DOI 10.1007/s10958-020-04992-x
Journal of Mathematical Sciences, Vol. 250, No. 1, October, 2020

METHOD FOR SOLVING THE NAVIER–STOKES AND EULER EQUATIONS OF MOTION FOR INCOMPRESSIBLE MEDIA

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UDC 517.957:532.51

We propose a method for solving the Navier–Stokes and Euler equations by introducing new unknowns and transforming the defining equations, which allows us to reduce the problem to simpler mathematical problems. Bibliography: 4 *titles.*

1 Introduction

The Navier–Stokes equations describe the motion of viscous media. In the case of an incompressible medium and in the presence of the external force potential, the Navier–Stokes equations can be written in the dimensionless variables as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial(p+\Phi)}{\partial x}+\frac{1}{\mathrm{Re}}\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\Big), \tag{1.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial(p+\Phi)}{\partial y}+\frac{1}{\mathrm{Re}}\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\Big), \tag{1.2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial(p+\Phi)}{\partial z}+\frac{1}{\mathrm{Re}}\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\Big). \tag{1.3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1.4}$$

where u, v, w are components of the velocity vector, p is the pressure depending on the coordinates x, y, z and time t, Φ is a given potential of external forces, and Re is the Reynolds number.

In the absence of viscous friction forces, the motion of an ideal medium is described by the Euler equations which can be written in the form (1.1)–(1.3) without terms proportional to $1/\mathrm{Re}$ on the right-hand sides; moreover, the order of equations is reduced, whereas the equation of continuity (1.4) is preserved.

Translated from *Problemy Matematicheskogo Analiza* **104**, 2020, pp. 11-20.

One of the main difficulties appearing in the study of the Navier–Stokes and Euler equations is that there are no general constructive methods for solving the equations. However, in spite of different orders of higher order derivatives, the structure of the Navier–Stokes equations is similar to that of the Euler equations; namely, both contain nonlinear convective terms combined with linear terms. Due to this fact, it is possible to apply the same approach to studying both systems. In this paper, we propose a constructive method for solving Equations (1.1)–(1.4). Using properties of the equations, we reduce (1.1)–(1.4) to a family of simpler mathematical problems.

2 The First Integral

2.1. Canonical form of equations.

Theorem 2.1. *Each of Equations* (1.1)–(1.4) *can be represented in the divergence form*

$$\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} + \frac{\partial R_i}{\partial z} + \frac{\partial S_i}{\partial t} = 0, \tag{2.1}$$

where P_i, Q_i, R_i, S_i are combinations of the unknowns and their first order derivatives.

Proof. 1. Equation (1.4) takes the form (2.1) for $P_4 = u$, $Q_4 = v$, $R_4 = w$, $S_4 = 0$.

2. We transform the nonlinear terms as follows:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial}{\partial x}\frac{u^2}{2} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - u\frac{\partial v}{\partial y} - u\frac{\partial w}{\partial z},$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial (uv)}{\partial x} + \frac{\partial}{\partial y}\frac{v^2}{2} + \frac{\partial (vw)}{\partial z} - v\frac{\partial u}{\partial x} - v\frac{\partial w}{\partial z},$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial (uw)}{\partial x} + \frac{\partial (vw)}{\partial y} + \frac{\partial}{\partial z}\frac{w^2}{2} - w\frac{\partial u}{\partial x} - w\frac{\partial v}{\partial y}.$$

By (1.4), the last two terms on the right-hand sides take the form

$$-u\frac{\partial v}{\partial y} - u\frac{\partial w}{\partial z} = u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{u^2}{2},$$

$$-v\frac{\partial u}{\partial x} - v\frac{\partial w}{\partial z} = v\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{v^2}{2},$$

$$-w\frac{\partial u}{\partial x} - w\frac{\partial v}{\partial y} = w\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\frac{w^2}{2}.$$

As a result, the nonlinear terms in (1.1)–(1.3) can be written as

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z}, \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z}. \end{aligned} \tag{2.2}$$

Since the linear terms in (1.1)–(1.3) are represented as the sums of the first order derivatives with respect to x, y, z, t, we can write (1.1)–(1.3) in the divergence form in view of (2.2):

$$
\begin{aligned}
&\frac{\partial}{\partial x}\Big(u^2+p+\Phi-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(uv-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial y}\Big)+\frac{\partial}{\partial z}\Big(uw-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial z}\Big)+\frac{\partial u}{\partial t}=0,\\
&\frac{\partial}{\partial x}\Big(uv-\frac{1}{\mathrm{Re}}\frac{\partial v}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(v^2+p+\Phi-\frac{1}{\mathrm{Re}}\frac{\partial v}{\partial y}\Big)+\frac{\partial}{\partial z}\Big(vw-\frac{1}{\mathrm{Re}}\frac{\partial v}{\partial z}\Big)+\frac{\partial v}{\partial t}=0,\\
&\frac{\partial}{\partial x}\Big(uw-\frac{1}{\mathrm{Re}}\frac{\partial w}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(vw-\frac{1}{\mathrm{Re}}\frac{\partial w}{\partial y}\Big)+\frac{\partial}{\partial z}\Big(w^2+p+\Phi-\frac{1}{\mathrm{Re}}\frac{\partial w}{\partial z}\Big)+\frac{\partial w}{\partial t}=0.
\end{aligned}
\tag{2.3}
$$

The theorem is proved. □

A similar assertion holds for the Euler equations. Moreover, each of the Navier–Stokes and Euler equations is linear with respect to P_i, Q_i, R_i, S_i.

2.2. Solution of the canonical equation. Equation (2.1) is said to be *canonical.*

Theorem 2.2. *Equation* (2.1) *has the general solution.*

Proof. 1. Equation (2.1) is linear. We consider partial solutions. For example, the two-term equation

$$\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}=0$$

generates one partial solution to Equation (2.1)

$$P_i=\frac{\partial \Psi_{i1}}{\partial y},\quad Q_i=-\frac{\partial \Psi_{i1}}{\partial x},\quad R_i=0,\quad S_i=0,$$

where Ψ_{i1} is an arbitrary twice continuously differentiable function of x, y, z, t. From (2.1) we can extract six different two-term equations

$$
\begin{aligned}
&\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}=0,\quad \frac{\partial P_i}{\partial x}+\frac{\partial R_i}{\partial z}=0,\quad \frac{\partial P_i}{\partial x}+\frac{\partial S_i}{\partial t}=0,\\
&\frac{\partial Q_i}{\partial y}+\frac{\partial R_i}{\partial z}=0,\quad \frac{\partial Q_i}{\partial y}+\frac{\partial S_i}{\partial t}=0,\quad \frac{\partial R_i}{\partial z}+\frac{\partial S_i}{\partial t}=0.
\end{aligned}
\tag{2.4}
$$

Solving each equation in (2.4), we obtain a particular solution to Equation (2.1). Thus, we have six different partial solutions

$$
\begin{aligned}
&P_i=\frac{\partial \Psi_{i1}}{\partial y},\quad Q_i=-\frac{\partial \Psi_{i1}}{\partial x},\quad R_i=0,\quad S_i=0,\\
&P_i=-\frac{\partial \Psi_{i2}}{\partial z},\quad Q_i=0,\quad R_i=\frac{\partial \Psi_{i2}}{\partial x},\quad S_i=0,\\
&P_i=\frac{\partial \Psi_{i3}}{\partial t},\quad Q_i=0,\quad R_i=0,\quad S_i=-\frac{\partial \Psi_{i3}}{\partial x},\\
&P_i=0,\quad Q_i=\frac{\partial \Psi_{i4}}{\partial z},\quad R_i=-\frac{\partial \Psi_{i4}}{\partial y},\quad S_i=0,\\
&P_i=0,\quad Q_i=-\frac{\partial \Psi_{i5}}{\partial t},\quad R_i=0,\quad S_i=\frac{\partial \Psi_{i5}}{\partial y},
\end{aligned}
\tag{2.5}
$$

$$P_i = 0, \quad Q_i = 0, \quad R_i = \frac{\partial \Psi_{i6}}{\partial t}, \quad S_i = -\frac{\partial \Psi_{i6}}{\partial z},$$

where Ψ_{i1}, $i = 1, \dots, 6$, are arbitrary twice continuously differentiable functions of x, y, z, t.

We consider the partial solutions as a four-dimensional vector and prove that these solutions are linearly independent. For this purpose we consider a linear combination with constant coefficients λ_k, $k = 1, \dots, 6$, and equate it to zero. We show that the coefficients are necessarily zero. For the first components we have

$$\lambda_1 \frac{\partial \Psi_{i1}}{\partial y} - \lambda_2 \frac{\partial \Psi_{i2}}{\partial z} + \lambda_3 \frac{\partial \Psi_{i3}}{\partial t} = 0.$$

We assume that there is at least one nonzero coefficient, for example, λ_1. Then

$$\frac{\partial \Psi_{i1}}{\partial y} = \frac{\lambda_2}{\lambda_1} \frac{\partial \Psi_{i2}}{\partial z} - \frac{\lambda_3}{\lambda_1} \frac{\partial \Psi_{i3}}{\partial t},$$

which contradicts the arbitrariness of Ψ_{ik}. Consequently, $\lambda_1 = 0$. In the same way, $\lambda_j = 0$ for $j = 2, \dots, 6$. Thus, the solutions are linearly independent.

2. We construct the general solution as a linear combination of six partial solutions. Since Ψ_{ik} are arbitrary, we can assume that $\lambda_k = 1$. Then the general solution to Equation (2.1) is defined by

$$\begin{aligned} P_i &= \frac{\partial \Psi_{i1}}{\partial y} - \frac{\partial \Psi_{i2}}{\partial z} + \frac{\partial \Psi_{i3}}{\partial t}, \quad Q_i = -\frac{\partial \Psi_{i1}}{\partial x} + \frac{\partial \Psi_{i4}}{\partial z} - \frac{\partial \Psi_{i5}}{\partial t}, \\ R_i &= \frac{\partial \Psi_{i2}}{\partial x} - \frac{\partial \Psi_{i4}}{\partial y} + \frac{\partial \Psi_{i6}}{\partial t}, \quad S_i = -\frac{\partial \Psi_{i3}}{\partial x} + \frac{\partial \Psi_{i5}}{\partial y} - \frac{\partial \Psi_{i6}}{\partial z}. \end{aligned} \tag{2.6}$$

The theorem is proved. □

Corollary 2.1. *A solution to Equation* (2.1) *is determined by the twice continuously differentiable functions* Ψ_{ij}, $j = 1, \dots, 6$, *of* x, y, z, t.

The aforesaid is also related to complete equations of the form (2.1), in particular, for (1.1)–(1.3), whereas Equation (1.4) is not complete because it does not contain the time-derivative. Since $C_3^2 = 3$, from (1.4) we can extract only three different two-term equations. Respectively, we have only three independent partial solutions. The general solution to Equation (1.4) is determined by the three functions Ψ_{4k}, $\Psi_{4(k+1)}$, $\Psi_{4(k+2)}$ as follows:

$$\begin{aligned} P_4 &= \frac{\partial \Psi_{4(k+2)}}{\partial y} - \frac{\partial \Psi_{4(k+1)}}{\partial z}, \\ Q_4 &= -\frac{\partial \Psi_{4(k+2)}}{\partial x} + \frac{\partial \Psi_{4k}}{\partial z}, \\ R_4 &= \frac{\partial \Psi_{4(k+1)}}{\partial x} - \frac{\partial \Psi_{4k}}{\partial y}. \end{aligned} \tag{2.7}$$

The above formulas for the solutions to Equation (2.1) determine relations between P_i, Q_i, R_i, S_i and, consequently, implicit relations between u, v, w, p. We study them in more detail.

We compare Equations (2.3) and formulas for solutions (2.6). Each equation in (2.3) has a solution of the form (2.6). Consequently, for every equation in (2.3) we have four expressions for

P_i, Q_i, R_i, S_i. Thus, we have in total 12 such expressions. Each of the expressions generated by one equation contains six arbitrary functions. For Equation (1.4) we additionally have three expressions for u, v, w with three arbitrary functions. Thus, we have 15 relations, where, together with the main unknowns u, v,w, p, there are 21 new functions of four variables. We enumerate these functions, starting with $j = 1$, and denote $\Psi_{1,j}$. Then we can write

$$\begin{cases} u^2 + p + \Phi - \dfrac{1}{\mathrm{Re}}\dfrac{\partial u}{\partial x} = \dfrac{\partial \Psi_{1,1}}{\partial y} - \dfrac{\partial \Psi_{1,2}}{\partial z} + \dfrac{\partial \Psi_{1,3}}{\partial t}, \\ uv - \dfrac{1}{\mathrm{Re}}\dfrac{\partial u}{\partial y} = -\dfrac{\partial \Psi_{1,1}}{\partial x} + \dfrac{\partial \Psi_{1,4}}{\partial z} - \dfrac{\partial \Psi_{1,5}}{\partial t}, \\ uw - \dfrac{1}{\mathrm{Re}}\dfrac{\partial u}{\partial z} = \dfrac{\partial \Psi_{1,2}}{\partial x} - \dfrac{\partial \Psi_{1,4}}{\partial y} + \dfrac{\partial \Psi_{1,6}}{\partial t}, \\ u = -\dfrac{\partial \Psi_{1,3}}{\partial x} + \dfrac{\partial \Psi_{1,5}}{\partial y} - \dfrac{\partial \Psi_{1,6}}{\partial z}, \end{cases} \tag{2.8}$$

$$\begin{cases} uv - \dfrac{1}{\mathrm{Re}}\dfrac{\partial v}{\partial x} = \dfrac{\partial \Psi_{1,7}}{\partial y} - \dfrac{\partial \Psi_{1,8}}{\partial z} + \dfrac{\partial \Psi_{1,9}}{\partial t}, \\ v^2 + p + \Phi - \dfrac{1}{\mathrm{Re}}\dfrac{\partial v}{\partial y} = -\dfrac{\partial \Psi_{1,7}}{\partial x} + \dfrac{\partial \Psi_{1,10}}{\partial z} - \dfrac{\partial \Psi_{1,11}}{\partial t}, \\ vw - \dfrac{1}{\mathrm{Re}}\dfrac{\partial v}{\partial z} = \dfrac{\partial \Psi_{1,8}}{\partial x} - \dfrac{\partial \Psi_{1,10}}{\partial y} + \dfrac{\partial \Psi_{1,12}}{\partial t}, \\ v = -\dfrac{\partial \Psi_{1,9}}{\partial x} + \dfrac{\partial \Psi_{1,11}}{\partial y} - \dfrac{\partial \Psi_{1,12}}{\partial z}, \end{cases} \tag{2.9}$$

$$\begin{cases} uw - \dfrac{1}{\mathrm{Re}}\dfrac{\partial w}{\partial x} = \dfrac{\partial \Psi_{1,13}}{\partial y} - \dfrac{\partial \Psi_{1,14}}{\partial z} + \dfrac{\partial \Psi_{1,15}}{\partial t}, \\ vw - \dfrac{1}{\mathrm{Re}}\dfrac{\partial w}{\partial y} = -\dfrac{\partial \Psi_{1,13}}{\partial x} + \dfrac{\partial \Psi_{1,16}}{\partial z} - \dfrac{\partial \Psi_{1,17}}{\partial t}, \\ w^2 + p + \Phi - \dfrac{1}{\mathrm{Re}}\dfrac{\partial w}{\partial z} = \dfrac{\partial \Psi_{1,14}}{\partial x} - \dfrac{\partial \Psi_{1,16}}{\partial y} + \dfrac{\partial \Psi_{1,18}}{\partial t}, \\ w = -\dfrac{\partial \Psi_{1,15}}{\partial x} + \dfrac{\partial \Psi_{1,17}}{\partial y} - \dfrac{\partial \Psi_{1,18}}{\partial z}, \end{cases} \tag{2.10}$$

$$u = \frac{\partial \Psi_{1,21}}{\partial y} - \frac{\partial \Psi_{1,20}}{\partial z}, \quad v = -\frac{\partial \Psi_{1,21}}{\partial x} + \frac{\partial \Psi_{1,19}}{\partial z}, \quad w = \frac{\partial \Psi_{1,20}}{\partial x} - \frac{\partial \Psi_{1,19}}{\partial y}. \tag{2.11}$$

2.3. The integral of the original equations. The functions $\Psi_{1,j}$, $j = 1, \ldots, 21$, on the right-hand sides of (2.8)–(2.11) are called *stream pseudofunctions* of the first order (cf. [1]–[3]). In the simplest case of $2D$ equations of steady flow, $\Psi_{1,21}$ is the usual stream function $\Psi(x, y)$. We consider $\Psi_{1,j}$ as new associated unknowns and preserve the unknowns u, v, w, p. We try to simplify Equations (2.8)–(2.11) and reduce them to the most convenient form in practice. The solution of this particular problem can be found in [2, 3]. After some transformations of (2.8)–(2.11) we obtain the nine equations

$$p = p_0 - \Phi - \frac{U^2}{2} - d - \frac{1}{3} \cdot \frac{\partial}{\partial t}\Big(\frac{\partial}{\partial x}(\Psi_2 - \Psi_1) + \frac{\partial}{\partial y}(\Psi_4 - \Psi_3) + \frac{\partial}{\partial z}(\Psi_6 - \Psi_5)\Big), \tag{2.12}$$

$$u^2 - v^2 + \frac{2}{\mathrm{Re}}\Big(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) = -\frac{\partial^2\Psi_{10}}{\partial x^2} + \frac{\partial^2\Psi_{10}}{\partial y^2} - \frac{\partial^2\Psi_{11}}{\partial z^2} - \frac{\partial^2\Psi_{12}}{\partial z^2}$$

$$+ \frac{\partial^2\Psi_{14}}{\partial x\partial z} + \frac{\partial^2\Psi_{15}}{\partial y\partial z} + \frac{\partial}{\partial t}\Big[-\frac{\partial\Psi_1}{\partial x} + \frac{\partial\Psi_3}{\partial y} + \frac{\partial(\Psi_5+\Psi_6)}{\partial z}\Big], \tag{2.13}$$

$$v^2 - w^2 + \frac{2}{\mathrm{Re}}\Big(-\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) = \frac{\partial^2\Psi_{10}}{\partial x^2} + \frac{\partial^2\Psi_{11}}{\partial x^2} - \frac{\partial^2\Psi_{12}}{\partial y^2} + \frac{\partial^2\Psi_{12}}{\partial z^2}$$

$$- \frac{\partial^2\Psi_{13}}{\partial x\partial y} - \frac{\partial^2\Psi_{14}}{\partial x\partial z} + \frac{\partial}{\partial t}\Big[\frac{\partial(\Psi_1+\Psi_2)}{\partial x} + \frac{\partial\Psi_4}{\partial y} - \frac{\partial\Psi_6}{\partial z}\Big], \tag{2.14}$$

$$uv - \frac{1}{\mathrm{Re}}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) = -\frac{\partial^2\Psi_{10}}{\partial x\partial y} + \frac{1}{2}\cdot\Big(\frac{\partial^2\Psi_{15}}{\partial x\partial z} + \frac{\partial^2\Psi_{14}}{\partial y\partial z} + \frac{\partial^2\Psi_{13}}{\partial z^2}\Big)$$

$$- \frac{1}{2}\frac{\partial}{\partial t}\Big[\frac{\partial\Psi_3}{\partial x} + \frac{\partial\Psi_1}{\partial y} + \frac{\partial(\Psi_8+\Psi_9)}{\partial z}\Big], \tag{2.15}$$

$$uw - \frac{1}{\mathrm{Re}}\Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big) = \frac{\partial^2\Psi_{11}}{\partial x\partial z} - \frac{1}{2}\cdot\Big(\frac{\partial^2\Psi_{15}}{\partial x\partial y} + \frac{\partial^2\Psi_{14}}{\partial y^2} + \frac{\partial^2\Psi_{13}}{\partial y\partial z}\Big)$$

$$+ \frac{1}{2}\frac{\partial}{\partial t}\Big[-\frac{\partial\Psi_5}{\partial x} + \frac{\partial(\Psi_9-\Psi_7)}{\partial y} + \frac{\partial\Psi_2}{\partial z}\Big], \tag{2.16}$$

$$vw - \frac{1}{\mathrm{Re}}\Big(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\Big) = -\frac{\partial^2\Psi_{12}}{\partial y\partial z} + \frac{1}{2}\Big(\frac{\partial^2\Psi_{14}}{\partial x\partial y} + \frac{\partial^2\Psi_{15}}{\partial x^2} - \frac{\partial^2\Psi_{13}}{\partial x\partial z}\Big)$$

$$+ \frac{1}{2}\cdot\frac{\partial}{\partial t}\Big[\frac{\partial(\Psi_7+\Psi_8)}{\partial x} + \frac{\partial\Psi_6}{\partial y} + \frac{\partial\Psi_4}{\partial z}\Big], \tag{2.17}$$

$$u = \frac{1}{2}\cdot\Big[\frac{\partial}{\partial y}\Big(-\frac{\partial\Psi_3}{\partial x} + \frac{\partial\Psi_1}{\partial y} + \frac{\partial\Psi_7}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(-\frac{\partial\Psi_5}{\partial x} + \frac{\partial\Psi_8}{\partial y} - \frac{\partial\Psi_2}{\partial z}\Big)\Big], \tag{2.18}$$

$$v = \frac{1}{2}\cdot\Big[\frac{\partial}{\partial x}\Big(\frac{\partial\Psi_3}{\partial x} - \frac{\partial\Psi_1}{\partial y} - \frac{\partial\Psi_7}{\partial z}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial\Psi_9}{\partial x} + \frac{\partial\Psi_6}{\partial y} - \frac{\partial\Psi_4}{\partial z}\Big)\Big], \tag{2.19}$$

$$w = \frac{1}{2}\cdot\Big[\frac{\partial}{\partial x}\Big(\frac{\partial\Psi_5}{\partial x} - \frac{\partial\Psi_8}{\partial y} + \frac{\partial\Psi_2}{\partial z}\Big) + \frac{\partial}{\partial y}\Big(-\frac{\partial\Psi_9}{\partial x} - \frac{\partial\Psi_6}{\partial y} + \frac{\partial\Psi_4}{\partial z}\Big)\Big]. \tag{2.20}$$

In (2.12)–(2.20), we used the simpler notation for the associated unknowns Ψ_k, where the subscript indicates the number of a pseudofunction without its order. The number of the unknowns Ψ_k is equal to 15, so that $k = 1, \ldots, 15$. In (2.12), p_0 is the additive pressure constant, U is the absolute value of the velocity vector, and d is the dissipative term defined by

$$\begin{aligned} &U = \sqrt{u^2+v^2+w^2}, \\ &d = -\frac{U^2}{6} + \frac{1}{3}\Big(-\Delta_{xy}\Psi_{10} + \Delta_{xz}\Psi_{11} - \Delta_{yz}\Psi_{12} - \frac{\partial^2\Psi_{13}}{\partial x\partial y} + \frac{\partial^2\Psi_{14}}{\partial x\partial z} - \frac{\partial^2\Psi_{15}}{\partial y\partial z}\Big), \end{aligned} \tag{2.21}$$

where Δ_{xy}, Δ_{xz}, Δ_{yz} denote the incomplete Laplace operators

$$\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta_{xz} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_{yz} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Equations (2.12)–(2.20) are relations between the main unknowns u, v, w, p, the associated unknowns Ψ_n, $n = 1, \dots, 15$, the Reynolds number Re, and the external force potential Φ. The derivatives with respect to u, v, w, p has the order less by 1 than their order in (1.1)–(1.4).

From (2.12)–(2.20) we easily obtain the first integral for the Euler equations. It suffices to put $1/\operatorname{Re} = 0$ in (2.13)–(2.17). One can show that the classical integrals of the Bernoulli, Euler–Bernoulli, and Lagrange–Cauchy equations governing the motion of an incompressible fluid are particular cases of (2.12)–(2.20) (cf. [4]).

Analyzing (2.12)–(2.20), we come to the following conclusion. We have the system of nine equations with 19 unknowns, i.e., the main unknowns u, v, w, p and the associated unknowns Ψ_n, where $n = 1, \dots, 15$. The fact that the number of unknowns exceeds the number of equations plays a positive role in particular problems since we have the possibility to model solutions and to satisfy some additional conditions, for example, the initial and boundary conditions.

Thus, we have constructed the first integral. Now, we find the main unknowns u, v, w, p.

3 Finding the Main Unknowns

We consider (2.12) and (2.18)–(2.20). We emphasize that these four relations determine the general structure of solutions to the Navier–Stokes equations. By (2.12), the pressure p is represented as the algebraic sum of the following four different terms: the external force potential Φ, the velocity pressure $U^2/2$, and the dissipative terms d and d_t, where d is defined in (2.21) and d_t is defined by

$$d_t = \frac{1}{3} \cdot \frac{\partial}{\partial t}\Big(\frac{\partial}{\partial x}(\Psi_2 - \Psi_1) + \frac{\partial}{\partial y}(\Psi_4 - \Psi_3) + \frac{\partial}{\partial z}(\Psi_6 - \Psi_5)\Big). \tag{3.1}$$

By (2.18)–(2.20), the velocities u, v, w are represented as a linear combination of the second order derivatives of the functions Ψ_k, $k = 1, \dots, 9$. We note that the linear combination is determined by only nine associated unknowns among 15 ones. To specify particular fragments of the linear combination and terms determining the pressure, it is necessary to solve the five nonlinear equations (2.13)–(2.17). Apparently, making transformations of (2.13)–(2.17), one can obtain two relations guaranteeing the validity of the remaining ones.

3.1. The prime generator of solutions. We consider (2.13)–(2.17). Denote by f_k, $k = 2, \dots, 6$, the sums of terms in (2.13)–(2.17) depending only on the six associated unknowns with numbers $10, \dots, 15$. We have

$$\begin{aligned}
f_2 &= -\frac{\partial^2\Psi_{10}}{\partial x^2} + \frac{\partial^2\Psi_{10}}{\partial y^2} - \frac{\partial^2\Psi_{11}}{\partial z^2} - \frac{\partial^2\Psi_{12}}{\partial z^2} + \frac{\partial^2\Psi_{15}}{\partial y\partial z} + \frac{\partial^2\Psi_{14}}{\partial x\partial z},\\
f_3 &= \frac{\partial^2\Psi_{10}}{\partial x^2} + \frac{\partial^2\Psi_{11}}{\partial x^2} - \frac{\partial^2\Psi_{12}}{\partial y^2} + \frac{\partial^2\Psi_{12}}{\partial z^2} - \frac{\partial^2\Psi_{13}}{\partial x\partial y} - \frac{\partial^2\Psi_{14}}{\partial x\partial z},\\
f_4 &= -\frac{\partial^2\Psi_{10}}{\partial x\partial y} + \frac{1}{2}\frac{\partial}{\partial z}\Big(- \frac{\partial\Psi_{15}}{\partial x} + \frac{\partial\Psi_{14}}{\partial y} + \frac{\partial\Psi_{13}}{\partial z}\Big),\\
f_5 &= \frac{\partial^2\Psi_{11}}{\partial x\partial z} + \frac{1}{2}\frac{\partial}{\partial y}\Big(- \frac{\partial\Psi_{15}}{\partial x} - \frac{\partial\Psi_{14}}{\partial y} - \frac{\partial\Psi_{13}}{\partial z}\Big),\\
f_6 &= -\frac{\partial^2\Psi_{12}}{\partial y\partial z} + \frac{1}{2}\frac{\partial}{\partial x}\Big(\frac{\partial\Psi_{14}}{\partial y} + \frac{\partial\Psi_{15}}{\partial x} - \frac{\partial\Psi_{13}}{\partial z}\Big).
\end{aligned} \tag{3.2}$$

Then we can write (2.13)–(2.17) as

$$\begin{aligned}
f_2 &= u^2 - v^2 + \frac{2}{\mathrm{Re}}\Big(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial t}\Big(\frac{\partial \Psi_1}{\partial x} - \frac{\partial \Psi_3}{\partial y} - \frac{\partial \Psi_5}{\partial z} - \frac{\partial \Psi_6}{\partial z}\Big),\\
f_3 &= v^2 - w^2 + \frac{2}{\mathrm{Re}}\Big(-\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) - \frac{\partial}{\partial t}\Big(\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial x} + \frac{\partial \Psi_4}{\partial y} - \frac{\partial \Psi_6}{\partial z}\Big),\\
f_4 &= uv - \frac{1}{\mathrm{Re}}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) + \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_8}{\partial z} + \frac{\partial \Psi_9}{\partial z}\Big),\\
f_5 &= uw - \frac{1}{\mathrm{Re}}\Big(-\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big) + \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_7}{\partial y} - \frac{\partial \Psi_9}{\partial y} - \frac{\partial \Psi_2}{\partial z}\Big),\\
f_6 &= vw - \frac{1}{\mathrm{Re}}\Big(-\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\Big) - \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial \Psi_7}{\partial x} + \frac{\partial \Psi_8}{\partial x} + \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z}\Big).
\end{aligned} \tag{3.3}$$

Equations (3.2) are linear inhomogeneous equations for Ψ_j, $j = 10,\dots,15$. Calculating the derivatives with respect to the coordinates in the last three equations in (3.2) and substituting the result into the first two equations in (3.2), we can obtain the zero right-hand sides. The obtained two equations have the form

$$\begin{aligned}
&\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_4}{\partial x^2} + \frac{\partial^2 f_4}{\partial y^2} + \frac{\partial^2 f_5}{\partial y \partial z} - \frac{\partial^2 f_6}{\partial x \partial z} = 0,\\
&\frac{\partial^2 f_3}{\partial y \partial z} + \frac{\partial^2 f_4}{\partial x \partial z} - \frac{\partial^2 f_5}{\partial x \partial y} - \frac{\partial^2 f_6}{\partial y^2} + \frac{\partial^2 f_6}{\partial z^2} = 0.
\end{aligned} \tag{3.4}$$

By (3.3), it is clear that the functions f_k depend on u, v, w and Ψ_j, $j = 1,\dots,9$. However, by (2.18)–(2.20), u, v, w can be also found from Ψ_j, $j = 1,\dots,9$. Thus, the functions f_k are determined by Ψ_j, $j = 1,\dots,9$, and (3.4) can be regarded as a system of two nonlinear equations of the fifth order with respect to Ψ_j containing $1/\mathrm{Re}$ as a parameter at the higher order derivative. Similar assertions are valid for the Euler equations with the only difference that we should put $1/\mathrm{Re} = 0$. Moreover, the order of higher derivatives in (3.4) is reduced from 5 to 4. Any collection of functions Ψ_j, $j = 1,\dots,9$, satisfying (3.4) generates a solution to Equations (1.1)–(1.4). Equations (3.4) can be regarded as a prime generator of solutions to Equations (1.1)–(1.4).

3.2. Finding the main unknowns. We describe an algorithm for finding u, v, w, p.

Step 1. Express f_k, $k = 2,\dots,6$, in terms of Ψ_j, $j = 1,\dots,9$, from (3.3) and (2.18)–(2.20).

Step 2. Solve Equations (3.4) for Ψ_j, $j = 1,\dots,9$.

Step 3. Find the main unknowns u, v, w from (2.18)–(2.20).

Step 4. Find the associated unknowns Ψ_j, $j = 10,\dots,15$, from the last three equations in (3.2).

Step 5. Find p from (2.12), taking into account (2.21) and (3.1).

Thus, the solution of Equations (1.1)–(1.4) is reduced to solving five simpler problems. As a result, all the main unknowns u, v, w, p can be found and Equations (1.1)–(1.4) can be solved.

For the Euler equations an analogous conclusion can be made with the only difference that we put $1/\mathrm{Re} = 0$ at all intermediate steps.

Remark 3.1. Analyzing the algorithm, we can conclude that the second and fourth steps are the most complicated. At the second step, we need to solve the system of two nonlinear equations (3.4) for nine unknowns Ψ_j, $j = 1, \dots, 9$. At the fourth step, we need to solve three linear equations (3.2) for six unknowns Ψ_j, $j = 10, \dots, 15$. Thus, to solve the Navier–Stokes equations (1.1)–(1.3), we need to solve nonlinear, as well as linear problems.

A deeper analysis of the obtained equations leads to the conclusion that the fourth and fifth steps can be considerably simplified. We prove the following assertion.

Theorem 3.1. *The main unknowns u, v, w, p are independent of the associated unknowns Ψ_i, $i = 10, \dots, 15$.*

Proof. 1. For u, v, w the assertion is obvious. Indeed, u, v, w are found by formulas (2.18)–(2.20) where the right-hand sides depend only on the associated unknowns Ψ_j, $j = 1, \dots, 9$.

2. We show that the function p defined by (2.12), is independent of Ψ_k, $k = 10, \dots, 15$. We consider the last equation in (3.2) and calculate term-by-term the double integral with respect to y and z, where for the sake of simplicity we assume that the additive functions appearing while integrating vanish:

$$\Psi_{12} = -\int_{y_0}^{y}\int_{z_0}^{z} f_6 dydz + \frac{1}{2}\left(\int_{z_0}^{z}\frac{\partial\Psi_{14}}{\partial x}dz + \int_{y_0}^{y}\int_{z_0}^{z}\frac{\partial^2\Psi_{15}}{\partial x^2}dydz - \int_{y_0}^{y}\frac{\partial\Psi_{13}}{\partial x}dy\right).$$

Similarly, for the third and fourth equations in (3.2) we calculate the double integrals with respect to x, z and x, y respectively. As a result, for the unknowns Ψ_{11} and Ψ_{10} we find

$$\Psi_{11} = \int_{x_0}^{x}\int_{z_0}^{z} f_5 dxdz + \frac{1}{2}\left(\int_{z_0}^{z}\frac{\partial\Psi_{15}}{\partial y}dz + \int_{x_0}^{x}\int_{z_0}^{z}\frac{\partial^2\Psi_{14}}{\partial y^2}dxdz + \int_{x_0}^{x}\frac{\partial\Psi_{13}}{\partial y}dx\right),$$

$$\Psi_{10} = -\int_{x_0}^{x}\int_{y_0}^{y} f_4 dxdy + \frac{1}{2}\left(-\int_{y_0}^{y}\frac{\partial\Psi_{15}}{\partial z}dy + \int_{x_0}^{x}\int_{y_0}^{y}\frac{\partial^2\Psi_{13}}{\partial z^2}dxdy + \int_{x_0}^{x}\frac{\partial\Psi_{14}}{\partial z}dx\right).$$

Using (2.21), we calculate the incomplete Laplace operators $\Delta_{xy}\Psi_{10}$, $\Delta_{xz}\Psi_{11}$, $\Delta_{yz}\Psi_{12}$:

$$\begin{aligned}\frac{\partial^2\Psi_{10}}{\partial x^2} + \frac{\partial^2\Psi_{10}}{\partial y^2} = &-\int_{y_0}^{y}\frac{\partial f_4}{\partial x}dy - \int_{x_0}^{x}\frac{\partial f_4}{\partial y}dx + \frac{1}{2}\left(\int_{y_0}^{y}\frac{\partial^3\Psi_{13}}{\partial z^2\partial x}dy + \int_{x_0}^{x}\frac{\partial^3\Psi_{13}}{\partial z^2\partial y}dx\right.\\ &\left.+\int_{x_0}^{x}\frac{\partial^3\Psi_{14}}{\partial y^2\partial z}dx + \frac{\partial^2\Psi_{14}}{\partial x\partial z} - \int_{y_0}^{y}\frac{\partial^3\Psi_{15}}{\partial x^2\partial z}dy - \frac{\partial^2\Psi_{15}}{\partial y\partial z}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\Psi_{11}}{\partial x^2} + \frac{\partial^2\Psi_{11}}{\partial z^2} = &\int_{z_0}^{z}\frac{\partial f_5}{\partial x}dz + \int_{x_0}^{x}\frac{\partial f_5}{\partial z}dx + \frac{1}{2}\left(\int_{x_0}^{x}\frac{\partial^3\Psi_{13}}{\partial z^2\partial y}dx + \frac{\partial^2\Psi_{13}}{\partial y\partial x}\right.\\ &\left.+\int_{x_0}^{x}\frac{\partial^3\Psi_{14}}{\partial y^2\partial z}dx + \int_{z_0}^{z}\frac{\partial^3\Psi_{14}}{\partial y^2\partial x}dz + \frac{\partial^2\Psi_{15}}{\partial y\partial z} + \int_{z0}^{z}\frac{\partial^3\Psi_{15}}{\partial x^2\partial y}dz\right),\end{aligned} \tag{3.5}$$

$$\frac{\partial^2\Psi_{12}}{\partial y^2}+\frac{\partial^2\Psi_{12}}{\partial z^2}=-\int\limits_{z_0}^{z}\frac{\partial f_6}{\partial y}dz-\int\limits_{y_0}^{y}\frac{\partial f_6}{\partial z}dy+\frac{1}{2}\Bigg(\int\limits_{z_0}^{z}\frac{\partial^3\Psi_{14}}{\partial y^2\partial x}dz+\frac{\partial^2\Psi_{14}}{\partial x\partial z}$$
$$-\int\limits_{y_0}^{y}\frac{\partial^3\Psi_{13}}{\partial z^2\partial x}dy-\frac{\partial^2\Psi_{13}}{\partial x\partial y}+\int\limits_{y_0}^{y}\frac{\partial^3\Psi_{15}}{\partial x^2\partial z}dy+\int\limits_{z_0}^{z}\frac{\partial^3\Psi_{15}}{\partial x^2\partial y}dz\Bigg).$$

We write the right-hand side of (2.21) taking into account the obtained relations. We note that all the terms with Ψ_k, $k=10,\dots,15$, cancel, and for d we get

$$d=-\frac{U^2}{6}+\frac{1}{3}\Bigg(\int\limits_{y_0}^{y}\frac{\partial f_4}{\partial x}dy+\int\limits_{x_0}^{x}\frac{\partial f_4}{\partial y}dx+\int\limits_{x_0}^{x}\frac{\partial f_5}{\partial z}dx+\int\limits_{z_0}^{z}\frac{\partial f_5}{\partial x}dz+\int\limits_{y_0}^{y}\frac{\partial f_6}{\partial z}dy+\int\limits_{z_0}^{z}\frac{\partial f_6}{\partial y}dz\Bigg). \quad (3.6)$$

The right-hand sides of (3.6) involve only U, f_4, f_5, f_6, depending on the unknowns Ψ_j, $j=1,\dots,9$, whereas the unknowns Ψ_k, $k=10,\dots,15$, are absent. The same is true for (2.12). Hence p is independent of Ψ_k, $k=10,\dots,15$, which is required. □

Thus, the fourth and fifth steps can be replaced with a single step of calculating the unknown p by formulas (2.12), (3.1), (3.6). Thereby the procedure is considerably simplified.

4 Examples

We consider examples of realization of the algorithm. We construct solutions to the Navier–Stokes equations in the case where a cascade of plane waves propagates in deep water without actions of external forces ($\Phi=0$). Such waves are the simplest ones in a liquid medium. From the physical point of view, such a wave process means that the wave propagates in the direction of some (wave) vector and the wave front is the plane orthogonal to this vector.

To simplify calculations, we construct the solutions in the complex form. The unknowns u, v, w are looked for as a combination of plane waves by formulas (2.18)–(2.20)

$$\begin{aligned}
u&=\frac{i}{2}\Big(Am_1e^{i(n_1x+m_1y+l_1z)}+Bl_2e^{i(n_2x+m_2y+l_2z)}\Big),\\
v&=\frac{i}{2}\Big(-An_1e^{i(n_1x+m_1y+l_1z)}+Cl_3e^{i(n_3x+m_3y+l_3z)}\Big),\\
w&=\frac{i}{2}\Big(-Bn_2e^{i(n_2x+m_2y+l_2z)}-Cm_3e^{i(n_3x+m_3y+l_3z)}\Big),
\end{aligned} \quad (4.1)$$

where i is the imaginary unity, (n_k,m_k,l_k), $k=1,2,3$, are wave vectors, and $A(t)$, $B(t)$, $C(t)$ are amplitudes depending on the time. The equation of continuity (1.4) follows from (4.1). Equations (3.4) are valid under the following two assumptions.

Assumption 1. The amplitudes satisfy the ordinary differential equations

$$\begin{aligned}
\frac{dA}{dt}&=-\frac{A}{\mathrm{Re}}(n_1^2+m_1^2+l_1^2),\\
\frac{dB}{dt}&=-\frac{B}{\mathrm{Re}}(n_2^2+m_2^2+l_2^2),\\
\frac{dC}{dt}&=-\frac{C}{\mathrm{Re}}(n_3^2+m_3^2+l_3^2).
\end{aligned} \quad (4.2)$$

It is easy to find the solution to the system (4.2)

$$A(t) = A(0)\exp\left\{-\frac{(n_1^2+m_1^2+l_1^2)t}{\mathrm{Re}}\right\},$$
$$B(t) = B(0)\exp\left\{-\frac{(n_2^2+m_2^2+l_2^2)t}{\mathrm{Re}}\right\}, \tag{4.3}$$
$$C(t) = C(0)\exp\left\{-\frac{(n_3^2+m_3^2+l_3^2)t}{\mathrm{Re}}\right\},$$

where $A(0)$, $B(0)$, $C(0)$ are arbitrary constants.

Assumption 2. The coordinates of the wave vectors satisfy a system of algebraic equations. Each solution, complex or real, to this system generates a solution to the Navier–Stokes equations.

We describe two variants of new solutions.

Solution 1: $n_1 = n_2 = n_3 = 1$, $m_1 = m_3 = 0$, $m_2 = \sqrt{5}$, $l_1 = l_3 = 3$, $l_2 = -2$. Then the velocities are defined by

$$u = -iB(0))\exp\left\{\frac{-10t}{\mathrm{Re}} + i(x+\sqrt{5}y-2z)\right\},$$
$$v = \frac{i}{2}(3C(0)-A(0))\exp\left\{\frac{-10t}{\mathrm{Re}} + i(x+3z)\right\}, \tag{4.4}$$
$$w = -\frac{iB(0)}{2}\exp\left\{\frac{-10t}{\mathrm{Re}} + i(x+\sqrt{5}y-2z)\right\}$$

and the pressure is defined by

$$p = p_0 + \frac{\sqrt{5}}{4}B(0)(A(0)-3C(0))\exp\left\{\frac{-20t}{\mathrm{Re}} + i(2x+\sqrt{5}y+z)\right\}. \tag{4.5}$$

Solution 2: $n_1 = n_2 = 0$, $n_3 = -i\sqrt{3}$, $m_1 = m_2 = m_3 = i\sqrt{3}$, $l_1 = l_2 = 1$, $l_3 = 2$. Then the velocities are defined by

$$u = \frac{i}{2}(i\sqrt{3}A(0)+B(0))\exp\left\{\frac{2t}{\mathrm{Re}} - \sqrt{3}y + iz\right\},$$
$$v = iC(0)\exp\left\{\frac{2t}{\mathrm{Re}} + \sqrt{3}x - \sqrt{3}y + 2iz\right\}, \tag{4.6}$$
$$w = \frac{\sqrt{3}}{2}C(0)\exp\left\{\frac{2t}{\mathrm{Re}} + \sqrt{3}x - \sqrt{3}y + 2iz\right\}$$

and the pressure is defined by

$$p = p_0 - \frac{1}{4}C(0)(i\sqrt{3}A(0)+B(0))\exp\left\{\frac{4t}{\mathrm{Re}} + \sqrt{3}x - 2\sqrt{3}y + 3iz\right\}. \tag{4.7}$$

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Submitted on November 2, 2019