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SOLVABILITY OF THE CAUCHY PROBLEM FOR A QUASILINEAR SYSTEM IN ORIGINAL COORDINATES

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We study the Cauchy problem for a system of quasilinear equations in the original coordinates by using the additional argument method. We obtain sufficient conditions for the existence and uniqueness of a local solution and show that the solution has the same x-smoothness as the initial function. We also obtain sufficient conditions for the existence and uniqueness of a global solution. Bibliography: 4 titles.

1 Introduction

We consider the system

$$\begin{aligned} &\partial_t u(t,x) + (a_1(t)u(t,x) + b_1(t)v(t,x))\partial_x u(t,x) = a_2 u(t,x) + b_2(t)v(t,x), \\ &\partial_t v(t,x) + (c_1(t)u(t,x) + g_1(t)v(t,x))\partial_x v(t,x) = g_2 v(t,x), \end{aligned} \tag{1.1}$$

where $u(t,x)$ and $v(t,x)$ are unknown functions, $a_1(t)$, $b_1(t)$, $b_2(t)$, $c_1(t)$, $g_1(t)$ are known functions, a_2 and g_2 are known constants. For the system (1.1) we consider the initial conditions

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x), \tag{1.2}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are known. The problem (1.1), (1.2) is considered in the domain $\Omega_T = \{(t,x) \mid 0 \leqslant t \leqslant T, x \in (-\infty,+\infty), T > 0\}$.

A similar problem was studied in [1]. In this paper, we get other sufficient conditions in the case of negative $a_1(t)$, $b_1(t)$, $c_1(t)$, $g_1(t)$ and nonnegative $b_2(t)$ on $[0,T]$. Using the additional argument method, we obtain a system of integral equations which is equivalent to the system considered in [1], but allowing one to prove estimates in a simper way.

By the additional argument method, we consider the extended characteristic system

$$\frac{d\eta_1(s,t,x)}{ds} = a_1(s)w_1(s,t,x) + b_1(s)w_3(s,t,x), \tag{1.3}$$

$$\frac{d\eta_2(s,t,x)}{ds} = c_1(s)w_4(s,t,x) + g_1(s)w_2(s,t,x), \tag{1.4}$$

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$$\frac{dw_1(s,t,x)}{ds} = a_2 w_1(s,t,x) + b_2(s) w_3(s,t,x), \tag{1.5}$$

$$\frac{dw_2(s,t,x)}{ds} = g_2 w_2(s,t,x), \tag{1.6}$$

$$w_3(s,t,x) = w_2(s,s,\eta_1), \quad w_4(s,t,x) = w_1(s,s,\eta_2), \tag{1.7}$$

$$\eta_1(t,t,x) = x, \quad \eta_2(t,t,x) = x, \tag{1.8}$$

$$w_1(0,t,x) = \varphi_1(\eta_1(0,t,x)), \quad w_2(0,t,x) = \varphi_2(\eta_2(0,t,x)). \tag{1.9}$$

The unknowns η_i, w_j, $i = 1,2$, $j = 1,\dots,4$, depend not only on t and x, but also on the additional variable s. Integrating (1.3)–(1.6) with respect to s and taking into account the conditions (1.7)–(1.9), we obtain the equivalent system of integral equations

$$\eta_1(s,t,x) = x - \int_s^t (a_1(\tau)w_1 + b_1(\tau)w_3)d\tau, \tag{1.10}$$

$$\eta_2(s,t,x) = x - \int_s^t (c_1(\tau)w_4 + g_1(\tau)w_2)d\tau, \tag{1.11}$$

$$w_1(s,t,x) = \varphi_1(\eta_1(0,t,x))\exp(a_2 s) + \int_0^s b_2(\tau)w_3\exp(a_2(s-\tau))d\tau, \tag{1.12}$$

$$w_2(s,t,x) = \varphi_2(\eta_2(0,t,x))\exp(g_2 s), \tag{1.13}$$

$$w_3(s,t,x) = w_2(s,s,\eta_1), \tag{1.14}$$

$$w_4(s,t,x) = w_1(s,s,\eta_2). \tag{1.15}$$

Substituting (1.10), (1.11) into (1.12)–(1.15), we get

$$\begin{aligned} w_1(s,t,x) = \varphi_1&\left(x - \int_0^t (a_1(\tau)w_1 + b_1(\tau)w_3)d\tau\right)\exp(a_2 s) \\ &+ \int_0^s b_2(\tau)w_3\exp(a_2(s-\tau))d\tau, \end{aligned} \tag{1.16}$$

$$w_2(s,t,x) = \varphi_2\left(x - \int_0^t (c_1(\tau)w_4(\tau,t,x) + g_1(\tau)w_2(\tau,t,x))d\tau\right)\exp(g_2 s), \tag{1.17}$$

$$w_3(s,t,x) = w_2\left(s,s,x - \int_s^t (a_1(\tau)w_1 + b_1(\tau)w_3)d\tau\right), \tag{1.18}$$

$$w_4(s,t,x) = w_1\left(s,s,x - \int_s^t (c_1(\tau)w_4 + g_1(\tau)w_2)d\tau\right). \tag{1.19}$$

Lemma 1.1. *Let $w_1(s,t,x)$, $w_2(s,t,x)$ satisfy the system of integral equations* (1.16)–(1.19)*. Assume that $w_1(s,t,x)$ and $w_2(s,t,x)$ are continuously differentiable and bounded, together with* *their first order derivatives. Then the pair* $u(t,x) = w_1(t,t,x)$*,* $v(t,x) = w_2(t,t,x)$ *is a solution to the problem* (1.1), (1.2) *on* Ω_{T_0}*,* $T_0 \leqslant T$*, where* T_0 *is a constant.*

Lemma 1.1 plays the key role in the additional argument method. It is proved in a standard way (cf., for example, [2]).

2 Existence of Local Solution

We introduce the notation

$$\Gamma_T = \{(s,t,x) \mid 0 \leqslant s \leqslant t \leqslant T, x \in (-\infty,+\infty), T > 0\},$$
$$C_\varphi = \max\{\sup_R |\varphi_i^{(l)}| \mid i = 1,2, l = 0.\dots,2\},$$
$$l = \max\{\sup|a_1(t)|, \sup|b_1(t)|, \sup b_2(t), \sup|c_1(t)|, \sup|g_1(t)|, |a_2|, |g_2|\},$$

where the supremum is taken over $[0,T]$. We also set $\|U\| = \sup\limits_{\Gamma_T}|U(s,t,x)|$, $\|f\| = \sup\limits_{\Omega_T}|f(t,x)|$ and introduce the spaces:

$\overline{C}^{1,2,2}(\Omega_T)$ is the space of functions that are differentiable with respect to t, twice differentiable with respect to x, have mixed second order derivatives, and are bounded, together with their derivatives on Ω_T,

$\overline{C}^2(R)$ is the space of functions that are continuous and bounded, together with their first and second order derivatives on R,

$C([0,T])$ is the space of continuous functions on $[0,T]$.

Theorem 2.1. *Assume that* $\varphi_1(x), \varphi_2(x) \in \overline{C}^2(R)$*,* $a_1(t), b_1(t), b_2(t), c_1(t), g_1(t) \in C([0,T])$*,* $a_1(t) < 0$*,* $b_1(t) < 0$*,* $b_2(t) \geqslant 0$*,* $c_1(t) < 0$*,* $g_1(t) < 0$*,* $t \in [0,T]$*, and* $\varphi_1'(x) \leqslant 0$*,* $\varphi_2'(x) \leqslant 0$*,* $x \in R$*. Then for all* $0 \leqslant t \leqslant T_2$*, where* $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$*, the Cauchy problem* (1.1), (1.2) *has a unique solution* $u(t,x), v(t,x) \in \overline{C}^{1,2,2}(\Omega_{T_2})$ *which can be found from the system of integral equations* (1.16)–(1.19)*.*

We divide the proof of Theorem 2.1 into two lemmas.

Lemma 2.1. *Assume that* $\varphi_1(x), \varphi_2(x) \in \overline{C}^2(R)$*,* $a_1(t), b_1(t), b_2(t), c_1(t), g_1(t) \in C([0,T])$*,* $a_1(t) < 0$*,* $b_1(t) < 0$*,* $b_2(t) \geqslant 0$*,* $c_1(t) < 0$*,* $g_1(t) < 0$*,* $t \in [0,T]$*. Then the system of integral equations* (1.16)–(1.19) *has a unique solution* $w_j \in \overline{C}^{1,1,1}(\Gamma_{T_2})$*,* $j = 1,\dots,4$*,* $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$*.*

Proof. The zeroth approximation to the solution to the system (1.16)–(1.19) is given by

$$w_{10}(s,t,x) = \varphi_1(x), \quad w_{20}(s,t,x) = \varphi_2(x), \quad w_{30}(s,t,x) = \varphi_2(x), \quad w_{40}(s,t,x) = \varphi_1(x).$$

The next approximations are defined by the recurrent sequence of systems of equations ($n = 1,2,\dots$)

$$w_{1n}(s,t,x) = \varphi_1\left(x - \int_0^t (a_1(\tau)w_{1n} + b_1(\tau)w_{3n})d\tau\right)\exp(a_2 s) + \int_0^s b_2(\tau)w_{3n}\exp(a_2(s-\tau))d\tau, \tag{2.1}$$

$$w_{2n}(s,t,x) = \varphi_2\left(x - \int\limits_0^t (c_1(\tau)w_{4n}(\tau,t,x) + g_1(\tau)w_{2n}(\tau,t,x))d\tau\right)\exp(g_2 s), \tag{2.2}$$

$$w_{3n}(s,t,x) = w_{2(n-1)}\left(s,s,x - \int\limits_s^t (a_1(\tau)w_{1n} + b_1(\tau)w_{3n})d\tau\right), \tag{2.3}$$

$$w_{4n}(s,t,x) = w_{1(n-1)}\Big(s,s,x - \int\limits_s^t (c_1(\tau)w_{4n} + g_1(\tau)w_{2n})d\tau\Big). \tag{2.4}$$

For the system (2.1)–(2.4) we define the zeroth approximation by $w_{jn}^0 = w_{j(n-1)}$, $j = 1,\dots,4$, and the next approximations by

$$\begin{aligned} w_{1n}^{k+1}(s,t,x) = \varphi_1&\left(x - \int\limits_0^t (a_1(\tau)w_{1n}^k + b_1(\tau)w_{3n}^k)d\tau\right)\exp(a_2 s) \\ &+ \int\limits_0^s b_2(\tau)w_{3n}^k \exp(a_2(s-\tau))d\tau, \end{aligned} \tag{2.5}$$

$$w_{2n}^{k+1}(s,t,x) = \varphi_2\left(x - \int\limits_0^t (c_1(\tau)w_{4n}^k(\tau,t,x) + g_1(\tau)w_{2n}^k(\tau,t,x))d\tau\right)\exp(g_2 s), \tag{2.6}$$

$$w_{3n}^{k+1}(s,t,x) = w_{2(n-1)}\left(s,s,x - \int\limits_s^t (a_1(\tau)w_{1n}^k + b_1(\tau)w_{3n}^k)d\tau\right), \tag{2.7}$$

$$w_{4n}^{k+1}(s,t,x) = w_{1(n-1)}\left(s,s,x - \int\limits_s^t (c_1(\tau)w_{4n}^k + g_1(\tau)w_{2n}^k)d\tau\right). \tag{2.8}$$

By the assumptions on coefficients, for all $0 \leqslant t \leqslant T_1$, where $T_1 = \min\{1/(20C_\varphi l), 1/(4l)\}$, we have $\|w_{jn}^k\| \leqslant 2C_\varphi$, $j = 1,\dots,4$. Further, the successive approximations (2.5)–(2.8) are bounded, continuous and converge to the solution to the system (2.1)–(2.4). Furthermore, $\|w_{jn}\| \leqslant 2C_\varphi$, $j = 1,\dots,4$. Differentiating (2.5)–(2.8) with respect to x, we get

$$\begin{aligned} w_{1nx}^{k+1}(s,t,x) = \varphi_1'&\left(x - \int\limits_0^t (a_1(\tau)w_{1n}^k + b_1(\tau)w_{3n}^k)d\tau\right)\left(1 - \int\limits_0^t (a_1(\tau)w_{1nx}^k + b_1(\tau)w_{3nx}^k)d\tau\right) \\ &\times \exp(a_2 s) + \int\limits_0^s b_2(\tau)w_{3nx}^k \exp(a_2(s-\tau))d\tau, \end{aligned} \tag{2.9}$$

$$\begin{aligned} w_{2n}^{k+1}(s,t,x) = \varphi_2'&\left(x - \int\limits_0^t (c_1(\tau)w_{4n}^k + g_1(\tau)w_{2n}^k)d\tau\right) \\ &\times \left(1 - \int\limits_0^t (c_1(\tau)w_{4nx}^k + g_1(\tau)w_{2nx}^k)d\tau\right)\exp(g_2 s), \end{aligned} \tag{2.10}$$

$$w_{3nx}^{k+1}(s,t,x) = w_{2(n-1)x}\left(1 - \int_s^t (a_1(\tau)w_{1nx}^k + b_1(\tau)w_{3nx}^k)d\tau\right), \tag{2.11}$$

$$w_{4nx}^{k+1}(s,t,x) = w_{1(n-1)x}\left(1 - \int_s^t (c_1(\tau)w_{4nx}^k + g_1(\tau)w_{2nx}^k)d\tau\right). \tag{2.12}$$

By the assumptions on coefficients, for all $0 \leqslant t \leqslant T_1$, where $T_1 = \min\{1/(20C_\varphi l), 1/(4l)\}$,

$$\|w_{1nx}^k\| \leqslant 4C_\varphi, \quad \|w_{2nx}^k\| \leqslant 4C_\varphi, \quad \|w_{3nx}^k\| \leqslant 6C_\varphi, \quad \|w_{4nx}^k\| \leqslant 6C_\varphi.$$

Differentiating (2.1)–(2.4) with respect to x, we get

$$\begin{aligned} w_{1nx} = \varphi_1'\left(x - \int_0^t (a_1(\tau)w_{1n} + b_1(\tau)w_{3n})d\tau\right)\left(1 - \int_0^t (a_1(\tau)w_{1nx} + b_1(\tau)w_{3nx})d\tau\right) \\ \times \exp(a_2 s) + \int_0^s b_2(\tau)w_{3nx}\exp(a_2(s-\tau))d\tau, \end{aligned} \tag{2.13}$$

$$\begin{aligned} w_{2nx} = \varphi_2'\left(x - \int_0^t (c_1(\tau)w_{4n} + g_1(\tau)w_{2n})d\tau\right) \\ \times \left(1 - \int_0^t (c_1(\tau)w_{4nx} + g_1(\tau)w_{2nx})d\tau\right)\exp(g_2 s), \end{aligned} \tag{2.14}$$

$$w_{3nx} = w_{2(n-1)x}\left(1 - \int_s^t (a_1(\tau)w_{1nx} + b_1(\tau)w_{3nx})d\tau\right), \tag{2.15}$$

$$w_{4nx} = w_{1(n-1)x}\left(1 - \int_s^t (c_1(\tau)w_{4nx} + g_1(\tau)w_{2nx})d\tau\right). \tag{2.16}$$

The successive approximations w_{1nx}^k, w_{2nx}^k, w_{3nx}^k, w_{4nx}^k converge to w_{1nx}, w_{2nx}, w_{3nx}, w_{4nx} as $k \to \infty$, and

$$\|\partial_x w_{1n}\| \leqslant 4C_\varphi, \quad \|\partial_x w_{2n}\| \leqslant 4C_\varphi, \quad \|\partial_x w_{3n}\| \leqslant 6C_\varphi, \quad \|\partial_x w_{4n}\| \leqslant 6C_\varphi.$$

The successive approximations (2.1)–(2.4) converge to the solution to the system (1.16)–(1.19), and $\|w_j\| \leqslant 2C_\varphi$, $j = 1, \dots, 4$. Differentiating twice the system (2.1)–(2.4) with respect to x and setting $\omega_j^n = w_{jnxx}$, $j = 1, \dots, 4$, we obtain the system of equations

$$\begin{aligned} \omega_1^n = -\varphi_1' \int_0^t (a_1(\tau)\omega_1^n + b_1(\tau)\omega_3^n)d\tau \exp(a_2 s) + \int_0^s b_2(\tau)\omega_3^n \exp(a_2(s-\tau))d\tau \\ + \varphi_1''\left(1 - \int_0^t (a_1(\tau)w_{1nx} + b_1(\tau)w_{3nx})d\tau\right)^2 \exp(a_2 s), \end{aligned} \tag{2.17}$$

$$\omega_2^n = -\varphi_2' \int_0^t (c_1(\tau)\omega_4^n + g_1(\tau)\omega_2^n)d\tau \exp(g_2 s)$$

$$+ \varphi_2'' \left(1 - \int_0^t (c_1(\tau)w_{4nx} + g_1(\tau)w_{2nx})d\tau\right)^2 \exp(g_2 s), \tag{2.18}$$

$$\omega_3^n = \omega_2^{n-1}\left(1 - \int_s^t (a_1(\tau)w_{1nx} + b_1(\tau)w_{3nx})d\tau\right)^2 - w_{2(n-1)x}\int_s^t (a_1(\tau)\omega_1^n + b_1(\tau)\omega_3^n)d\tau, \tag{2.19}$$

$$\omega_4^n = \omega_1^{n-1}\left(1 - \int_s^t (c_1(\tau)w_{4nx} + g_1(\tau)w_{2nx})d\tau\right)^2 - w_{1(n-1)x}\int_s^t (c_1(\tau)\omega_4^n + g_1(\tau)\omega_2^n)d\tau. \tag{2.20}$$

For all $0 \leqslant t \leqslant T_2$, where $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$, the following estimates hold:

$$\|\omega_i^n\| \leqslant 25C_\varphi, \quad i = 1, 2, \quad \|\omega_3^n\| \leqslant 124C_\varphi, \quad \|\omega_4^n\| \leqslant 124C_\varphi.$$

Denote $q_n = \begin{pmatrix} w_{1nx} \\ w_{2nx} \end{pmatrix}$, $p_n = \sum_{j=1}^4 \|w_{j(n+1)} - w_{jn}\|$ and introduce the norm $\|q_n\| = \|w_{1nx}\| + \|w_{2nx}\|$. Using induction, for all $0 \leqslant t \leqslant T_2$, where $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$, we find

$$\sum_{n=0}^N \|q_{n+1} - q_n\| \leqslant 2\|q_1 - q_0\| + 0.9\sum_{n=1}^N p_n,$$

where $\sum_{n=1}^N p_n$ are bounded for any N. Consequently, the partial sums $\sum_{n=0}^N \|q_{n+1} - q_n\|$ are bounded for any N and the series $\sum_{n=0}^\infty \|q_{n+1} - q_n\|$ converges. Therefore, $w_{inx} \to w_{ix} = \partial_x w_i$, $i = 1, 2$. Further, $w_{3nx} \to w_{3x} = \partial_x w_3$ and $w_{4nx} \to w_{4x} = \partial_x w_4$. Consequently, $w_{jnx} \to w_{jx} = \partial_x w_j$, $j = 1, \dots, 4$, where the functions $\partial_x w_j$ are continuous with respect to all its arguments on Γ_{T_2}, $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$. The following estimates hold:

$$\|\partial_x w_i\| \leqslant 4C_\varphi, \quad i = 1, 2, \quad \|\partial_x w_3\| \leqslant 6C_\varphi, \ \|\partial_x w_4\| \leqslant 6C_\varphi.$$

Similarly, w_j, $j = 1, \dots, 4$, have continuous bounded t-derivatives on Γ_{T_2}. The uniqueness of a solution is proved in the same way as in [3]. □

Lemma 2.2. *Assume that* $\varphi_1(x), \varphi_2(x) \in \overline{C}^2(R)$, $a_1(t), b_1(t), b_2(t), c_1(t), g_1(t) \in C([0, T])$, $a_1(t) < 0$, $b_1(t) < 0$, $b_2(t) \geqslant 0$, $c_1(t) < 0$, $g_1(t) < 0$, $t \in [0, T]$, $\varphi_1'(x) \leqslant 0$, $\varphi_2'(x) \leqslant 0$, $x \in R$. *Then the functions* $\{w_j\}$, $j = 1, \dots, 4$, *solving the system* (1.16)–(1.19) *have the continuous bounded derivatives* $\dfrac{\partial^2 w_j}{\partial x^2}$, $\dfrac{\partial^2 w_j}{\partial x \partial t}$, $j = 1, \dots, 4$ *on* Γ_{T_2}, *where* $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$.

Proof. As proved in [1], the following inequalities hold on Γ_{T_2} :

$$\left| \int_s^t (a_1(\tau)w_{1n} + b_1(\tau)w_{3n})d\tau \right| \leqslant 0.16,$$

$$\left| \int_s^t (c_1(\tau)w_{4n} + g_1(\tau)w_{2n})d\tau \right| \leqslant 0.16,$$

where $T_2 = \min\{(1/(25C_\varphi l), 1/(10l)\}$. We fix $x_0 \in R$ and consider the set $\Omega_{x_0} = \{x \mid x_0 - 0.16 \leqslant x \leqslant x_0 + 0.16\}$. Let $x_1, x_2 \in \Omega_{x_0}$. We prove that

$$|\eta_{1n}(s,t,x_1) - \eta_{1n}(s,t,x_2)| \leqslant |x_1 - x_2|, \tag{2.21}$$

$$|\eta_{2n}(s,t,x_1) - \eta_{2n}(s,t,x_2)| \leqslant |x_1 - x_2|, \tag{2.22}$$

where

$$\eta_{1n}(s,t,x) = x - \int_s^t (a_1(\tau)w_{1n}(\tau,t,x) + b_1(\tau)w_{3n}(\tau,t,x))d\tau,$$

$$\eta_{2n}(s,t,x) = x - \int_s^t (c_1(\tau)w_{4n}(\tau,t,x) + g_1(\tau)w_{2n}(\tau,t,x))d\tau.$$

We assume that

$$w_{1(n-1)x} \leqslant 0, \quad w_{2(n-1)x} \leqslant 0. \tag{2.23}$$

For all $n \in N$ on Γ_{T_2}, where $T_2 = \min\{1/(25C_\varphi l), 1/(10l)\}$, we have

$$1 - \int_s^t (a_1(\tau)w_{1nx} + b_1(\tau)w_{3nx})d\tau > 0, \quad 1 - \int_s^t (c_1(\tau)w_{4nx} + g_1(\tau)w_{2nx})d\tau > 0. \tag{2.24}$$

From (2.15), (2.23), (2.24) it follows that $w_{3nx} \leqslant 0$. From (2.16), (2.23), (2.24) it follows that $w_{4nx} \leqslant 0$. Since $w_{3nx} \leqslant 0$, from (2.13), (2.24), and the conditions $b_2(t) \geqslant 0$, $t \in [0,T]$, $\varphi_1'(x) \leqslant 0$, $x \in R$, we find $w_{1nx} \leqslant 0$.

From (2.14), (2.24), and the conditions $\varphi_2'(x) \leqslant 0$, $x \in R$, we find $w_{2nx} \leqslant 0$.

Since $w_{1nx} \leqslant 0$, $w_{2nx} \leqslant 0$, $w_{3nx} \leqslant 0$, $w_{4nx} \leqslant 0$, we have

$$1 - \int_s^t (a_1(\tau)w_{1nx} + b_1(\tau)w_{3nx})d\tau \leqslant 1,\ 1 - \int_s^t (c_1(\tau)w_{4nx} + g_1(\tau)w_{2nx})d\tau \leqslant 1. \tag{2.25}$$

By (2.24), (2.25) and the finite increment formula, we obtain (2.21) and (2.22).

Arguing in the same way as in [4], we can prove the x-equicontinuity of ω_1^n and ω_2^n for $x \in \Omega_{x_0}$, which implies the x-equicontinuity of ω_1^n and ω_2^n at any point $x_0 \in R$.

We consider the system of equations

$$\widetilde{\omega}_1^n = -\varphi_1'(\eta_1(0,t,x)) \int_0^t (a_1(\tau)\widetilde{\omega}_1^n + b_1(\tau)\widetilde{\omega}_3^n)d\tau \exp(a_2 s) + \int_0^s b_2(\tau)\widetilde{\omega}_3^n \exp(a_2(s-\tau))d\tau$$
$$+ \varphi_1''\left(1 - \int_0^t (a_1(\tau)w_{1x} + b_1(\tau)w_{3x})d\tau\right)^2 \exp(a_2 s),$$

$$\widetilde{\omega}_2^n = -\varphi_2'(\eta_2(0,t,x)) \int_0^t (c_1(\tau)\widetilde{\omega}_4^n + g_1(\tau)\widetilde{\omega}_2^n)d\tau \exp(g_2 s)$$
$$+ \varphi_2''\left(1 - \int_0^t (c_1(\tau)w_{4x} + g_1(\tau)w_{2x})d\tau\right)^2 \exp(g_2 s),$$

$$\widetilde{\omega}_3^n = \widetilde{\omega}_2^{n-1}\left(1 - \int\limits_s^t (a_1(\tau)w_{1x} + b_1(\tau)w_{3x})d\tau\right)^2 - w_{2x}(s,s,\eta_1(s,t,x)) \int\limits_s^t (a_1(\tau)\widetilde{\omega}_1^n + b_1(\tau)\widetilde{\omega}_3^n)d\tau,$$

$$\widetilde{\omega}_4^n = \widetilde{\omega}_1^{n-1}\left(1 - \int\limits_s^t (c_1(\tau)w_{4x} + g_1(\tau)w_{2x})d\tau\right)^2 - w_{1x}(s,s,\eta_2(s,t,x)) \int\limits_s^t (c_1(\tau)\widetilde{\omega}_4^n + g_1(\tau)\widetilde{\omega}_2^n)d\tau.$$

On Γ_{T_2}, the following estimates hold:

$$\|\widetilde{\omega}_1^n\| \leqslant 2C_\varphi, \ \|\widetilde{\omega}_2^n\| \leqslant 2C_\varphi, \quad \|\widetilde{\omega}_3^n\| \leqslant 3C_\varphi, \quad \|\widetilde{\omega}_4^n\| \leqslant 3C_\varphi.$$

Further, $\widetilde{\omega}_j^n \to \widetilde{\omega}_j$, $j = 1, \dots, 4$, on Γ_{T_2} and

$$\|\widetilde{\omega}_1\| \leqslant 2C_\varphi, \quad \|\widetilde{\omega}_2\| \leqslant 2C_\varphi, \quad \|\widetilde{\omega}_3\| \leqslant 3C_\varphi, \quad \|\widetilde{\omega}_4\| \leqslant 3C_\varphi.$$

We show that ω_j^n converge to $\widetilde{\omega}_j$, $j = 1, \dots, 4$, as $n \to \infty$ on Γ_{T_2}. On Γ_{T_2}, we have

$$|\omega_1^n - \widetilde{\omega}_1| \leqslant |R_1^n| + 0.14(\|\omega_1^n - \widetilde{\omega}_1\| + \|\omega_3^n - \widetilde{\omega}_3\|),$$
$$\|\omega_3^n - \widetilde{\omega}_3\| \leqslant |R_2^n| + |\omega_2^{n-1} - \widetilde{\omega}_2| + 0.16(\|\omega_1^n - \widetilde{\omega}_1\| + \|\omega_3^n - \widetilde{\omega}_3\|),$$

where

$$R_1^n = \Bigg| (\varphi_1''(\eta_{1n}(0,t,x) - \varphi_1''(\eta_1(0,t,x)))\eta_{1nx}^2(s,t,x) + \varphi_1''(\eta_1(0,t,x))[\eta_{1nx}^2(0,t,x) - \eta_{1x}^2(0,t,x)]$$
$$- (\varphi_1'(\eta_{1n}(0,t,x)) - \varphi_1'(\eta_1(0,t,x))) \int\limits_0^t (a_1(\tau)\omega_1^n(\tau,t,x) + b_1(\tau)\omega_3^n(\tau,t,x))d\tau \Bigg| \exp(a_2 s),$$

$$R_2^n = \Bigg| \omega_2^{n-1}(s,s,\eta_{1n}(s,t,x))[\eta_{1nx}^2(s,t,x) - \eta_{1x}^2(s,t,x)] - \int\limits_s^t (a_1(\tau)\omega_1^n(\tau,t,x) + b_1(\tau)\omega_3^n(\tau,t,x))d\tau$$
$$\times \left[w_{2(n-1)x}(s,s,\eta_{1n}(s,t,x)) - w_{2x}(s,s,\eta_1(s,t,x))\right] \Bigg|,$$

$$\eta_{1n}(s,t,x) = x - \int\limits_s^t (a_1(\tau)w_{1n}(\tau,t,x) + b_1(\tau)w_{3n}(\tau,t,x))d\tau,$$

$$\eta_{2n}(s,t,x) = x - \int\limits_s^t (c_1(\tau)w_{4n}(\tau,t,x) + g_1(\tau)w_{2n}(\tau,t,x))d\tau.$$

Since all functions in R_1^n and R_2^n are uniformly continuous, equicontinuous, and bounded, for any ε there exists N such that $|R_1^n| < \varepsilon$ and $|R_2^n| < \varepsilon$ for $n \geqslant N$. Consequently, for $n \geqslant N$

$$\begin{aligned} &\|\omega_1^n - \widetilde{\omega}_1\| \leqslant 1.2\varepsilon + 0.2\|\omega_3^n - \widetilde{\omega}_3\|, \\ &\|\omega_3^n - \widetilde{\omega}_3\| \leqslant 1.2\varepsilon + 1.2\|\omega_2^{n-1} - \widetilde{\omega}_2\| + 0.2\|\omega_1^n - \widetilde{\omega}_1\|. \end{aligned} \tag{2.26}$$

Hence for $n \geqslant N$

$$\|\omega_1^n - \widetilde{\omega}_1\| \leqslant \frac{4}{3}\varepsilon + \frac{1}{3}\|\omega_2^{n-1} - \widetilde{\omega}_2\|. \tag{2.27}$$

Similarly, for $n \geqslant N$

$$\|\omega_2^n - \widetilde{\omega}_2\| \leqslant \frac{4}{3}\varepsilon + \frac{1}{3}\|\omega_1^{n-1} - \widetilde{\omega}_1\|. \tag{2.28}$$

Adding (2.27) and (2.28), we get

$$\|\omega_1^n - \widetilde{\omega}_1\| + \|\omega_2^n - \widetilde{\omega}_2\| \leqslant \frac{8}{3}\varepsilon + \frac{1}{3}(\|\omega_2^{n-1} - \widetilde{\omega}_2\| + \|\omega_1^{n-1} - \widetilde{\omega}_1\|).$$

The following inequality can be proved by induction:

$$\|\omega_1^{N+k} - \widetilde{\omega}_1\| + \|\omega_2^{N+k} - \widetilde{\omega}_2\| \leqslant \left(\frac{1}{3}\right)^k (\|\omega_1^N - \widetilde{\omega}_1\| + \|\omega_2^N - \widetilde{\omega}_2\|) + 4\varepsilon$$

for $n \geqslant N$. Consequently, $\omega_1^{N+k} \to \widetilde{\omega}_1$, $\omega_2^{N+k} \to \widetilde{\omega}_2$ as $N \to \infty$, $k \to \infty$. From (2.26) it follows that $\omega_3^n \to \widetilde{\omega}_3$ as $n \to \infty$. Similarly, $\omega_4^n \to \widetilde{\omega}_4$ as $n \to \infty$ and

$$\|\widetilde{\omega}_1\| \leqslant 2C_\varphi, \quad \|\widetilde{\omega}_2\| \leqslant 2C_\varphi, \quad \|\widetilde{\omega}_3\| \leqslant 3C_\varphi, \quad \|\widetilde{\omega}_4\| \leqslant 3C_\varphi.$$

Thus, $w_{jnxx} \to w_{jxx} = \widetilde{\omega}_j$, where the functions $\dfrac{\partial^2 w_j}{\partial x^2}$, $j = 1, \ldots, 4$, are continuous and bounded on Γ_{T_2}. Furthermore, they have continuous bounded derivatives $\dfrac{\partial^2 w_j}{\partial x \partial t}$, $j = 1, \ldots, 4$ on Γ_{T_2}. □

3 Existence of Global Solution

Theorem 3.1. *Assume that* $\varphi_1(x), \varphi_2(x) \in \overline{C}^2(R)$, $a_1(t), b_1(t), b_2(t), c_1(t), g_1(t) \in C([0,T])$, $a_1(t) < 0$, $b_1(t) < 0$, $b_2(t) \geqslant 0$, $c_1(t) < 0$, $g_1(t) < 0$, $t \in [0,T]$; $\varphi_1'(x) \leqslant 0$, $\varphi_2'(x) \leqslant 0$, $x \in R$. *Then for any* $T > 0$ *the Cauchy problem* (1.1), (1.2) *has a unique solution* $u(t,x), v(t,x) \in \overline{C}^{1,2,2}(\Omega_T)$ *that can be found from* (1.16)–(1.19).

Proof. Differentiating (1.1) with respect to x and denoting $p(t,x) = u_x(t,x)$, $q(t,x) = v_x(t,x)$, we obtain the system of equations

$$\begin{aligned}
&\partial_t p + (a_1(t)u(t,x) + b_1(t)v(t,x))\partial_x p = -a_1(t)p^2 - b_1(t)pq + a_2 p + b_2(t)q, \\
&\partial_t q + (c_1(t)u(t,x) + g_1(t)v(t,x))\partial_x q = -g_1(t)q^2 - c_1(t)pq + g_2 q, \\
&p(0,x) = \varphi_1'(x), \quad q(0,x) = \varphi_2'(x).
\end{aligned} \tag{3.1}$$

We add (1.10)–(1.15) and the equations

$$\begin{aligned}
\frac{d\gamma_1(s,t,x)}{ds} &= -a_1(s)\gamma_1^2 - b_1(s)\gamma_1\gamma_2(s,s,\eta_1) + a_2\gamma_1 + b_2(s)\gamma_2(s,s,\eta_1), \\
\frac{d\gamma_2(s,t,x)}{ds} &= -g_1(s)\gamma_2^2 - c_1(s)\gamma_1(s,s,\eta_2)\gamma_2 + g_2\gamma_2
\end{aligned} \tag{3.2}$$

with the conditions $\gamma_1(0,t,x)=\varphi_1'(\eta_1)$ and $\gamma_2(0,t,x)=\varphi_2'(\eta_2)$. We write (3.2) in the form

$$\gamma_1(s,t,x)=\varphi_1'(\eta_1)\exp\Bigg(-\int\limits_0^s(a_1(\tau)\gamma_1+b_1(\tau)\gamma_2(\tau,\tau,\eta_1)-a_2)d\tau\Bigg)$$
$$+\int\limits_0^s b_2(\tau)\gamma_2(\tau,\tau,\eta_1)\exp\Bigg(-\int\limits_\tau^s(a_1(\tau)\gamma_1+b_1(\tau)\gamma_2(\nu,\nu,\eta_1)-a_2)d\nu\Bigg)d\tau, \tag{3.3}$$
$$\gamma_2(s,t,x)=\varphi_2'(\eta_2)\exp\Bigg(-\int\limits_0^s(g_1(\tau)\gamma_2+c_1(\tau)\gamma_1(\tau,\tau,\eta_2)-g_2)d\tau\Bigg).$$

Using the method of successive approximations, we establish the existence of continuous solution to the system (3.3) on Γ_{T_2}, where $T_2=\min\{1/(25C_\varphi l),1/(10l)\}$. We define the successive approximations

$$\gamma_1^{n+1}=\varphi_1'(\eta_1)\exp\Bigg(-\int\limits_0^s(a_1(\tau)\gamma_1^n+b_1(\tau)\gamma_2^n(\tau,\tau,\eta_1)-a_2)d\tau\Bigg)$$
$$+\int\limits_0^s b_2(\tau)\gamma_2^n(\tau,\tau,\eta_1)\exp\Bigg(-\int\limits_\tau^s(a_1(\tau)\gamma_1^n+b_1(\tau)\gamma_2^n(\nu,\nu,\eta_1)-a_2)d\nu\Bigg)d\tau, \tag{3.4}$$
$$\gamma_2^{n+1}=\varphi_2'(\eta_2)\exp\Bigg(-\int\limits_0^s(g_1(\tau)\gamma_2^n+c_1(\tau)\gamma_1^n(\tau,\tau,\eta_2)-g_2)d\tau\Bigg);$$

moreover, $\gamma_1^0=\varphi_1'(\eta_1)$, $\gamma_2^0=\varphi_2'(\eta_2)$. On Γ_{T_2}, we have

$$|\gamma_i^{n+1}|\leqslant 2C_\varphi,\quad |\eta_{ix}|\leqslant 1,\quad |\,\gamma_{ix}^{n+1}|\leqslant 5C_\varphi,\quad i=1,2.$$

The successive approximations $\{\gamma_i^n\}$, $i=1,2$, converge to a continuous solution to the system (3.3) on Γ_{T_2} since

$$\|\gamma_1^{n+1}-\gamma_1^n\|+\|\gamma_2^{n+1}-\gamma_2^n\|\leqslant 0.6(\|\gamma_1^n-\gamma_1^{n-1}\|+\|\gamma_2^n-\gamma_2^{n-1}\|).$$

On Γ_{T_2}, we have $|\gamma_i|\leqslant 2C_\varphi$, $i=1,2$. As in [1], we can prove the existence of a continuously differentiable solution to the problem (3.3). Consequently,

$$\gamma_1(t,t,x)=p(t,x)=\frac{\partial u}{\partial x},\quad \gamma_2(t,t,x)=q(t,x)=\frac{\partial v}{\partial x}.$$

As in [4], we can prove that for all t and x on Ω_T

$$\|v\|\leqslant C_\varphi\exp(|g_2|T),\quad \|u\|\leqslant C_\varphi\exp(|a_2|T)(1+Tl\exp(|g_2|T)). \tag{3.5}$$

From (3.3) it follows that $\gamma_1\leqslant 0$ and $\gamma_2\leqslant 0$ on Γ_T. Consequently,

$$\|\gamma_2\|\leqslant C_\varphi\exp(|g_2|T),\quad \|\gamma_1\|\leqslant C_\varphi\exp(|a_2|T)(1+Tl\exp(|g_2|T)).$$

Since

$$\gamma_1(t,t,x)=\frac{\partial u}{\partial x},\quad \gamma_2(t,t,x)=\frac{\partial v}{\partial x},$$

for all t and x on Ω_T the following estimates hold:

$$\begin{aligned} &\|\partial_x v\| \leqslant C_\varphi \exp(|g_2|T), \\ &\|\partial_x u\| \leqslant C_\varphi \exp(|a_2|T)(1 + Tl \exp(|g_2|T)). \end{aligned} \tag{3.6}$$

As in [4], for all t and x we obtain the estimates

$$|\partial^2_{x^2} u| \leqslant E_{11} \operatorname{ch}(T\sqrt{C_{12}C_{21}}) + E_{21}\sqrt{\frac{C_{12}}{C_{21}}} \operatorname{sh}(T\sqrt{C_{12}C_{21}}), \tag{3.7}$$

$$|\partial^2_{x^2} v| \leqslant E_{21} \operatorname{ch}(T\sqrt{C_{12}C_{21}}) + E_{11}\sqrt{\frac{C_{21}}{C_{12}}} \operatorname{sh}(T\sqrt{C_{12}C_{21}}), \tag{3.8}$$

where E_{11}, E_{21}, C_{12}, C_{21} are constants. Owing to the global estimates (3.5), (3.6)–(3.8), we can extend the solution to any given interval $[0,\ T]$. We take $u(T_0, x)$ and $v(T_0, x)$ for the initial values. Using Theorem 2.1, we extend the solution to the interval $[T_0,\ T_1]$. Then for the initial values we take $u(T_1, x)$, $v(T_1, x)$. Using Theorem 2.1, we extend the solution to the interval $[T_1,\ T_2]$. In particular, $u(T_k, x), v(T_k, x) \in \overline{C}^2(R)$ satisfy the estimate

$$|u(T_k, x)| \leqslant C_\varphi \exp(|a_2|T)(1 + Tl \exp(|g_2|T)), \quad |v(T_k, x)| \leqslant C_\varphi \exp(|g_2|T),$$

$$|\partial_x u(T_k, x)| \leqslant C_\varphi \exp(|a_2|T)(1 + Tl \exp(|g_2|T)), \quad |\partial_x v(T_k, x)| \leqslant C_\varphi \exp(|g_2|T).$$

The second order derivatives satisfy the estimates (3.7) and (3.8), where T can be taken for t. As a result, we can extend the solution to any given interval $[0,\ T]$ in finitely many steps.

The uniqueness of a solution to the Cauchy problem (1.1), (1.2) is proved with the help of estimates similar to those used in the proof of the convergence of successive approximations. □

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