DOI 10.1007/s10958-020-04985-w
*Journal of Mathematical Sciences, Vol. 249, No. 6, September, 2020*

# CONSTRUCTION OF HOMOTOPIC INVARIANTS OF MAPS FROM SPHERES TO COMPACT CLOSED MANIFOLDS

**I. S. Zubov**
State Socio-Humanitarian University
30, Zelenaya St., Kolomna 140410, Russia
reestr_rr@mail.ru

UDC 517.925 + 512.77

*We study the homotopic classifications of maps from circles and spheres to manifolds and compare the classical approach to define the Hopf invariant with the approach based on Chen's iterated integrals. Bibliography: 5 titles.*

## 1 Iterated Integrals

Let $\omega_1, \omega_2, \dots, \omega_r$ be differential 1-forms on a smooth manifold $M$. Denote by $P_{x_0}(M)$ the space of paths $\gamma : I = [0,1] \to M$ with starting point $x_0 = \gamma(0)$. By an *iterated integral* of $\omega_1, \omega_2, \cdots \omega_r$ we mean a function $\int \omega_1 \dots \omega_r : P_{x_0}(M) \to \mathbb{R}$ on the path space $P_{x_0}(M)$ such that for a path $\gamma$ it is inductively defined by the equality

$$\int\limits_{\gamma} \omega_1 \dots \omega_r = \int\limits_{\gamma} \Bigg( \int\limits_{\gamma^\tau} \omega_1 \dots \omega_{r-1} \Bigg) \omega_r, \quad \gamma^\tau = \gamma(\tau t), \quad t \in [0,1].$$

We note that $\int\limits_{\gamma} \omega_1$ is a usual curvilinear integral. For complex-valued differential 1-forms $\omega_1 \dots \omega_r$ the iterated integrals take the values in $\mathbb{C}$. We denote by $P_{x_0}^{x_1}(M)$. the space of paths $\gamma$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$ and by $B_s(M)$ the space of iterated integrals $\int\limits_{\gamma} \omega_1 \dots \omega_r$ of length $r \leqslant s$. We consider some properties of iterated integrals.

**1.1.** The product of iterated integrals is a linear combination of iterated integrals.

Translated from *Problemy Matematicheskogo Analiza* **103**, 2020, pp. 101-112.

**Theorem 1.1.** *The product of iterated integrals of length $k$ and $l$ respectively is equal to the sum of iterated integrals of length $k+l$*

$$\int\limits_\gamma \omega_1 \cdots \omega_k \cdot \int\limits_\gamma \omega_{k+1} \cdots \omega_{k+l} = \sum_{\sigma \in S_{k,l}} \int\limits_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)},$$

*where the sum is taken over all shuffles $(k,l)$ in the permutation group $S_{n+k}$.*

**Proof.** Let $\gamma^*\omega_i = f_i(t)dt$, $i = 1, \ldots, k+l$. By the definition of iterated integrals,

$$\int\limits_\gamma \omega_1 \cdots \omega_k \cdot \int\limits_\gamma \omega_{k+1} \cdots \omega_{k+l} = \int\limits_{[0,1]} \gamma^*\omega_1 \cdots \gamma^*\omega_k \cdot \int\limits_{[0,1]} \gamma^*\omega_{k+1} \cdots \gamma^*\omega_{k+l}$$

$$= \int\limits_{C_k} \prod_{i=1}^{k} f_i(t_i)dt_1 \cdots dt_k \cdot \int\limits_{C_l} \prod_{i=k+1}^{k+l} f_i(t_i)dt_{k+1} \cdots dt_{k+l} = \int\limits_{C_k \times C_l = \bigcup\limits_{\sigma \in S_{k,l}} \sigma(C_{k+l})} \prod_{i=1}^{k+l} f_i(t_i)dt_1 \cdots dt_{k+l},$$

where the last equality holds in view of the Fubini theorem. Since the integral over the union of measurable sets is equal to the sum of integrals over each set in this union provided that the intersection of these sets has zero measure, we have

$$\int\limits_{C_k \times C_l = \bigcup\limits_{\sigma \in S_{k,l}} \sigma(C_{k+l})} \prod_{i=1}^{k+l} f_i(t_i)dt_1 \cdots dt_{k+l} = \sum_{\sigma \in S_{k,l}} \int\limits_{\sigma(C_{k+l})} \prod_{i=1}^{k+l} f_i(t_i)dt_1 \cdots dt_{k+l}$$

$$= \sum_{\sigma \in S_{k,l}} \int\limits_\gamma \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(k+l)},$$

where $C_n$ is an $n$-dimensional standard simplex. □

**1.2.** We consider differential forms $\omega_1, \ldots, \omega_r$ on $M^n$ and a path $\gamma : [0,1] \to M^n$. We set $\gamma'(\tau) = \gamma(t(\tau))$, where $t(\tau) : [0,1] \to [0,1]$ is a change of variable. If $t(\tau)$ is monotonically increasing, then the equivalence class of paths, up to a change of variable, is called an *oriented curve.* We use the induction on $r$ to prove the invariance property under a differentiable monotonically increasing change of variable

$$\int\limits_{\gamma'} \omega_1 \cdots \omega_r = \int\limits_\gamma \omega_1 \cdots \omega_r.$$

For $r = 1$ we have the curvilinear integral $\int\limits_\gamma \omega_1$ independent of the path parametrization.

For $r \geqslant 1$ we write the definition of an $r$-iterated integral with $t = t_{r-1}$ in the recurrent form

$$\int\limits_\gamma \omega_1 \cdots \omega_r = \int\limits_0^1 \Bigg( \int\limits_{\gamma^t} \omega_1 \cdots \omega_{r-1} \Bigg) f_r(t)dt,$$

where $f_r(t)dt = \gamma^*(\omega_r)$ and the path $\gamma^t$ is defined by $\gamma^\tau(t) = \gamma(t\tau)$, $t \in [0,1]$. This is possible because of its representation formula in the form of a repeated integral.

Multiplication by an equivalence class of paths is an associative operation. For example, to identify the products of three paths $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$, we can use

$$t(\tau) = \begin{cases} 2\tau, & 0 \leqslant \tau \leqslant \dfrac{1}{4}, \\ \tau^2 + \dfrac{3}{2}\tau - \dfrac{1}{4}, & \dfrac{1}{4} \leqslant \tau \leqslant \dfrac{1}{2}, \\ \dfrac{5}{2}\tau - \dfrac{3}{2}, & \dfrac{1}{2} \leqslant \tau \leqslant 1. \end{cases}$$

**1.3.** The product $\alpha \cdot \beta : [0,1] \to M^n$ of two paths $\alpha : [0,1] \to M^n$ and $\beta : [0,1] \to M^n$ such that $\alpha(1) = \beta(0)$ is defined by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & 0 \leqslant t \leqslant 1/2, \\ \beta(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

**Theorem 1.2.** *Let $\alpha$ and $\beta$ be two paths such that their product $\gamma = \alpha \cdot \beta :\ [0,1] \to M^n$ is defined. Then*

$$\int\limits_{\gamma=\alpha\cdot\beta} \omega_1 \cdots \omega_r = \int\limits_{\alpha} \omega_1 \cdots \omega_r + \sum_{k=1}^{r-1} \int\limits_{\alpha} \omega_1 \cdots \omega_k \int\limits_{\beta} \omega_{k+1} \cdots \omega_r + \int\limits_{\beta} \omega_1 \cdots \omega_r.$$

**Proof.** We argue by induction. For the base step we use the additivity property of usual curvilinear integrals. Further, let us consider the path $\gamma^t = (\alpha\beta)^t$. By the definition of $\gamma^t$,

$$(\alpha \cdot \beta)^t = \begin{cases} \alpha^t, & 0 \leqslant t \leqslant 1/2, \\ \alpha \cdot (\beta)^t, & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Therefore, for any 1-differential form $\omega$

$$((\alpha \cdot \beta)^t)^*\omega = \begin{cases} (\alpha^t)^*\omega = f(2t)dt, & 0 \leqslant t \leqslant 1/2, \\ (\alpha \cdot \beta^t)^*\omega = g(2t-1)dt, & 1/2 \leqslant t \leqslant 1, \end{cases}$$

where $f(t)dt = (\alpha)^*\omega$ and $g(t)dt = (\beta)^*\omega$. By the additivity property of usual integrals,

$$\int\limits_{\alpha\cdot\beta} \omega_1 \cdots \omega_r = \int\limits_0^1 \Bigg( \int\limits_{(\alpha\cdot\beta)^t} \omega_1 \cdots \omega_{r-1} \Bigg) (\alpha \cdot \beta)^t \omega_r$$

$$= \int\limits_0^{\frac{1}{2}} \Bigg( \int\limits_{(\alpha)^t} \omega_1 \cdots \omega_{r-1} \Bigg) f_r(2t)dt + \int\limits_{\frac{1}{2}}^1 \Bigg( \int\limits_{\alpha\cdot(\beta)^t} \omega_1 \cdots \omega_{r-1} \Bigg) g_r(2t-1)dt$$

which can be written in the form

$$\int\limits_{\alpha} \omega_1 \cdots \omega_r + \int\limits_{\frac{1}{2}}^{1} \Bigg( \sum_{k=0}^{r-1} \int\limits_{\alpha} \omega_1 \cdots \omega_k \int\limits_{\beta^t} \omega_{k+1} \cdots \omega_{r-1} \Bigg) g_r(2t-1)dt$$

$$= \int\limits_{\alpha} \omega_1 \cdots \omega_r + \sum_{k=1}^{r-1} \int\limits_{\alpha} \omega_1 \cdots \omega_k \int\limits_{\beta} \omega_{k+1} \cdots \omega_r + \int\limits_{\beta} \omega_1 \cdots \omega_r$$

in view of the definition of $r$-iterated integrals. □

**1.4.** We introduce the path $\gamma^{-1}$ by $\gamma^{-1}(t) = \gamma(1-t)$, $t \in [0,1]$. Arguing as in Subsection 1.2, we obtain the following assertion.

**Theorem 1.3.**

$$\int\limits_{\gamma^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int\limits_{\gamma} \omega_r \cdots \omega_1.$$

**1.5.** We introduce the definition of a pin.

**Definition 1.1.** By a *pin* we mean a path of the form $\gamma\gamma^{-1}$. By a *pin insertion* we mean that a path $\alpha$ is represented as $\alpha = \beta_1\gamma\gamma^{-1}\beta_2$. We say that a *pin is removed* if $\gamma\gamma^{-1}$ is removed from the above representation.

**Theorem 1.4.** *An iterated integral is independent of insertions or removals of pins.*

**Proof.** It suffices to prove the equality $\int\limits_{\gamma\gamma^{-1}} \omega_1 \cdots \omega_r = 0$, $r \geqslant 1$, which is valid in view of Subsections 1.1, 1.3, and 1.4 and can be proved by induction on $r$. □

**1.6.** A *loop* is a path $\gamma$ such that $\gamma(0) = \gamma(1)$. We denote by $\Omega_{x_0}(M)$ the space of paths with marked point $x_0$. If we consider loops up to an insertion or removal of pins, then an equivalence relation is defined on the loop space. The set of equivalence classes is a topological space equipped with the quotient topology generated by the compact-open topology in the original loop space. We denote by $\overline{\Omega_{x_0}(M)}$ the corresponding quotient space. It is a topological group with the group operation induced by the product of loops in the original loop space. In particular, the product of equivalence classes is associative and, in terms of loops, we can say that the product of loops, regarded up to a monotone change of variable, is an associative operation on loops. A connected component of the identity element in the group $\overline{\Omega_{x_0}(M)}$ is a normal divisor in the group. The corresponding quotient group is a group isomorphic to the fundamental group of the manifold $M^n$. By Subsection 1.5, the iterated integrals are continuous functions on $\overline{\Omega_{x_0}(M)}$ (in fact, differentiable functions of the same smoothness order as the space of differential forms and the space of loops under consideration).

**1.7.** The value of 2-iterated integral $\int \omega_1\omega_2$ over the commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ of two loops $\alpha$ and $\beta$ is calculated via 1-iterated integrals by

$$\int\limits_{[\alpha,\beta]} \omega_1\omega_2 = \int\limits_{\alpha} \omega_1 \int\limits_{\beta} \omega_2 - \int\limits_{\beta} \omega_1 \int\limits_{\alpha} \omega_2.$$

We place the brackets in the commutator of two loops $\alpha$ and $\beta$ as $[\alpha,\beta]=(\alpha\beta)(\alpha^{-1}\beta^{-1})=(\alpha\beta)((\beta\alpha)^{-1})$ and note that the iterated integral is independent of the bracket disposition in the product of loops. By properties of iterated integrals, a direct computation shows that

$$\int\limits_{[\alpha,\beta]}\omega_1\omega_2=\int\limits_{(\alpha\beta)((\beta\alpha)^{-1})}\omega_1\omega_2=\int\limits_{\alpha\beta}\omega_1\omega_2+\int\limits_{(\beta\alpha)^{-1}}\omega_1\omega_2+\int\limits_{\alpha\beta}\omega_1\int\limits_{(\beta\alpha)^{-1}}\omega_2$$

$$=\int\limits_{\alpha}\omega_1\omega_2+\int\limits_{\beta}\omega_1\omega_2+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2+\int\limits_{\beta\alpha}\omega_2\omega_1-\Big(\int\limits_{\alpha}\omega_1+\int\limits_{\beta}\omega_1\Big)\Big(\int\limits_{\alpha}\omega_2+\int\limits_{\beta}\omega_2\Big)$$

$$=\int\limits_{\alpha}\omega_1\omega_2+\int\limits_{\beta}\omega_1\omega_2+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2+\int\limits_{\beta}\omega_2\omega_1+\int\limits_{\alpha}\omega_2\omega_1+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2$$

$$-\Big(\int\limits_{\alpha}\omega_1+\int\limits_{\beta}\omega_1\Big)\Big(\int\limits_{\alpha}\omega_2+\int\limits_{\beta}\omega_2\Big)$$

$$=\Big(\int\limits_{\alpha}\omega_1\omega_2+\int\limits_{\alpha}\omega_2\omega_1\Big)+\Big(\int\limits_{\beta}\omega_1\omega_2+\int\limits_{\beta}\omega_2\omega_1\Big)+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2$$

$$-\Big(\int\limits_{\alpha}\omega_1+\int\limits_{\beta}\omega_1\Big)\Big(\int\limits_{\alpha}\omega_2+\int\limits_{\beta}\omega_2\Big)$$

$$=\int\limits_{\alpha}\omega_1\int\limits_{\alpha}\omega_2+\int\limits_{\beta}\omega_1\int\limits_{\beta}\omega_2+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2-\Big(\int\limits_{\alpha}\omega_1+\int\limits_{\beta}\omega_1\Big)\Big(\int\limits_{\alpha}\omega_2+\int\limits_{\beta}\omega_2\Big)$$

$$=\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2-\int\limits_{\beta}\omega_1\int\limits_{\alpha}\omega_2.$$

**1.8.** The value of 2-iterated integral $\int\omega_1\omega_2$ over the product $\gamma=\prod\limits_{i=1}^{m}[\alpha_i,\beta_i]$ of commutators of loops is equal to the sum of the values of 2-iterated integrals over the factors, i.e.,

$$\int\limits_{\prod\limits_{i=1}^{m}[\alpha_i,\beta_i]}\omega_1\omega_2=\sum_{i=1}^{m}\int\limits_{[\alpha_i,\beta_i]}\omega_1\omega_2.$$

By Subsection 1.7,

$$\int\limits_{\prod\limits_{i=1}^{m}[\alpha_i,\beta_i]}\omega_1\omega_2=\sum_{i=1}^{m}\Big(\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2-\int\limits_{\beta}\omega_1\int\limits_{\alpha}\omega_2\Big).$$

We prove this assertion by induction on $m$. The induction step can be easily obtained from Subsection 1.3 and the induction assumption. We consider the base case, i.e., $m=2$. Let

$\gamma = \gamma_1\gamma_2$, where $\gamma_1 = [\alpha_1, \beta_1]$ and $\gamma_2 = [\alpha_2, \beta_2]$. According to Subsection 1.3, we can write

$$\int\limits_{\gamma_1\gamma_2} \omega_1\omega_2 = \int\limits_{\gamma_1} \omega_1\omega_2 + \int\limits_{\gamma_2} \omega_1\omega_2 + \int\limits_{\gamma_1} \omega_1 \int\limits_{\gamma_2} \omega_2.$$

By properties of usual curvilinear integrals, for $i = 1, 2$

$$\int\limits_{\gamma_i} \omega_i = \int\limits_{\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}} \omega_i = \int\limits_{\alpha_i} \omega_i + \int\limits_{\beta_i} \omega_i - \int\limits_{\alpha_i} \omega_i - \int\limits_{\beta_i} \omega_i = 0.$$

Then we obtain the required equality

$$\int\limits_{\gamma_1\gamma_2} \omega_1\omega_2 = \int\limits_{\gamma_1} \omega_1\omega_2 + \int\limits_{\gamma_2} \omega_1\omega_2.$$

The general case is treated in a similar way.

The definition of iterated integrals and their properties, except for that indicated in Subsection 1.4, are generalized to the case of matrix-valued differential 1-forms. One of the most important properties of iterated integrals concerns the differentiability property and, in particular, the Stokes formula. Iterated integrals of 1-forms are functions on the loop space $\Omega_{x_0}(M^n)$ with a given starting point $x_0$ in the manifold $M^n$ or, in other words, are differential 0-forms on $\Omega_{x_0}(M^n)$. Thus, we have the following assertion.

**1.9.** For iterated integrals of differential 1-forms on the loop space $\Omega_{x_0}M = P_{x_0}^{x_0}(M)$ we have the differentiation formula (cf. [1])

$$d\int\limits_{\gamma} \omega_1 \cdots \omega_q = -\sum_{i=1}^{q} \int\limits_{\gamma} \omega_1 \cdots \omega_{i-1} d\omega_i \omega_{i+1} \dots \omega_q$$

$$- \sum_{i=1}^{q-1} (-1)^i \int\limits_{\gamma} \omega_1 \cdots \omega_{i-1}(\omega_i \wedge \omega_{i+1})\omega_{i+2}\omega_{i+2} \cdots \omega_q \tag{1.1}$$

and the Stokes formula

$$\int\limits_{C} (d\int \omega_1 \cdots \omega_q) = \int\limits_{\partial C} \left( \int \omega_1 \cdots \omega_q \right) = \int\limits_{C(1)} \omega_1 \cdots \omega_q - \int\limits_{C(0)} \omega_1 \cdots \omega_q,$$

where the path $C : [0; 1] \to P_{x_0}^{x_1}(M)$ is a singular simplex in the space $P_{x_0}^{x_1}(M)$. This simplex defines a homotopy between the paths $\gamma_1$ and $\gamma_2$; moreover, $C(0) = \gamma_1$ and $C(1) = \gamma_2$ in the path space $P_{x_0}^{x_1}(X_n)$.

We denote by $B_s(M)$ the vector space of iterated integrals on $M$ of length at most $s$ and by $\eta_x$ a constant path at a point $x$ in $M$, i.e., $\eta_x(t) = x$ for all $t$. If $r \geqslant 1$, then

$$\left\langle \int \omega_1 \dots \omega_r, \eta_x \right\rangle = 0$$

for all $x \in M$. Thus, the estimation on the constant path $\eta_x$ defines a linear functional $\varepsilon : B_s(M) \to R$, $I \to \langle I, \eta_x \rangle$ independent of $x$. If

$$I = \lambda + \sum a_i \int \omega_i + \sum a_{ij} \int \omega_i\omega_j + \cdots ,$$

then $\varepsilon(I) = \lambda$. Denote by $\overline{B}_\varepsilon(M)$ the kernel of the map $\varepsilon$. There are iterated integrals of length at most $s$ with zero constant term. We have the natural inclusion $i : R \to B_s(M)$ such that $\varepsilon \circ i = \mathrm{id}$. As a result, we obtain the natural decomposition into the direct sum $B_s(M) \cong R \oplus \overline{B}_s(M)$. For loops $\alpha, \beta \in PM$ attached to the point $x$ we have a differential form on the commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. We note that a constant loop $\eta_x$ at $x$ is often denoted by 1. For paths $\alpha\eta_x$, $\eta_x\alpha$ and an iterated integral $I$ we can write

$$I(\alpha) = I(\alpha\eta_x) = I(\eta_x\alpha).$$

We recall that the classical curvilinear integral satisfies the conditions

$$\left\langle \int \omega, [\alpha, \beta] \right\rangle = 0, \quad \left\langle \int \omega, (\alpha - 1)(\beta - 1) \right\rangle = 0,$$

where $\alpha$ and $\beta$ are loops attached to the point $x$. The following assertion generalizes this fact.

**Lemma 1.1.** *Assume that $\omega_1, \dots, \omega_r \in E^1(M)$ and $x \in M$. Assume that $\alpha_1, \alpha_2, \dots, \alpha_s$ are loops on $M$ attached to the point $x$.*

(a) *If $I \in B_r$ and $r < s$, then $\langle I, (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_s) \rangle = 0$, where 1 denotes $\eta_x$, i.e., a constant path at the point $x$.*

(b) *If $I \in \overline{B}_r$ and $r < s$, then $\langle I, [\alpha_1[\alpha_2[\dots[\alpha_{s-1}]\dots]]] \rangle = 0$.*

## 2 Iterated Integrals of Differential Forms of Arbitrary Degree

Let $\omega_1, \omega_2, \dots, \omega_r$ be differential forms of $\deg \omega_i = p_i$ on a compact closed manifold $M^n$. The iterated integral $\int \omega_1 \cdots \omega_r$ is a differential form of degree $p_1 + p_2 + \dots + p_r - r$ on the path space $P(M)$ which will be denoted by $P_{x_0}^{x_1}$ if the starting $x_0$ and ending $x_1$ points are given. In the case $x_0 = x_1$, we deal with the loop space $\Omega_{x_0}$. $P(M)$.

We define an iterated integral on the path space $P(M)$. Differential forms of degree $n$ on the path space are forms on convex sets $U, V \subset \mathbb{R}^n$, which will be denoted by $\omega_U$ and $\omega_V$ respectively. We consider maps $\alpha : U \to P(M)$ and $\beta : V \to P(M)$. There exists a differentiable map $f : U \to V$ such that $\beta \circ f = \alpha$ and $f^*\omega_V = \omega_U$. We consider the suspension map (cf. [1]) $\varphi_\alpha : I \times U \to M$. We also consider a differential form $\omega$ defined on $U$ and such that $\deg \omega = p > 1$. We have $\varphi_\alpha^* = dt \wedge \omega' + \omega''$. Denote

$$\omega'(t) = \int_0^t \omega' dt.$$

By definition, a 1-iterated integral is represented as

$$\int \omega_1 = \int_0^1 \omega_1'(t_1) dt_1.$$

In this case, we obtain a differential form on $U$ of degree $p-1$. For a 2-iterated integral we have

$$\int \omega_1\omega_2 = \int_0^1 \left( \int_0^{t_2} \omega_1 \wedge \omega_2'(t_2) \right) dt_2.$$

We define an $r$-iterated integral by induction:

$$\int \omega_1 \ldots \omega_r = \int_0^1 \left( \int_0^{t_r} \omega_1 \ldots \omega_{r-1} \wedge \omega_r'(t_r) \right) dt_r.$$

For iterated integrals of differential forms of an arbitrary degree the following differentiation formula holds (cf. [1])

$$d \int \omega_1\omega_2 \cdots \omega_r = \sum_{i=1}^{r} (-1)^i \int J\omega_1 \cdots J\omega_{i-1} d\omega_i \omega_{i+1} \cdots \omega_r$$
$$- \sum_{i=1}^{r-1} J\omega_1 \cdots J\omega_{i-1}(J\omega_i \wedge \omega_{i+1})\omega_{i+2} \cdots \omega_r,$$

where $J\omega_i = (-1)^{\deg \omega_i} \cdot \omega_i$. This formula is a generalization of formula (1.1) for iterated integrals of differential 1-forms. To obtain the Stokes formula, we restrict the path space $P(M)$ to the loop space $\Omega_{x_0}(M)$ with the marked point $x_0$. Then the Stokes formulas can be written as

$$\left\langle d \int \omega_1\omega_2 \cdots \omega_r, C \right\rangle = \left\langle \int \omega_1\omega_2 \cdots \omega_r, \partial C \right\rangle.$$

## 3 Detection of Homotopically Nontrivial Elements of Fundamental Groups of One-Dimensional Complex Manifolds

In this section, we consider holomorphic and meromorphic differential 1-forms on one-dimensional complex manifolds.

The homotopic nontriviality of loops on one-dimensional complex manifolds is determined by nonzero values of the homotopy periods on the loops. By homotopy periods we mean iterated integrals depending only on the homotopy class of loops. Thus, the homotopy period is well defined on the element of the fundamental group corresponding to the loop. The homotopy periods define functions on the fundamental group of a one-dimensional complex manifold. We consider one-dimensional complex manifolds whose fundamental groups are given by finitely many generators and finitely many relations (finitely presented groups).

**Proposition 3.1.** *The intersection of terms of the lower central series of the finitely presented fundamental group of a one-dimensional complex manifold is the unit group.*

**Proof.** 1. If a one-dimensional complex manifold $C$ is not compact, then its fundamental group is a free group, $\pi_1(C, x_0) = F_n$, $n > 0$, with finitely many generators $\bigcap_{k=1}^{\infty} \Gamma_k F_n = \{e\}$.

2. If a one-dimensional complex manifold is closed, then its fundamental group is either the trivial group $G = \{e\}$ or a group with finitely many generators and one relation. The

fundamental group of the manifold $X = C\backslash\{x\}$ obtained from $C$ by removing one point is a free group $F_{2g}$, where $g$ is the genius of the manifold $C$. The embedding $i : X \to C$ induces an epimorphism $\pi_1(X) \to \pi_1(C) \to 1$ of fundamental groups and an epimorphism $\Gamma_k\pi_1(X) \to \Gamma_k\pi_1(C) \to 1$, $k = 1, 2, \ldots$, of their lower central series. Since $\bigcap\limits_{k=1}^{\infty} \Gamma_k\pi_1(X) = \{e\}$, it follows that $\bigcap\limits_{k=1}^{\infty} \Gamma_k\pi_1(C) = \{e\}$. Thus, the finitely presented fundamental group $\pi_1(C)$ of a one-dimensional complex manifold is the trivial intersection of terms of the lower central series. □

The group $\pi_1(C)$ is a residually nilpotent group. Thus, for every nontrivial element $g$ of the fundamental group $\pi_1(C)$ there exists a maximal natural number $r$ such that $g$ has the nonzero image in the quotient group $\Gamma_r(C)/\Gamma_{r+1}(C)$.

We choose a system of canonical loops $a_1, \ldots, a_g, b_1, \ldots, b_g$ on $C$, cut $C$ along these loops, and transform this one-dimensional complex manifold to a $2g$-polygon. We can choose loops starting at the point $x_0 \in C$ and representing the generators of the fundamental group $\pi_1(C, x_0)$ (we preserve the notation). Any iterated integral $\int \omega_1 \cdots \omega_r$, $r \geqslant 1$, of holomorphic 1-forms on a one-dimensional complex manifold is a homotopy period. Indeed, let two loops $\gamma_1, \gamma_2 \in \Omega_{x_0}(C)$ are homotopic, i.e., there exists a map $h : [0, 1] \to \Omega_{x_0}(C)$ such that $h(0) = \gamma_1$ and $h(1) = \gamma_2$. By properties of iterated integrals,

$$\int\limits_{\gamma_2} \omega_1 \cdots \omega_r - \int\limits_{\gamma_1} \omega_1 \cdots \omega_r = \int\limits_{\partial h} \omega_1 \cdots \omega_r = \int\limits_{h} d \int \omega_1 \cdots \omega_r$$

$$= -\int\limits_{h} \sum_{i=1}^{r} \int \omega_1 \cdots d\omega_i \cdots \omega_r - \sum_{i=1}^{r-1} \int \omega_1 \cdots (\omega_i \wedge \omega_{i+1}) \cdots \omega_r = 0.$$

The last equality is valid because $d\omega_i = 0$, $i = 1, \ldots, r$, and $\omega_i \wedge \omega_{i+1} = 0$, $i = 1, \ldots, r-1$, for holomorphic forms on a one-dimensional complex manifold. Hence

$$\int\limits_{\gamma_1} \omega_1 \cdots \omega_r = \int\limits_{\gamma_2} \omega_1 \cdots \omega_r,$$

i.e., the iterated integral $\int \omega_1 \cdots \omega_r$ of holomorphic forms is a homotopy period. As known, there is a holomorphic 1-form $\omega_i$ on $C$ such that

$$\operatorname{Re} \int\limits_{a_i} \omega_i = 1, \quad \operatorname{Re} \int\limits_{a_j} \omega_i = 0, \quad j \neq i, \ j = 1, \ldots, r, \quad \operatorname{Re} \int\limits_{b_j} \omega_i = 0, \quad j = 1, \ldots, r,$$

for $b$-periods. Similarly, there is a holomorphic form $\omega_j$ such that $\operatorname{Re} \int\limits_{b_j} \omega_j = 1$. The remaining periods of this form vanish. Such a 1-form detects the homotopic nontriviality of the generators $a_1, \ldots, a_g, b_1, \ldots, b_g$.

Using the Chen isomorphism theorem in the class of smooth differential 1-forms

$$H^0(B_r, x_0) = \operatorname{Hom}(\mathbb{R}[\pi_1(M, x_0)]/J^{r+1}, \mathbb{R}),$$

where $H^0(B_r, x_0)$ are homotopy periods of $r$-iterated integrals, we can prove the homotopic nontriviality of any element of $\pi_1(M, x_0)$ with the help of iterated integrals. In the class of holomorphic (or antiholomorphic) 1-forms, the problem is more complicated.

The nontriviality of elements $[a_i, a_j] = a_i a_j a_i^{-1} a_j^{-1}$ can be recognized with the help of holomorphic 1-forms $\omega_1, \omega_2, \dots, \omega_g$ with periods

$$\int\limits_{a_i} \omega_j = 2\pi\sqrt{-1}\,\delta_{ij},$$

where $\delta_{ij}$ is the Kronecker symbol. Indeed,

$$\int\limits_{[a_i,a_j]} \omega_i\omega_j = \begin{vmatrix} \int\limits_{a_i} \omega_i & \int\limits_{a_i} \omega_j \\ \int\limits_{a_j} \omega_i & \int\limits_{a_j} \omega_j \end{vmatrix} = \begin{vmatrix} 2\pi\sqrt{-1} & 0 \\ 0 & 2\pi\sqrt{-1} \end{vmatrix} = -4\pi^2 \neq 0.$$

Since $\int\limits_{[a_i,a_j]} \omega_i\omega_j$ is a nonzero homotopy period, the loop commutator $[a_i, a_j]$ is not homotopic to a constant loop. Similarly, the loop commutator $[b_i, b_j]$ is not homotopic to a constant loop either. We will prove this assertion below for the products of loop commutators

$$[a_{i_1}, a_{i_2}] \cdot [a_{i_3}, a_{i_4}] \cdot \ldots \cdot [a_{i_s}, a_{i_{i+1}}],$$
$$[b_{i_1}, b_{i_2}] \cdot [b_{i_3}, b_{i_4}] \cdot \ldots \cdot [b_{i_s}, b_{i_{i+1}}].$$

We choose a basis of holomorphic forms $\omega_1, \dots, \omega_g$ on the one-dimensional complex manifold $C$ such that the $a$-periods $\int\limits_{a_i} \omega_j$ are purely imaginary and the $b$-periods $\int\limits_{b_i} \omega_j$ yield a complex $g \times g$-matrix with negative definite real part.

For the loop commutators $[a_i, b_j]$ we can choose a pair of 1-forms from the above sets so that the 2-iterated integral $\int\limits_{[a_i,b_j]} \omega_k\omega_l$ is a complex number with nonzero imaginary part. Consequently, the commutator $[a_i, b_j]$ is not homotopic to a constant loop.

Analyzing properties of iterated integrals and matrices of periods of holomorphic 1-forms, we can assert that for any $\gamma \in \Gamma_r \pi_1(C)$ there exists an $r$-iterated integral of only holomorphic 1-forms (or antiholomorphic 1-forms) which does not vanish on the loop $\gamma$, $\int\limits_{\gamma} \omega_{i_1} \cdots \omega_{i_r} \neq 0$. This fact means that $\gamma$ is not homotopic to a constant loop.

By the residual nilpotence and the above assumption, we can conclude that if for $\gamma \in \pi_1(C)$ and all iterated integrals

$$\int\limits_{\gamma} \omega_{i_1} \cdots \omega_{i_r} = 0, \quad r > 0,$$

then $\gamma$ is homotopic to a constant loop.

## 4 Homotopic Classification of Maps

One of the oldest problems in algebraic topology is the classification problem of maps from a sphere to a sphere, up to a homotopy $f : S^m \to S^n$. To study this problem, it is useful to consider the auxiliary space $C_f = D^{m+1} \bigcup\limits_f S^n$, called the *map cone*. In this space, there are cells of dimension 0, $n$, $m+1$ corresponding to a point, the interior of a disc $D^{m+1}$ and the disc ${}^0D^n = S^n \backslash \{\mathrm{pt}\}$ respectively. In particular, for $m = n$ we have the space $D^{n+1} \bigcup\limits_f S^n$. The $n$-dimensional homologies $H_n(C_f, \mathbb{Z})$ of this space are isomorphic to $\mathbb{Z}_{|d|}$. The number $d$ is a homotopic invariant of the map $f$, called the *degree* of $f$. This construction was originally used for a circle, but was later generalized to higher dimensions.

Another interesting classical case concerns the maps $f : S^{2n-1} \to S^n$, $n \geqslant 1$. In this case, the map cone is $C_f = D^{2n} \bigcup\limits_f S^n$. There are cells of dimension 0, $n$, $2n$. In this situation, it is of interest to consider the cohomologies $H^n(C_f, \mathbb{Z}) = \mathbb{Z}$ and $H^{2n}(C_f, \mathbb{Z}) = \mathbb{Z}$. We note that the generators of these groups correspond to the cells $e^n$, $e^{2n}$ in the cell partition of the cone $C_f$.

We recall that cohomologies possess the multiplicative structure. We denote by $x_n$ and $x_{2n}$ the generators of $H^n(C_f, \mathbb{Z}) = \mathbb{Z}$ and $H^{2n}(C_f, \mathbb{Z}) = \mathbb{Z}$ respectively. Then the product $x_n \cup x_n$ belongs to the cohomology group $H^{2n}(C_f, \mathbb{Z})$. Therefore (cf. [2]), $x_n \cup x_n = h(f)x_{2n}$, where $h(f)$, called the *Hopf invariant* of $f : S^{2n-1} \to S^n$, is an integer depending only on the homotopy class of the map.

## 5 Iterated Integrals and Definition of the Hopf Invariant

We consider a map $f : S^{2n-1} \to S^n$ and regard the sphere $S^{2n-1}$ as the suspension $S^{2n-1} = \Sigma S^{2n-2}$ over the sphere $S^{2n-2}$. We fix a point $x_0 \in S^{2n-2}$ and for each $x \in S^{2n-2}$ define a loop $\alpha_x : I \to S^n$ such that $\alpha_x(0) = \alpha_x(1) = x_0$. With each $x \in S^{2n-2}$ we associate a loop on the sphere. Thus, we obtain the map $\varphi : S^{2n-2} \to \Omega_{x_0} S^n$ in the loop space $\Omega_{x_0} S^n$ with the marked point $x_0$.

Let $\omega_n$ be the volume differential form $\int\limits_{S^n} \omega_n = 1$ on $S^n$. We consider the induced form $f^*\omega_n$ on $S^{2n-1}$. As known, all forms of degree from 1 to $2n-2$ on $S^{2n-1}$ are exact. We consider a form $\psi$ on $S^{2n-1}$ such that $d\psi = f^*\omega_n$ (cf. [3, 4]) and the outer product $\psi \wedge f^*\omega_n$ which is a form of degree $2n-1$ on $S^{2n-1}$. We denote by $h_f$ the result of integrating this form over the sphere and note that it is a homotopic invariant of $f$ (cf. [3, 4]).

Now, we introduce the number invariant in another way by using the Chen theory of iterated integrals. We consider the 2-iterated integral $\int \omega_n \omega_n$ which is a differential form of degree $2n-2$ on the loop space with a marked point in $\Omega_{x_0} S^n$. The map $\varphi_f : S^{2n-2} \to \Omega_{x_0} S^n$ defines a singular $(2n-2)$-dimensional chain in the loop space $\Omega_{x_0} S^n$. Then we integrate this form over the singular chain and denote by $\widetilde{h}_f$ the obtained result:

$$\left\langle \int \omega_n \omega_n, \varphi_f(S^{2n-2}) \right\rangle = \widetilde{h}_f.$$

Analyzing the definition of $h_f$ and $\widetilde{h}_f$, we conclude that they are identical (cf. [5]), $h_f = \widetilde{h}_f$, and depend only on the homotopy class of $f$ (cf. [3, 4]). It is also known (cf. [3]) that $h_f$ coincides with the Hopf invariant defined via links (cf. [4]).

Let $M^n$ be a compact closed oriented manifold of dimension $n$, and let $\omega_n$ be the volume form on $M^n$. We extend the above definition to the map $f : S^{2n-1} \to M^n$. Indeed, deg $f^*\omega_n = n$. Moreover, there exists a differential form $\psi$ such that $d\psi = f^*\omega_n$. Then

$$h_f(M) = \int\limits_{S^{2n-1}} \psi \wedge f^*\omega_n$$

or $S^{2n-2} \xrightarrow{\varphi_f} \Omega_{x_0} M^n$, where $S^{2n-1}$ is regarded as the suspension over the sphere $S^{2n-2}$. We set

$$\widetilde{h}_f(M) = \left\langle \int \omega_n \omega_n, \varphi_f(S^{2n-2}) \right\rangle.$$

Then $h_f(M)$ and $\widetilde{h}_f(M)$ are homotopic invariants; moreover, $h_f(M) = \widetilde{h}_f(M)$. This generalizes the Hopf invariant of $f : S^{2n-1} \to S^n$ to a larger class of maps $f : S^{2n-1} \to M^n$ from $(2n-1)$-dimensional spheres to $n$-dimensional manifolds.

## References


1. R. M. Hain, "Iterated integrals and homotopy periods," *Mem. Am. Math. Soc.* **291** (1984).
2. A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge (2002).
3. H. Whitney, *Geometric Integration Theory*, Princeton Univ. Press, Princeton, NJ (1957).
4. J. H. C. Whitehead, "An expression of Hopf's invariant as an integral," *Proc. Natl. Acad. Sci. USA* **33**, 117–123 (1947).
5. K. T. Chen, "Iterated integrals of differential forms and loop space homology," *Ann. Math.* **97**, 217–246 (1973).


Submitted on November 2, 2019