1$) consists in the following identity

$$\int_0 \left(\frac{C_p\left(\overline{B}_r^{x_0} \setminus D,\, B_{2r}^{x_0})\right)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = \infty. \tag{1.7}$$

For the Laplace equation this assertion is the classical result due to Wiener [4, 5]. In the case of linear divergence-form uniformly elliptic equations with measurable coefficients, the Wiener criterion was obtained in [6]. In the case $p \neq 2$, a sufficient condition in the form (1.7) for the regularity of a boundary point was found in [7], where for equation of the form (1.1) without small parameter ε the estimate for the modulus of continuity of a solution at a regular boundary point was also proved. The estimates obtained in [7] were generalized to a large class of quasilinear elliptic equations of the p-Laplacian type in [8]. The necessity of the condition (1.7) for the regularity of a boundary point in the case of p-Laplacian type equations was established in [9] for $n - 1 < p \leqslant n$ and in [10] in the general case.

As was already mentioned, for equation of the form (1.1) without small parameter ε the estimate for the modulus of continuity for solutions to the Dirichlet problem at a regular boundary point was obtained in [7]. In this paper, we prove an analogous estimate for the modulus of continuity of a solution to the Dirichlet problem (1.6) for Equation (1.1) with constants independent of ε in the case where the boundary point $x_0 \in \partial D$ lies on the phase interface Σ, i.e., $x_0 \in \partial D \cap \Sigma$.

We note that for any fixed ε the regularity criterion for a boundary point $x_0 \in \Sigma$ coincides with that for the classical p-Laplacian. In particular, the condition (1.7) is necessary and sufficient for the regularity of a boundary point. However, the known from [7, 8] estimates for the boundary modulus of continuity considerably degenerate as $\varepsilon \to 0$.

Similar questions for interior estimates for degenerate linear elliptic equations were considered in [11]–[14]. For linear parabolic equations that degenerate with respect to a small parameter on a part of the domain it was proved in [15] that the solution is Hölder on the interface Σ, whereas the upper Nash–Aronson type estimates were obtained in [16]. The results of [15] were generalized to the case of parabolic p-Laplacian type equations in [17, 18]. Similar phenomena arise in the study of the $p(x)$-Laplacian with variable exponent [19].

For $x_0 \in \partial D \cap \Sigma$ we set

$$\gamma(r) = \left(\frac{C_p((\overline{B}_r^{x_0} \cap \{x_n \leqslant 0\}) \setminus D,\, B_{2r}^{x_0}))}{r^{n-p}} \right)^{\frac{1}{p-1}}$$

Theorem 1.1. *If for* $x_0 \in \partial D \cap \Sigma$ *the condition* (1.7) *holds, then for* $0 < r \leqslant \rho/4 \leqslant \operatorname{diam} D/4$ *the solution* u_f *to the Dirichlet problem* (1.6) *satisfies the estimate*

$$\operatorname*{ess\,sup}_{D \cap B_r^{x_0}} |u_f - f(x_0)| \leqslant 2 \operatorname*{osc}_{\partial D \cap B_\rho^{x_0}} f + \operatorname*{osc}_{\partial D} f \cdot \exp\left(-C \int_r^\rho \gamma(t) t^{-1}\, dt \right), \tag{1.8}$$

where the positive constant C *depends only on* n *and* p.

If in a neighborhood of a boundary point x_0, the domain is symmetric with respect to the hyperplane Σ, then the obtained estimate coincides with the classical estimate obtained in [7].

The proof of Theorem 1.1 is based on the weak type Harnack inequality (cf. Lemma 2.5) for nonnegative bounded supersolutions to Equation (1.1). Throughout the paper, B_R is an open ball with radius R and a fixed center located on Σ and the half-balls are denoted by $B_R^{(1)} = B_R \cap \{x_n > 0\}$ and $B_R^{(2)} = B_R \cap \{x_n < 0\}$.

2 The Weak Type Harnack Inequality for Nonnegative Supersolutions

Let $|E|$ be the n-dimensional Lebesgue measure of a measurable set $E \subset R^n$, and let

$$\fint_{E} f\,dx = \frac{1}{|E|}\int_{|E|} f\,dx.$$

We use the Sobolev embedding theorem

$$\Bigg(\fint_{B_R} |\varphi|^{pk} dx\Bigg)^{1/k} \leqslant C(n,p)R^p \fint_{B_R} |\nabla\varphi|^p dx, \quad \varphi \in C_0^\infty(B_R), \quad k = \frac{n}{n-1}. \tag{2.1}$$

In what follows, w is a nonnegative supersolution to Equation (1.1) in B_{4R}, $\widetilde{w}$ is the even extension of w from $B_{4R}^{(2)}$ to $B_{4R}^{(1)}$ with respect to the hyperplane Σ, and

$$v = \begin{cases} \min(w, \widetilde{w}) & \text{in} \quad B_{4R}^{(1)}, \\ w & \text{in} \quad B_{4R}^{(2)}. \end{cases} \tag{2.2}$$

Lemma 2.1. *For any $q > 0$ the following estimate holds*:

$$\inf_{B_R} v \geqslant C(n,p,q)\Bigg(\fint_{B_{3R}} v^{-q}(x) dx\Bigg)^{-1/q}. \tag{2.3}$$

Proof. Without loss of generality we assume that the supersolution w is positive. Otherwise, we consider the function $w + \delta$ and pass to the limit as $\delta \to 0$ in the estimate (2.3). Taking the test function $\varphi = v^\gamma \eta^p$ in (1.4), where $\gamma < 1 - p$ and the cut-off function $\eta \in C_0^\infty(B_{4R})$ is radially symmetric and such that $0 \leqslant \eta \leqslant 1$, we obtain the estimate

$$|\gamma| \int_{B_{4R}} \omega_\varepsilon |\nabla w|^p w^{\gamma-1}\eta^p\,dx \leqslant p \int_{B_{4R}} \omega_\varepsilon |\nabla w|^{p-1}|\nabla\eta|\eta^{p-1}\,dx.$$

Applying the Young inequality to the integrand on the right-hand side, we find

$$\int_{B_{4R}} \omega_\varepsilon |\nabla w|^p w^{\gamma-1}\eta^p\,dx \leqslant C(p) \int_{B_{4R}} \omega_\varepsilon w^{\gamma+p-1}|\nabla\eta|^p\,dx.$$

By (1.2), the choice of γ, and the definition (2.2) of v, we have

$$\int_{B_{4R}^{(2)}} |\nabla v|^p v^{\gamma-1}\eta^p\,dx = \int_{B_{4R}^{(2)}} |\nabla w|^p w^{\gamma-1}\eta^p\,dx \leqslant C(p) \int_{B_{4R}} v^{\gamma+p-1}|\nabla\eta|^p\,dx. \tag{2.4}$$

We proceed with a similar estimate in the half-ball $B_{4R}^{(1)}$. Setting

$$G_R = B_{4R}^{(1)} \cap \{w < \widetilde{w}\} \tag{2.5}$$

and assuming that G_R is not empty, we substitute the function

$$\varphi = (w^\gamma - \widetilde{w}^\gamma)_+ \, \eta^p = \begin{cases} (w^\gamma - \widetilde{w}^\gamma)\, \eta^p & \text{in } G_R, \\ 0 & \text{in } B_{4R} \setminus G_R, \end{cases}$$

into (1.4); here η and γ are the same as above. As a result, using (1.2), we get

$$|\gamma| \int\limits_{G_R} |\nabla w|^p w^{\gamma-1} \eta^p \, dx \leqslant |\gamma| \int\limits_{G_R} |\nabla w|^{p-1} |\nabla \widetilde{w}| \widetilde{w}^{\gamma-1} \eta^p \, dx + p \int\limits_{G_R} |\nabla w|^{p-1} (w^\gamma - \widetilde{w}^\gamma) |\nabla \eta| \eta^{p-1} \, dx.$$

Further, applying the Young inequality to the integrands on the right-hand side of this inequality and recalling the definition (2.5) of the set G_R, we find

$$\int\limits_{G_R} |\nabla w|^p w^{\gamma-1} \eta^p \, dx \leqslant C(p) \Bigg(\int\limits_{G_R} |\nabla \widetilde{w}|^p \widetilde{w}^{\gamma-1} \eta^p \, dx + \int\limits_{G_R} w^{\gamma+p-1} |\nabla \eta|^p \, dx \Bigg).$$

Adding the integral

$$\int\limits_{B_{4R}^{(1)} \setminus G_R} |\nabla \widetilde{w}|^p w^{\gamma-1} \eta^p \, dx$$

to both sides of the last inequality and recalling the definition (2.2) of v, we obtain the estimate

$$\int\limits_{B_{4R}^{(1)}} |\nabla v|^p v^{\gamma-1} \eta^p \, dx \leqslant C(p) \Bigg(\int\limits_{B_{4R}^{(1)}} |\nabla \widetilde{w}|^p \widetilde{w}^{\gamma-1} \eta^p \, dx + \int\limits_{B_{4R}^{(1)}} v^{\gamma+p-1} |\nabla \eta|^p \, dx \Bigg).$$

From properties of the even extension of w, the radially symmetry of the cut-ff function η, and the relations (2.4) it follows that

$$\int\limits_{B_{4R}^{(1)}} |\nabla v|^p v^{\gamma-1} \eta^p \, dx \leqslant C(p) \int\limits_{B_{4R}} v^{\gamma+p-1} |\nabla \eta|^p \, dx. \tag{2.6}$$

Adding both sides of (2.4) and (2.6), we find

$$\int\limits_{B_{4R}} |\nabla v|^p v^{\gamma-1} \eta^p \, dx \leqslant C(p) \int\limits_{B_{4R}} v^{\gamma+p-1} |\nabla \eta|^p \, dx,$$

which implies

$$\int\limits_{B_{4R}} |\nabla (v^{(\gamma+p-1)/p} \eta)|^p \, dx \leqslant C(p) |\gamma + p - 1|^p \int\limits_{B_{4R}} v^{\gamma+p-1} |\nabla \eta|^p \, dx. \tag{2.7}$$

In the above consideration, the set G_R is assumed to be nonempty. If G_R is empty, then $v = \widetilde{w}$ in $B_{4R}^{(1)}$ and (2.7) immediately follows from (2.4) and properties of the even extension of $\widetilde{w}$.

From (2.7) and the Sobolev embedding theorem (2.1) it follows that

$$\left(\fint_{B_{4R}} v^{k(\gamma+p-1)} \eta^{kp}\, dx \right)^{1/k} \leqslant C(p)|\gamma+p-1|^p R^p \fint_{B_{4R}} v^{\gamma+p-1} |\nabla \eta|^p\, dx. \tag{2.8}$$

For $R \leqslant \rho < r \leqslant 3R$ we take a cut-off function $\eta \in C_0^\infty(B_r)$ in (2.8) such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in B_ρ and $|\nabla \eta| \leqslant Cr(R(r-\rho))^{-1}$. Then

$$\left(\fint_{B_\rho} v^{k(\gamma+p-1)}\, dx \right)^{1/k} \leqslant C(p)|\gamma+p-1|^p \Big(\frac{r}{r-\rho} \Big)^p \fint_{B_r} v^{\gamma+p-1}\, dx. \tag{2.9}$$

We iterate this estimate. For $j = 0, 1, \ldots$ we denote $r_j = R + 2^{-j+1}R$, $\chi_j = -qk^j$ and substitute $r = r_j$, $\rho = r_{j+1}$, $\gamma = \chi_j - p + 1$ into (2.9). As a result, for

$$\Phi_j = \left(\fint_{B_{r_j}} v^{\chi_j}\, dx \right)^{1/\chi_j}$$

we obtain the recurrent relation

$$\Phi_j \leqslant C^{1/|\chi_j|} (2^j |\chi_j|)^{p/|\chi_j|} \Phi_{j+1},$$

which implies (cf. [20]) the required estimate (2.3). □

Due to the following assertion we can obtain an auxiliary weak type Harnack inequality for the function v defined in (2.2). Below, B_r^z denotes an open ball of radius r and center z.

Lemma 2.2. *For any ball $B_{2r}^z \subset B_{4R}$ the following estimate holds*:

$$\int_{B_r^z} |\nabla \ln v|^p\, dx \leqslant C(p) r^{n-p}. \tag{2.10}$$

Proof. Let $\eta \in C_0^\infty(B_{2r}^z)$ be a radially symmetric cut-off function nonincreasing with respect to the distance from its argument to z and such that $\eta = 1$ in B_r^z and $|\nabla \eta| \leqslant Cr^{-1}$. Substituting the test function $\varphi = w^{1-p}\eta^p$ into the integral inequality (1.4), we find, as in (2.4),

$$\int_{B_{2r}^z \cap B_{4R}^{(2)}} |\nabla \ln v|^p \eta^p\, dx = \int_{B_{2r}^z \cap B_{4R}^{(2)}} |\nabla \ln w|^p \eta^p\, dx \leqslant C(n,p) r^{n-p}. \tag{2.11}$$

If $B_r^z \subset B_{4R}^{(2)}$, then the required estimate (2.10) is proved. Let $B_r^z \cap B_{4R}^{(1)}$ be nonempty. To prove a similar estimate in $B_r^z \cap B_{4R}^{(1)}$, we first assume that the set G_R defined by (2.5) is not empty and choose the test function

$$\varphi = (w^{1-p} - \widetilde{w}^{1-p})_+ \eta^p = \begin{cases} \left(w^{1-p} - \widetilde{w}^{1-p}\right) \eta^p & \text{in } G_R, \\ 0 & \text{in } B_{4R} \setminus G_R. \end{cases}$$

Then it is easy to see that

$$(p-1)\int\limits_{G_R} |\nabla \ln w|^p \eta^p \, dx \leqslant (p-1)\int\limits_{G_R} |\nabla w|^{p-1} |\nabla \ln \widetilde{w}| \widetilde{w}^{1-p} \eta^p \, dx$$

$$+ p \int\limits_{G_R} |\nabla \ln w|^{p-1} |\nabla \eta| \eta^{p-1} \, dx.$$

Since $w \leqslant \widetilde{w}$ on G_R, we can use the Cauchy inequality to find

$$\int\limits_{G_R} |\nabla \ln w|^p \eta^p \, dx \leqslant C(p)\Bigg(\int\limits_{G_R} |\nabla \ln \widetilde{w}|^p \eta^p \, dx + \int\limits_{G_R} |\nabla \eta|^p \, dx \Bigg). \tag{2.12}$$

Adding the integral

$$\int\limits_{(B^z_{2r} \setminus G_R) \cap B^{(1)}_{4R}} |\nabla \ln \widetilde{w}|^p \eta^p \, dx$$

to both sides of the estimate (2.12) and using the definition (2.2) of v, we get

$$\int\limits_{B^z_{2r} \cap B^{(1)}_{4R}} |\nabla \ln v|^p \eta^p \, dx \leqslant C(p)\Bigg(\int\limits_{B^z_{2r} \cap B^{(1)}_{4R}} |\nabla \ln \widetilde{w}|^p \eta^p \, dx + \int\limits_{B^z_{2r} \cap B^{(1)}_{4R}} |\nabla \eta|^p \, dx \Bigg). \tag{2.13}$$

We first consider the case where the center of the ball B^z_r is located in $\overline{B}^{(2)}_{4R}$. Then in view of (2.11), properties of the even extension of the function w, the choice of the center of B^z_r, and properties of the cut-off function η, we have

$$\int\limits_{B^z_{2r} \cap B^{(1)}_{4R}} |\nabla \ln \widetilde{w}|^p \eta^p \, dx \leqslant C(n,p) r^{n-p}$$

and, in view of (2.13),

$$\int\limits_{B^z_{2r} \cap B^{(1)}_{4R}} |\nabla \ln v|^p \eta^p \, dx \leqslant C(n,p) r^{n-p}.$$

Therefore, from (2.11) we obtain the estimate

$$\int\limits_{B^z_{2r}} |\nabla \ln v|^p \eta^p \, dx \leqslant C(n,p) r^{n-p},$$

which implies (2.10). Now, we consider the case where G_R is not empty and the center of the ball B^z_r is located in $B^{(1)}_{4R}$. We denote by Q^z_r the image of the ball B^z_r under the mirror extension with respect to the hyperplane Σ. We introduce the function $\widetilde{\eta}(x_1, \ldots, x_{n-1}, x_n) = \eta(x_1, \ldots, x_{n-1}, -x_n)$. By (2.11) and the choice of the cut-off function η, we have

$$\int\limits_{B^z_{2r} \cap B^{(1)}_{4R}} |\nabla \ln \widetilde{w}|^p \eta^p \, dx = \int\limits_{Q^z_{2r} \cap B^{(2)}_{4R}} |\nabla \ln w|^p \widetilde{\eta}^p \, dx \leqslant C(n,p) r^{n-p}.$$

Now, (2.10) follows from (2.13) and (2.11). If the set G_R is empty, then $v = \widetilde{w}$ in $B^{(1)}_{4R}$ and (2.10) follows from (2.11). □

By the John–Nirenberg lemma (cf. [20, 21]) or by the embedding theorem for functions with gradient in the Morrey space [22, Theorem 7.21], we obtain the following assertion from Lemma 2.2.

Corollary 2.1. *There exist positive constants q_0 and C depending only on n and p such that*

$$\Bigg(\fint_{B_{3R}} v^{-q_0}\,dx\Bigg)^{-1/q_0} \geqslant C\Bigg(\fint_{B_{3R}} v^{q_0}\,dx\Bigg)^{1/q_0}. \tag{2.14}$$

Now, from (2.3) and (2.14) we obtain the following auxiliary weak type Harnack inequality.

Lemma 2.3. *There exists positive constants q_0 and C depending only on n and p and such that for $q \in (0, q_0]$*

$$\inf_{B_R} v \geqslant C\Bigg(\fint_{B_{3R}} v^q\,dx\Bigg)^{1/q}. \tag{2.15}$$

Our next goal is to specify q_0 in (2.15). Since the arguments of the proof of Lemma 2.1 do not lead to the key estimate (2.4) in the case $\gamma > 1-p$, we need to use some other test functions. We first prove an intermediate gradient estimate.

Lemma 2.4. *For $0 < \beta_0 \leqslant \beta < p-1$ under the condition that*

$$\varepsilon p^{-p/(p-1)}(p-1)^2 \leqslant \frac{\beta_0}{4} \tag{2.16}$$

for a nonnegative radially symmetric function $\eta \in C_0^\infty(B_{4R})$ the following estimate holds:

$$\int_{B_{4R}} |\nabla v|^p v^{-\beta-1}\eta^p\,dx \leqslant C(p)\beta_0^{-p/(p-1)} \int_{B_{4R}} v^{p-\beta-1}|\nabla\eta|^p\,dx. \tag{2.17}$$

Proof. We choose the following test function in (1.4):

$$\varphi = \min(\widetilde{w}^{p-1-\beta}w^{1-p}, w^{-\beta})\eta^p = \begin{cases} \widetilde{w}^{p-1-\beta}w^{1-p}\eta^p & \text{in } B_{4R}^{(1)} \setminus G_R, \\ w^{-\beta}\eta^p & \text{in } G_R \cup B_{4R}^{(2)}, \end{cases}$$

where the set G_R is defined by (2.5). Then we obtain the inequality

$$\begin{aligned}
&(p-1)\int_{B_{4R}^{(1)}\setminus G_R} \omega_\varepsilon|\nabla w|^p w^{-p}\widetilde{w}^{p-1-\beta}\eta^p\,dx + \beta\int_{G_R\cup B_{4R}^{(2)}} \omega_\varepsilon|\nabla w|^p w^{-\beta-1}\eta^p\,dx \\
&\leqslant (p-1-\beta)\int_{B_{4R}^{(1)}\setminus G_R} \omega_\varepsilon|\nabla w|^{p-1}|\nabla\widetilde{w}|w^{1-p}\widetilde{w}^{p-\beta-2}\eta^p\,dx \\
&+ p\int_{B_{4R}^{(1)}\setminus G_R} \omega_\varepsilon|\nabla w|^{p-1}w^{1-p}\widetilde{w}^{p-\beta-1}|\nabla\eta|\eta^{p-1}\,dx + p\int_{G_R\cup B_{4R}^{(2)}} \omega_\varepsilon|\nabla w|^{p-1}w^{-\beta}|\nabla\eta|\eta^{p-1}\,dx.
\end{aligned} \tag{2.18}$$

Successively applying the Young inequality to each of the integrands on the right-hand side of (2.18), we obtain

$$|\nabla w|^{p-1}|\nabla\widetilde{w}|w^{1-p}\widetilde{w}^{p-\beta-2} \leqslant p^{-1}\delta_1|\nabla w|^p w^{-p}\widetilde{w}^{p-\beta-1} + (p-1)p^{-1}\delta_1^{-1/(p-1)}|\nabla\widetilde{w}|^p\widetilde{w}^{-\beta-1}, \tag{2.19}$$

$$|\nabla w|^{p-1}w^{1-p}\widetilde{w}^{p-\beta-1}|\nabla\eta|\eta^{p-1} \leqslant p^{-1}\delta_2|\nabla w|^p w^{-p}\widetilde{w}^{p-\beta-1}\eta^p + (p-1)p^{-1}\delta_2^{-1/(p-1)}\widetilde{w}^{p-\beta-1}|\nabla\eta|^p, \tag{2.20}$$

$$|\nabla w|^{p-1}w^{-\beta}|\nabla\eta|\eta^{p-1} \leqslant p^{-1}\delta_3|\nabla w|^p w^{-\beta-1}\eta^p + (p-1)p^{-1}\delta_3^{-1/(p-1)}w^{-\beta+p-1}|\nabla\eta|^p, \tag{2.21}$$

where $\delta_i > 0$, $i = 1, 2, 3$. Inserting $\delta_1 = (p-1)p(p-1-\beta)^{-1}/2$, $\delta_2 = (p-1)/2$, $\delta_3 = \beta/2$, into (2.19)–(2.21) and taking into account these estimates in (2.18), we find

$$\frac{\beta}{2}\int\limits_{B_{4R}^{(2)}} |\nabla w|^p w^{-\beta-1}\eta^p\omega_\varepsilon\,dx \leqslant (p-1)\int\limits_{B_{4R}^{(1)}\setminus G_R} ((p-1-\beta)p^{-1}\delta_1^{-1/(p-1)}|\nabla\widetilde{w}|^p\widetilde{w}^{-\beta-1}\eta^p$$
$$+\delta_2^{-1/(p-1)}\widetilde{w}^{-\beta+p-1}|\nabla\eta|^p)\,\omega_\varepsilon dx + \int\limits_{B_{4R}^{(2)}\cup G_R} (p-1)\delta_3^{-1/(p-1)}w^{-\beta+p-1}|\nabla\eta|^p\omega_\varepsilon\,dx.$$

Taking into account the definition (1.2) of the weight, we have

$$\frac{\beta}{2}\int\limits_{B_{4R}^{(2)}} |\nabla w|^p w^{-\beta-1}\eta^p\,dx \leqslant \varepsilon C_1(p,\beta)\int\limits_{B_{4R}^{(1)}\setminus G_R} |\nabla\widetilde{w}|^p\widetilde{w}^{-\beta-1}\eta^p\,dx + \varepsilon C_2(p)\int\limits_{B_{4R}^{(1)}\setminus G_R} \widetilde{w}^{-\beta+p-1}|\nabla\eta|^p\,dx$$
$$+\varepsilon C_3(p,\beta)\int\limits_{G_R} w^{-\beta+p-1}|\nabla\eta|^p\,dx + C_3(p,\beta)\int\limits_{B_{4R}^{(2)}} w^{-\beta+p-1}|\nabla\eta|^p\,dx,$$

where

$$C_1(p,\beta) = 2^{1/(p-1)}p^{-p/(p-1)}(p-1)^{(p-2)/(p-1)}(p-1-\beta)^{p/(p-1)},$$
$$C_2(p) = 2^{1/(p-1)}(p-1)^{(p-2)/(p-1)}, \quad C_3(p,\beta) = 2^{1/(p-1)}(p-1)\beta^{-1/(p-1)}.$$

For ε such that

$$\varepsilon C_1(p,\beta) \leqslant \beta_0/4 \leqslant \beta/4, \tag{2.22}$$

using the evenness of the extension of $\widetilde{w}$ and the definition of v (recall that $v = w$ in $B_{4R}^{(2)} \cup G_R$ and $v = \widetilde{w}$ in$B_{4R}^{(1)} \setminus G_R$), we find

$$\frac{\beta}{4}\int\limits_{B_{4R}^{(2)}} |\nabla w|^p w^{-\beta-1}\eta^p\,dx \leqslant \varepsilon C_2(p)\int\limits_{B_{4R}^{(1)}\setminus G_R} v^{-\beta+p-1}|\nabla\eta|^p\,dx$$
$$+\varepsilon C_3(p,\beta)\int\limits_{G_R} v^{-\beta+p-1}|\nabla\eta|^p\,dx + C_3(p,\beta)\int\limits_{B_{4R}^{(2)}} v^{-\beta+p-1}|\nabla\eta|^p\,dx.$$

Thus, we obtain the estimate

$$\int\limits_{B_{4R}^{(2)}} |\nabla w|^p w^{-\beta-1}\eta^p\,dx \leqslant C(p)\beta_0^{-p/(p-1)} \int\limits_{B_{4R}} v^{-\beta+p-1}|\nabla\eta|^p\,dx. \tag{2.23}$$

In particular,

$$\int\limits_{B_{4R}^{(1)}\setminus G_R} |\nabla\widetilde{w}|^p \widetilde{w}^{-\beta-1}\eta^p\,dx \leqslant C(p)\beta_0^{-p/(p-1)} \int\limits_{B_{4R}} v^{p-\beta-1}|\nabla\eta|^p\,dx. \tag{2.24}$$

Now, we substitute the test function

$$\varphi = (w^{-\beta} - \widetilde{w}^{-\beta})_+\eta^p = \begin{cases} (w^{-\beta} - \widetilde{w}^{-\beta})\eta^p & \text{in } G_R, \\ 0 & \text{in } B_{4R}\setminus G_R, \end{cases}$$

into (1.4). Then

$$\beta\int\limits_{G_R} |\nabla w|^p w^{-\beta-1}\eta^p\,dx \leqslant \beta\int\limits_{G_R} |\nabla w|^{p-1}|\nabla\widetilde{w}|\widetilde{w}^{-\beta-1}\eta^p\,dx + p\int\limits_{G_R} |\nabla w|^{p-1}w^{-\beta}|\nabla\eta|\eta^{p-1}\,dx.$$

Applying the Young inequality to the integrands on the right-hand side of the last estimate and using the definition (2.5) of the set G_R, we get

$$\int\limits_{G_R} |\nabla w|^p w^{-\beta-1}\eta^p\,dx \leqslant C(p)\beta_0^{-p/(p-1)}\Bigg(\int\limits_{G_R} |\nabla\widetilde{w}|^p\widetilde{w}^{-\beta-1}\eta^p\,dx + \int\limits_{G_R} w^{-\beta+p-1}|\nabla\eta|^p\,dx\Bigg).$$

Therefore, in view of (2.23) and the definition of $\widetilde{w}$, we have

$$\int\limits_{G_R} |\nabla w|^p w^{-\beta-1}\eta^p\,dx \leqslant C(p)\beta_0^{-p/(p-1)}\Bigg(\int\limits_{B_{4R}} v^{-\beta+p-1}|\nabla\eta|^p\,dx + \int\limits_{G_R} w^{-\beta+p-1}|\nabla\eta|^p\,dx\Bigg). \tag{2.25}$$

Adding (2.23), (2.24), (2.25) and using the definition (2.2) of v, we find

$$\int\limits_{B_{4R}} |\nabla v|^p v^{-\beta-1}\eta^p\,dx \leqslant C(p)\beta_0^{-p/(p-1)} \int\limits_{B_{4R}} v^{-\beta+p-1}|\nabla\eta|^p\,dx,$$

which means the required estimate (2.17). It remains to note that (2.16) implies (2.22). □

Now, we proceed by proving the weak type Harnack inequality.

Lemma 2.5. *Let $0 < \beta_0 < p-1$, and let the condition* (2.16) *hold. Then for $r \leqslant (p-\beta_0-1)k$, where $k = n/(n-1)$,*

$$\inf_{B_R} v \geqslant C(n,p,\beta_0)\Bigg(\fint\limits_{B_{5R/2}} v^r\,dx\Bigg)^{1/r}. \tag{2.26}$$

Proof. Let $\beta_0 \leqslant \beta < p-1$. From (2.17) we find

$$\int\limits_{B_{4R}} |\nabla(v^{(-\beta+p-1)/p}\eta|^p\,dx \leqslant C(n,p)\beta_0^{-p/(p-1)} \int\limits_{B_{4R}} v^{-\beta+p-1}|\nabla\eta|^p\,dx$$

and, by the Sobolev embedding theorem (2.1),

$$\Bigg(\ \fint\limits_{B_{4R}} v^{k(-\beta+p-1)}\eta^{kp}\,dx\Bigg)^{1/k} \leqslant C(n,p)\beta_0^{-p/(p-1)}R^p \fint\limits_{B_{4R}} v^{-\beta+p-1}|\nabla\eta|^p\,dx. \tag{2.27}$$

If $(p-\beta_0-1)k \leqslant q_0$, then the required inequality (2.26) is a consequence of (2.15) and the Hölder inequality. Let $q > q_0$. We choose $q_1 \in (q_0/k, q_0]$ such that $q_1 k^m = (p-\beta_0-1)k$ for some $m \in \mathbb{N}$. For $s > 0$ we introduce the functional

$$\Phi(s, B_r, v) = \Bigg(\ \fint\limits_{B_r} v^s\,dx\Bigg)^{1/s}$$

and show that

$$\Phi((p-\beta_0-1)k, B_{5R/2}, v) \leqslant C(n,p,\beta_0)\Phi(q_1, B_{3R}, v) \tag{2.28}$$

For this purpose for $5R/2 \leqslant \rho < R \leqslant 3R$ we take in (2.27) a cut-off function $\eta \in C_0^\infty(B_r)$ that is equal to 1 in B_ρ and such that $|\nabla\eta| \leqslant Cr(R(r-\rho))^{-1}$. We set $\beta = p-1-\theta$. As a result, we obtain the estimate

$$\Phi(k\theta, B_\rho, v) \leqslant \big(C(n,p)\beta_0^{-p/(p-1)}\big)^{1/\theta}\left(\frac{r}{r-\rho}\right)^{p/\theta}\Phi(\theta, B_r, v), \tag{2.29}$$

is valid for $0 < \theta \leqslant p-1-\beta_0$. Now, for $j = 0, 1, \dots, m-1$ we set $r_j = 3R - 2^{-j-1}R$. Taking $r = r_{j+1}$, $\rho = r_j$, $\theta = (p-\beta_0-1)k^{-j}$ in (2.29), we iterate the obtained relation. As a result, we obtain the relation (2.28). Now, by the choice of q_1, the estimate (2.15), and the Hölder inequality $\Phi(q_1, B_{3R}, v) \leqslant \Phi(q_0, B_{3R}, v)$ we obtain the required estimate (2.26) for $r = (p-\beta_0-1)k$. For $r < (p-\beta_0-1)k$ the estimate (2.26) again follows from the Hölder inequality $\Phi(r, B_{5R/2}v) \leqslant \Phi((p-\beta_0-1)k, B_{5R/2}, v)$ □

From Lemmas 2.4 and 2.5 we obtain the key estimate.

Lemma 2.6. *There is $\varepsilon_0 = \varepsilon_0(n,p)$ such that for $\varepsilon \leqslant \varepsilon_0$*

$$\fint\limits_{B_{2R}} |\nabla v|^{p-1}\,dx \leqslant C(n,p)R^{1-p}\big(\inf_{B_R} v\big)^{p-1}, \tag{2.30}$$

$$\fint\limits_{B_{2R}} v^{p-1}\,dx \leqslant C(n,p)\big(\inf_{B_R} v\big)^{p-1}. \tag{2.31}$$

Proof. Let $\beta \in (0, p-1)$. Using the Hölder inequality, we obtain the estimate

$$\fint\limits_{B_{4R}} |\nabla v|^{p-1}\eta^p\,dx \leqslant \Bigg(\ \fint\limits_{B_{4R}} |\nabla v|^p v^{-\beta-1}\eta^p\,dx\Bigg)^{(p-1)/p}\Bigg(\ \fint\limits_{B_{4R}} v^{(\beta+1)(p-1)}\eta^p\,dx\Bigg)^{1/p}. \tag{2.32}$$

We choose β_0 such that

$$(\beta_0+1)(p-1)=(p-\beta_0-1)k, \tag{2.33}$$

which implies

$$\beta_0=\frac{p-1}{n+(p-1)(n-1)}. \tag{2.34}$$

In (2.32), we take $\beta=\beta_0$ from (2.34) and use the inequality in Lemma 2.4 to estimate the first term on the right-hand side of (2.32). Under such a choice of β_0, the condition (2.16) takes the form

$$\varepsilon\leqslant\varepsilon_0(n,p)=\frac{p^{p/(p-1)}}{4n(p-1)+4(p-1)^2(n-1)}.$$

We have

$$\fint_{B_{4R}}|\nabla v|^{p-1}\eta^p\,dx\leqslant\Bigg(\fint_{B_{4R}}v^{-\beta_0+p-1}|\nabla\eta|^p\,dx\Bigg)^{(p-1)/p}\Bigg(\fint_{B_{4R}}v^{(-\beta_0+p-1)k}\eta^p\,dx\Bigg)^{1/p}. \tag{2.35}$$

Choosing $\eta\in C_0^\infty(B_{5R/2})$ such that $\eta=1$ in B_{2R} and $|\nabla\eta|\leqslant 8R^{-1}$, applying the estimate from Lemma 2.5 to the integrals on the right-hand side of (2.35), and using (2.33) again, we arrive at the required estimate (2.30). The inequality (2.31) under the same bound on ε follows from the same estimate in Lemma 2.5 and the definition of β_0 since $(\beta_0+1)(p-1)>p-1$. □

3 The Oscillation Lemma

We derive boundary estimates for the modulus of continuity.

For the solutions u_f to the Dirichlet problem (1.5) with smooth boundary function h defined on $\overline{D}$ and nonnegative on ∂D we set $m=\inf\limits_{\partial D\cap B_{4R}^{x_0}}h$, $x_0\in\partial D\cap\Sigma$ and

$$u_m=\begin{cases}\min(u,m), & \text{in}\quad D\cap B_{4R}^{x_0},\\ m, & \text{in}\quad B_{4R}^{x_0}\setminus D.\end{cases}$$

Lemma 3.1. *The function u_m is a bounded nonnegative supersolution to Equation* (1.1) *in the ball $B_{4R}^{x_0}$.*

Proof. Let $\varphi\in C_0^\infty(B_{4R}^{x_0})$, $\varphi\geqslant 0$. Then $(u-l)_-\varphi\in W_0^{1,2}(D\cap B_{4R}^{x_0})$ for $l\leqslant m$. We consider the function $T_\delta(s)=\delta^{-1}((s-m)_--(s-(m-\delta))_-)$, $\delta>0$. It is clear that $T_\delta(s)=1$ for $s\leqslant m-\delta$, $T_\delta(s)=0$ for $s\geqslant m$, the function $T_\delta(\cdot)$ is nonincreasing, and $T_\delta(u)\varphi\in W_0^{1,p}(B_{4R}^{x_0}\cap D)$. Taking $T_\delta(u)\varphi$ for a test function in the definition of the solution (1.3), we get

$$\int_{B_{4R}^{x_0}}\omega_\varepsilon|\nabla u|^{p-2}T_\delta(u)\nabla u\cdot\nabla\varphi\,dx=-\int_{B_{4R}^{x_0}}\omega_\varepsilon|\nabla u|^pT_\delta'(u)\varphi\,dx\geqslant 0.$$

Passing to the limit as $\delta\to 0$, we find

$$\int_{B_{4R}^{x_0}}\omega_\varepsilon|\nabla u_m|^{p-2}\nabla u_m\cdot\nabla\varphi\,dx\geqslant 0,$$

which is equivalent to the definition of a supersolution to Equation (1.1) in the ball $B_{4R}^{x_0}$. □

Further, we set $w = u_m$ and introduce v by formula (2.2) in the ball $B^{x_0}_{4R}$, i.e., $v = \min(u_m, \widetilde{u_m})$.

Lemma 3.2. *The function v is supersolution to Equation* (1.1) *in the ball $B^{x_0}_{4R}$.*

Proof. Let $\varphi \in C_0^\infty(B^{x_0}_{4R})$. We set

$$T_\delta(s) = \begin{cases} 1, & s \leqslant 0, \\ 1 - s\delta^{-1}, & 0 < s < \delta, , \\ 0, & s \geqslant \delta. \end{cases} \quad \widetilde{T}_\delta(s) = 1 - T_\delta(s).$$

Choosing in the definition of the supersolution (1.4) the test function $T_\delta(w - \widetilde{w})\varphi$ for w and using the monotonicity of the flow $|\xi|^{p-2}\xi$, we find

$$\int\limits_{B^{x_0}_{4R}} \omega_\varepsilon T_\delta(w - \widetilde{w})|\nabla w|^{p-2}\nabla w \cdot \nabla\varphi \, dx \geqslant - \int\limits_{B^{x_0}_{4R}} \omega_\varepsilon T'_\delta(w - \widetilde{w})\varphi|\nabla w|^{p-2}\nabla w \cdot \nabla(w - \widetilde{w}) \, dx$$

$$\geqslant - \int\limits_{B^{x_0}_{4R}} \omega_\varepsilon T'_\delta(w - \widetilde{w})\varphi|\nabla \widetilde{w}|^{p-2}\nabla \widetilde{w} \cdot \nabla(w - \widetilde{w}) \, dx := J.$$

For the integral J on the right-hand side we use the definition of $\widetilde{T}_\delta$ and the fact that $\widetilde{w}$ is a supersolution to Equation (1.1) in $B^{x_0}_{4R} \cap \{x_n > 0\}$ and $w - \widetilde{w} = 0$ in $B^{x_0}_{4R} \cap \{x_n < 0\}$. Then

$$J = \int\limits_{B^{x_0}_{4R} \cap \{x_n > 0\}} \omega_\varepsilon \widetilde{T}'_\delta(w - \widetilde{w})\varphi|\nabla \widetilde{w}|^{p-2}\nabla \widetilde{w} \cdot \nabla(w - \widetilde{w}) \, dx$$

$$\geqslant - \int\limits_{B^{x_0}_{4R} \cap \{x_n > 0\}} \omega_\varepsilon \widetilde{T}_\delta(w - \widetilde{w})|\nabla \widetilde{w}|^{p-2}\nabla \widetilde{w} \cdot \nabla\varphi \, dx.$$

Consequently,

$$\int\limits_{B^{x_0}_{4R}} \omega_\varepsilon T_\delta(w - \widetilde{w})|\nabla w|^{p-2}\nabla w \cdot \nabla\varphi \, dx + \int\limits_{B^{x_0}_{4R} \cap \{x_n > 0\}} \omega_\varepsilon \widetilde{T}_\delta(w - \widetilde{w})|\nabla \widetilde{w}|^{p-2}\nabla \widetilde{w} \cdot \nabla\varphi \, dx \geqslant 0.$$

Passing to the limit as $\delta \to 0$, we find

$$\int\limits_{\{w \leqslant \widetilde{w}\}} \omega_\varepsilon |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi \, dx + \int\limits_{\{w > \widetilde{w}\}} \omega_\varepsilon |\nabla \widetilde{w}|^{p-2}\nabla \widetilde{w} \cdot \nabla\varphi \, dx \geqslant 0.$$

Hence

$$\int\limits_{B^{x_0}_{4R}} \omega_\varepsilon |\nabla \min(w, \widetilde{w})|^{p-2}\nabla \min(w, \widetilde{w}) \cdot \nabla\varphi \, dx \geqslant 0,$$

which is equivalent to the definition of a supersolution in the ball $B^{x_0}_{4R}$. □

In what follows, $\varepsilon_0(n, p)$ is the same as in Lemma 2.6. We recall that the function $\gamma(\cdot)$ was introduced before Theorem 1.1.

Lemma 3.3. *Let $\varepsilon \leqslant \varepsilon_0(n,p)$. Then*

$$\inf_{B_R^{x_0}} u \geqslant C(n,p)m\gamma(R). \tag{3.1}$$

Proof. Inserting the test function $\varphi = (m-v)\eta^p$, into the integral inequality (1.4) for the supersolution v in the domain $B_{2R}^{x_0}$, where the function $\eta \in C_0^\infty(B_{2R}^{x_0})$ is radially symmetric, $\eta = 1$ in $B_R^{x_0}$, and $|\nabla\eta| \leqslant 4R^{-1}$, we find

$$\int\limits_{B_{2R}^{x_0}} |\nabla v|^p \eta^p \,\omega_\varepsilon\, dx \leqslant 4R^{-1} m \int\limits_{B_{2R}^{x_0}} |\nabla v|^{p-1} \,\omega_\varepsilon\, dx.$$

In particular, by the choice of the weight ω_ε and the definition of v, we have

$$\int\limits_{B_{2R}^{x_0}\cap\{x_n<0\}} |\nabla w|^p \eta^p \, dx \leqslant 4R^{-1} m \int\limits_{B_{2R}^{x_0}} |\nabla v|^{p-1} \, dx. \tag{3.2}$$

We recall that $w = u_m$. Let $G_R = \{w < \widetilde{w}\} \cap B_{2R}^{x_0}$. Inserting the test function

$$\varphi = (\widetilde{w} - w)_+ \eta^p = \begin{cases} (\widetilde{w} - w)\eta^p & \text{in} \quad G_R, \\ 0 & \text{in} \quad B_{2R}^{x_0} \setminus G_R, \end{cases}$$

into the integral inequality (1.4) for the supersolution v, where the cut-off function η is the same as above, we find

$$\int\limits_{G_R} |\nabla w|^p \eta^p \, dx \leqslant \int\limits_{G_R} |\nabla w|^{p-1} |\nabla \widetilde{w}| \eta^p \, dx + p \int\limits_{G_R} |\nabla w|^{p-1} \widetilde{w} |\nabla\eta| \eta^{p-1} \, dx.$$

Since $\widetilde{w} = \widetilde{u}_m \leqslant m$, applying the Young inequality to the integrand in the first term on the right-hand side of this estimate, we find

$$\int\limits_{G_R} |\nabla w|^p \eta^p \, dx \leqslant C(p) \Bigg(\int\limits_{G_R} |\nabla \widetilde{w}|^p \eta^p \, dx + R^{-1} m \int\limits_{G_R} |\nabla w|^{p-1} \, dx \Bigg).$$

Taking into account (3.2) and properties of the even extension $\widetilde{w}$ of w, we find

$$\int\limits_{(B_{2R}^{x_0}\cap\{x_n<0\})\cup G_R} |\nabla w|^p \eta^p \, dx \leqslant C(p) R^{-1} m \int\limits_{B_{2R}^{x_0}} |\nabla v|^{p-1} \, dx. \tag{3.3}$$

Furthermore, in view of (3.2) and the definition of $\widetilde{w}$, we have

$$\int\limits_{(B_{2R}^{x_0}\cap\{x_n>0\})\setminus G_R} |\nabla \widetilde{w}|^p \eta^p \, dx \leqslant C(p) R^{-1} m \int\limits_{B_{2R}^{x_0}} |\nabla v|^{p-1} \, dx. \tag{3.4}$$

Adding (3.3) and (3.4) and recalling the definition of v, we obtain the relation

$$\int\limits_{B_{2R}^{x_0}} |\nabla v|^p \eta^p \, dx \leqslant C(p) R^{-1} m \int\limits_{B_{2R}^{x_0}} |\nabla v|^{p-1} \, dx$$

which, together with the estimate $v^p \leqslant m v^{p-1}$, implies

$$\int\limits_{B^{x_0}_{2R}} |\nabla(\eta v)|^p \eta^p \, dx \leqslant C(p)\Bigg(R^{-1} m \int\limits_{B^{x_0}_{2R}} |\nabla v|^{p-1} \, dx + R^{-p} m \int\limits_{B^{x_0}_{2R}} v^{p-1} \, dx \Bigg). \tag{3.5}$$

Using (2.30) and (2.31) to estimate the integrals on the right-hand side of (3.5) and multiplying both sides of the obtained relation by m^{-p}, we get

$$\int\limits_{B^{x_0}_{2R}} |\nabla(\eta v m^{-1})|^p \eta^p \, dx \leqslant C(n,p) m^{1-p} R^{n-p} \big(\inf_{B^{x_0}_R} v\big)^{p-1}.$$

Since $\eta v m^{-1} \geqslant 1$ on $(\overline{B}^{x_0}_R \cap \{x_n \leqslant 0\}) \setminus D$, from the definition of capacity we have

$$C_p((\overline{B}^{x_0}_R \cap \{x_n \leqslant 0\}) \setminus D, B^{x_0}_{2R}) \leqslant C(n,p) m^{1-p} R^{n-p} \big(\inf_{B^{x_0}_R} v\big)^{p-1}.$$

Now, the required inequality (3.1) follows from the definition of v and γ. □

Lemma 3.4. *Let $\varepsilon \leqslant \varepsilon_0(n,p)$, and let $C = C(n,p)$ be the same constant as in Lemma* 3.3. *Then*

$$\operatorname*{ess\,osc}_{D \cap B^{x_0}_R} u \leqslant (1 - C\gamma(R)) \operatorname*{ess\,osc}_{D \cap B^{x_0}_{4R}} u + C\gamma(R) \operatorname*{osc}_{\partial D \cap B^{x_0}_{4R}} h.$$

Proof. We set $H_R = \sup\limits_{\partial D \cap B^{x_0}_R} h$, $h_R = \inf\limits_{\partial D \cap B^{x_0}_R} h$, $M_R = \sup\limits_{D \cap B^{x_0}_R} u$, $m_R = \inf\limits_{D \cap B^{x_0}_R} u$. Applying the estimate (3.1) to $M_{4R} - u$ and $u - m_{4R}$, we get

$$(M_{4R} - H_{4R}) C\gamma(R) \leqslant M_{4R} - M_R, \quad (h_{4R} - m_{4R}) C\gamma(R) \leqslant m_R - m_{4R},$$

where the constant C depends only on n and p. Adding these inequalities, we obtain the estimate

$$M_R - m_R \leqslant (1 - C\gamma(R))(M_{4R} - m_{4R}) + C\gamma(R)(H_{4R} - h_{4R}),$$

which implies the required assertion. □

4 Proof of the Main Result

In this section, we prove Theorem 1.1. We recall that $\varepsilon_0 = \varepsilon_0(n,p)$ is a positive number defined in Lemma 2.6. If $\varepsilon \in [\varepsilon_0, 1]$, then the assertion of the theorem (and even stronger estimates) follows from the results of [7, 8]. In what follows, we assume that $\varepsilon \in (0, \varepsilon_0)$.

It suffices to prove the required estimate for a smooth boundary function. We set $\xi(r) = \operatorname*{ess\,osc}\limits_{D \cap B^{x_0}_r} u$, $\xi_j = \xi(4^{-j} R)$, $\gamma_j = \gamma(4^{-j} R)$. Successively applying Lemma 3.4 and taking into account that $\operatorname*{ess\,osc}\limits_{D} u \leqslant \operatorname*{osc}\limits_{\partial D} f$ by the maximum principle, we obtain the iterated relation

$$\xi_k \leqslant \prod_{j=1}^{k} (1 - C\gamma_j) \operatorname*{osc}_{\partial D} f + C \sum_{j=1}^{k} \gamma_j \prod_{l=j+1}^{k} (1 - C\gamma_j) \operatorname*{osc}_{\partial D \cap B^{x_0}_\rho} f.$$

We assume that the constant C in Lemmas 3.3 and 3.4 is sufficiently small; namely,

$$C \leqslant (C_p(\overline{B}_1^{x_0}, B_2^{x_0}))^{1/(1-p)}/4.$$

Since $\gamma(r) \leqslant (C_p(\overline{B}_1^{x_0}, B_2^{x_0}))^{1/(p-1)}$ for any r, we have $C\gamma(r) \leqslant 1/4$. Since the logarithmic function is convex for $x \in [0,1/2]$, we have the inequality $\ln(1-x) \geqslant -2x\ln 2$ which implies

$$\prod_{j=k_1}^{k_2}(1-C\gamma_j) = \exp\biggl(\sum_{j=k_1}^{k_2}\ln(1-C\gamma_j)\biggr) \leqslant \exp\biggl(-\widetilde{C}\sum_{j=k_1}^{k_2}\gamma_j\biggr),$$

where $\widetilde{C} = C\ln 4$. Therefore,

$$\xi_k \leqslant \exp\biggl(-\widetilde{C}\sum_{j=1}^{k}\gamma_j\biggr)\operatorname*{osc}_{\partial D} f + \widetilde{C}\sum_{j=1}^{k}\gamma_j\exp\biggl(-\widetilde{C}\sum_{l=j+1}^{k}\gamma_l\biggr)\operatorname*{osc}_{\partial D\cap B_\rho^{x_0}} f.$$

Since $1-e^{-x} \geqslant x/2$ for $x \in [0,1]$ in view of the concavity of the function $1-e^{-x}$, we have

$$\exp\biggl(-\widetilde{C}\sum_{l=j+1}^{k}\gamma_l\biggr) - \exp\biggl(-\widetilde{C}\sum_{l=j}^{k}\gamma_l\biggr) \geqslant \frac{\widetilde{C}\gamma_j}{2}\exp\biggl(-\widetilde{C}\sum_{l=j+1}^{k}\gamma_l\biggr),$$

which implies

$$\widetilde{C}\sum_{j=1}^{k}\gamma_j\exp\biggl(-\widetilde{C}\sum_{l=j+1}^{k}\gamma_l\biggr) \leqslant 2\sum_{j=1}^{k}\biggl(\exp\biggl(-\widetilde{C}\sum_{l=j+1}^{k}\gamma_l\biggr) - \exp\biggl(-\widetilde{C}\sum_{l=j}^{k}\gamma_l\biggr)\biggr) \leqslant 2.$$

Hence

$$\xi_k \leqslant \exp\biggl(-\widetilde{C}\sum_{j=1}^{k}\gamma_j\biggr)\operatorname*{osc}_{\partial D} f + 2\operatorname*{osc}_{\partial D\cap B_\rho^{x_0}} f. \tag{4.1}$$

By the definition of $\gamma(r)$, for $t \in [4^{-j-1}\rho, 4^{-j}\rho]$ we have $\gamma(t) \leqslant 4^{(n-p)/(p-1)}\gamma_j$. Consequently,

$$\sum_{j=1}^{k}\gamma_j \geqslant \frac{4^{(p-n)/(p-1)}}{\ln 4}\int\limits_{4^{-1-k}\rho}^{\rho}\gamma(t)t^{-1}\,dt.$$

The last estimate and (4.1) for $r \in (4^{-k-1}\rho, 4^{-k}\rho]$, $k \in \mathbb{N}$, imply

$$\begin{aligned}\xi(r) \leqslant \xi_k &\leqslant \exp\biggl(-C(n,p)\int\limits_{4^{-1-k}\rho}^{\rho}\gamma(t)t^{-1}\,dt\biggr)\operatorname*{osc}_{\partial D} f + 2\operatorname*{osc}_{\partial D\cap B_\rho^{x_0}} f\\ &\leqslant \exp\biggl(-C(n,p)\int\limits_{r}^{\rho}\gamma(t)t^{-1}\,dt\biggr)\operatorname*{osc}_{\partial D} f + 2\operatorname*{osc}_{\partial D\cap B_\rho^{x_0}} f.\end{aligned}$$

Theorem 1.1 is proved.

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