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REFINED MODEL OF THERMOELASTOPLASTIC BENDING OF LAYERED PLATES WITH REGULAR STRUCTURES. I. STATEMENT OF THE PROBLEM

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We formulate the problem of quasistatic thermoelastoplastic bending of layered plates with regular structures in the geometrically linear statement. The mechanical behavior of isotropic layers is described by the deformation-type relations of thermoelastoplasticity with regard for their different tensile and compression resistances. The linearized governing relations of layered media are deduced with the help of the method of variable parameters of elasticity. The obtained equations enable us to describe, with different degrees of accuracy, the stress-strain state of these plates by taking into account their weakened resistance to transverse shears. Note that the relations of traditional nonclassical Reissner and Reddy theories follow from these equations as particular cases. Within the framework of the proposed refined theories and Reddy theory, the force boundary conditions for tangential stresses are satisfied on the front surfaces. The boundary conditions for normal stresses are not satisfied on these surfaces. The variations of deflections across the thickness of the structures are not taken into account. The three-dimensional equilibrium equations and the boundary conditions imposed on the end surface of the plate are reduced to two-dimensional relations by the method of weighted residuals. As weight functions, we use homogeneous polynomials in the transverse coordinate.

Keywords: layered composite plate, regular structure, deformation-type thermoelastoplasticity, bent plate, Reissner theory, Reddy theory, refined theory of bending.

At present, thin-walled elements, such as plates and shells, made of composite materials (CM) [16, 25, 30], including materials with layered structures [24, 26, 36, 37], are used in the engineering practice more and more extensively. In the case where the anisotropy of CM is strongly pronounced, the analysis of bending behavior of linearly elastic plates should be carried out by taking into account their weakened resistance to transverse shear [2, 3, 7, 13, 33, 38]. However, as shown in [20, 21], under the conditions of elastoplastic bending deformation of layered plates with regular structures, it is also necessary to take into account their weakened resistance to transverse shears even if the analyzed CM exhibits weakly pronounced anisotropy under the conditions of linearly elastic behavior (e.g., if it is a metal-composite substance). This problem is complicated, in particular, by the fact that some materials of the phases of compositions (e.g., certain grades of steels) have weak heat sensitivities in sufficiently wide temperature ranges [5], whereas the other materials (polymers or alloys of light metals) are highly heat-sensitive and, moreover, the strength and stiffness characteristics of these materials, as a rule, rapidly worsen as temperature increases [5, 18]. The indicated specific character of the behaviors of materials of the layers of compositions at high temperatures may lead to the situations in which the analyzed composite products subjected to intense thermal and force loads behave in a quite different way than it is usually expected at moderate temperatures.

The investigations of the problems of bending deformation of layered plates within the framework of the classical theory that does not take into account transverse shears in these plates, were carried out, e.g., in [11, 39].

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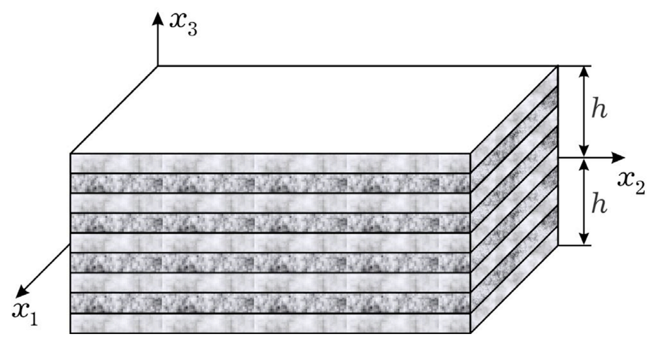

Fig. 1

The weakened resistance of CM plates to transverse shear is usually taken into account either within the framework of the Reissner–Mindlin theory [7, 29, 34, 35], or in the second version of the Timoshenko theory [2, 3, 13, 21] (or in the Reddy theory [31, 33], as it is called in the English-language literature), or within the framework of the Kulikov theory based on the hypothesis of broken line [10, 15]. The asymptotic methods of numerical analyses are used less frequently [14, 17]. In the case of bending of linearly elastic CM plates and shells, it was shown [2, 3] that the solutions obtained on the basis of the Reddy theory do not require additional refinement, i.e., it is not necessary to apply the theories of higher orders of accuracy [1, 27, 28, 31, 32, 36]. However, it remains unclear whether the solutions obtained within the framework of the Reddy theory in the cases of elastoplastic or nonlinear elastic bending of layered plates with regular structures, especially at high temperatures, are sufficiently accurate for the engineering purpose or it is necessary to refine these solutions and, in particular, whether the solutions obtained in [21] are sufficiently accurate. In addition, it is of interest to determine the accuracy of the Reissner theory in the indicated cases of bending of CM plates because, among all nonclassical theories, this theory is most extensively used for the practical purposes [1, 7, 13]. By analyzing an example of layered beam-wall with regular structure, it was shown [22] that the refinement of the corresponding nonclassical theories is indeed necessary if the analyzed beam has a sufficiently large relative height (of about 1/7 or greater). It is also necessary to take into account the fact that beams-walls are characterized by the realization of generalized plane stressed states, whereas bent plates are characterized by the presence of complex stress-strain states (SSS). The last may result in the appearance of certain specific features of deformation of thin-walled structural elements of this kind that are not observed in beams-walls. Furthermore, the influence of thermal action was not taken into account in [22].

Thus, the present work is devoted to the construction of a refined model of bending of layered plates with regular structures, which has the classical theory, Reissner theory, and Reddy theory as particular cases. We also perform the comparative analysis of solutions obtained on the basis of these theories and more accurate relations in the cases of both thermoelastic and thermoelastoplastic deformation of bent plates. These investigations are required, in particular, to clarify the question whether the contemporary finite elements constructed on the basis of the Reissner [1, 4] and Reddy [23] theories enable one to give adequate description of the mechanical behavior of the layered elements of thin-walled products with regular structures.

1. Refined Model of Bending Deformation of a Layered Plate with Regular Structure

In a Cartesian coordinate system x_i, we consider a plate of thickness $2h$ formed by regularly arranged layers of small thickness parallel to the reference plane Ox_1x_2. This plane coincides with the median plane of the thin-walled element and the Ox_3-axis is oriented in the transverse direction (see Fig. 1).

To describe the weakened resistance of this plate to transverse shears, we represent the strains ε_{i3} in the form of finite sums of power series in the variable x_3:

$$\varepsilon_{i3}(\mathbf{r}) = \beta\left[\frac{x_3+h}{2h}\varepsilon_{i3}^{(+)}(\mathbf{x}) - \frac{x_3-h}{2h}\varepsilon_{i3}^{(-)}(\mathbf{x})\right] + \frac{h^2-\beta x_3^2}{h^2}\sum_{k=0}^{K} x_3^k \varepsilon_{i3}^{(k)}(\mathbf{x}),$$

$$\mathbf{x}\in G, \quad |x_3| \le h, \qquad \mathbf{x} = \{x_1, x_2\}, \qquad \mathbf{r} = \{x_1, x_2, x_3\}, \tag{1}$$

$$i = 1, 2,$$

where $\varepsilon_{i3}^{(\pm)}$ and $\varepsilon_{i3}^{(k)}$ are functions of two variables x_1 and x_2 (the problem is considered in the quasistatic statement) that should be determined and have the following sense: $\varepsilon_{i3}^{(\pm)}$ are the transverse shear strains on the top and bottom $(x_3 = \pm h)$ front surfaces of the plate. For $\varepsilon_{i3}^{(\pm)} \equiv 0$, the functions $\varepsilon_{i3}^{(0)}$ determine the transverse shear strains in the median plane $x_3 = 0$; $\varepsilon_{i3}^{(k)}$, $1 \le k \le K$, are unknown expansion coefficients with dimensions 1/m^k; K is the number of terms preserved in the partial sums of the power series; G is the domain occupied by the plate (in plan), and β is the switching parameter.

Note that, for $\beta = 0$ and $\varepsilon_{i3}^{(k)}(\mathbf{x}) \equiv 0$, $0 \le k \le K$, $i = 1, 2$, we get the relations based on the hypotheses of the classical theory [11, 39]. For $K = 0$ and $\beta = 1$, expression (1) yields the relations of the Reddy theory [2, 3, 13, 21, 33]. Moreover, for $K \ge 1$ and $\beta = 1$, we get the refinement of the Reddy theory and, for $K = 0$ and $\beta = 0$, expression (1) gives the relations of the Reissner theory [1, 7, 34, 35].

In view of the kinematic assumption traditionally used for thin-walled structural elements, the variations of deflection u_3 in the transverse direction x_3 can be neglected [2, 3, 7, 11, 13, 21, 29, 34, 35, 39]:

$$u_3(\mathbf{r}) = u_3^0(\mathbf{x}), \qquad \mathbf{x}\in G, \quad |x_3| \le h. \tag{2}$$

By using the differential Cauchy relations [2, 7, 21], in view of (1) and (2), we obtain

$$u_i(\mathbf{r}) = u_i^0(\mathbf{x}) - x_3\partial_i u_3^0(\mathbf{x})$$

$$+ 2\sum_{k=0}^{K}\frac{x_3^{k+1}}{h^2}\left(\frac{h^2}{k+1} - \frac{\beta x_3^2}{k+3}\right)\varepsilon_{i3}^{(k)}(\mathbf{x})$$

$$+ \frac{\beta x_3}{h}\left(\frac{x_3}{2} + h\right)\varepsilon_{i3}^{(+)}(\mathbf{x})$$

$$+ - \frac{\beta x_3}{h}\left(\frac{x_3}{2} - h\right)\varepsilon_{i3}^{(-)}(\mathbf{x}), \tag{3}$$

$$\mathbf{x}\in G, \quad |x_3| \le h, \quad i = 1, 2,$$

$$
\begin{aligned}
\varepsilon_{12}(\mathbf{r}) &= \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) = \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) - x_3 \partial_1 \partial_2 u_3^0 \\
&\quad + \sum_{k=0}^{K} \frac{x_3^{k+1}}{h^2}\left(\frac{h^2}{k+1} - \frac{\beta x_3^2}{k+3}\right)\left(\partial_1 \varepsilon_{23}^{(k)} + \partial_2 \varepsilon_{13}^{(k)}\right) \\
&\quad + \frac{\beta x_3}{2h}\left(\frac{x_3}{2} + h\right)\left(\partial_1 \varepsilon_{23}^{(+)} + \partial_2 \varepsilon_{13}^{(+)}\right) \\
&\quad - \frac{\beta x_3}{2h}\left(\frac{x_3}{2} - h\right)\left(\partial_1 \varepsilon_{23}^{(-)} + \partial_2 \varepsilon_{13}^{(-)}\right), \\
\varepsilon_{ii}(\mathbf{r}) &= \partial_i u_i = \partial_i u_i^0 - x_3 \partial_i^2 u_3^0 \\
&\quad + 2\sum_{k=0}^{K} \frac{x_3^{k+1}}{h^2}\left(\frac{h^2}{k+1} - \frac{\beta x_3^2}{k+3}\right)\partial_i \varepsilon_{i3}^{(k)} \\
&\quad + \frac{\beta x_3}{h}\left(\frac{x_3}{2} + h\right)\partial_i \varepsilon_{i3}^{(+)} \\
&\quad - \frac{\beta x_3}{h}\left(\frac{x_3}{2} - h\right)\partial_i \varepsilon_{i3}^{(-)}, \\
&\qquad \mathbf{x} \in G, \quad |x_3| \le h,
\end{aligned}
\tag{4}
$$

where u_i are the displacements of points of the analyzed plate in the tangential directions x_i, $i = 1, 2$, u_i^0 are the displacements of points of the median plane $x_3 = 0$, and ∂_i is the operator of differentiation with respect to the variable x_i.

Hence, in relations (1)–(4), the functions u_i^0, u_3^0, $\varepsilon_{i3}^{(\pm)}$, $\varepsilon_{i3}^{(k)}$, $i = 1, 2$, $0 \le k \le K$, depending only on two variables x_1 and x_2, are unknown.

As in [20], we can assume that the materials of all layers are isotropic and homogeneous and that their thermoelastoplastic behavior is described by the following generalized deformation-type quasilinear equations [8, 9]:

$$
\begin{aligned}
\sigma_{ii}^{(n)} - \sigma_0^{(n)} &= 2g_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right)\left(\varepsilon_{ii}^{(n)} - \varepsilon_0^{(n)}\right), \\
\sigma_{ij}^{(n)} &= 2g_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right)\varepsilon_{ij}^{(n)}, \qquad j \ne i, \quad i, j = 1, 2, 3, \\
\sigma_0^{(n)} &= K_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right)\left(\varepsilon_0^{(n)} - \alpha_n(\Theta - \Theta_0)\right), \\
T_n &= g_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right)\Gamma_n, \quad 1 \le n \le N.
\end{aligned}
\tag{5}
$$

Here, $\sigma_0^{(n)}$ and $\varepsilon_0^{(n)}$ are, respectively, the mean normal stress and mean linear strain in the material of the *n*th phase of layered composition; T_n and Γ_n are, respectively, the intensity of tangential stresses and the intensity of shear strains in the *n*th component of the composition {see (4) in [20]}; $g_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right)$ is a function known from the experiments and playing the role the proportionality coefficient between T_n and Γ_n; $K_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right)$ is the tripled modulus of volumetric expansion of the material of the *n*th phase of composition also known from the experiments; $\alpha_n = \alpha_n(\Theta)$ is the coefficient of linear thermal expansion of the material of the *n*th component of composition; Θ is the temperature of the layered composition; Θ_0 is the temperature of its natural state, and N is the number of families of regularly arranged layers. The dependence of the functions g_n and K_n in (5) on the argument $\varepsilon_0^{(n)}$ makes it possible, in the general case, to take into account the effect of different resistances of the material of the *n*th phase of composition [9], whereas the dependence on Θ reflects the heat sensitivity of the material.

It is reasonable to represent relations (5) in the matrix form

$$\boldsymbol{\sigma}_n = \mathbf{A}_n \boldsymbol{\varepsilon}_n + \mathbf{p}_n, \quad n = 1, 2, \ldots, N, \tag{6}$$

where

$$\begin{aligned}
\boldsymbol{\sigma}_n &= \left\{\sigma_{11}^{(n)}, \sigma_{22}^{(n)}, \sigma_{33}^{(n)}, \sigma_{23}^{(n)}, \sigma_{31}^{(n)}, \sigma_{12}^{(n)}\right\}^*, \\
\boldsymbol{\varepsilon}_n &= \left\{\varepsilon_{11}^{(n)}, \varepsilon_{22}^{(n)}, \varepsilon_{33}^{(n)}, \varepsilon_{23}^{(n)}, \varepsilon_{31}^{(n)}, \varepsilon_{12}^{(n)}\right\}^*, \\
\mathbf{p}_n &= p^{(n)}\{1,1,1,0,0,0\}^*, \\
p^{(n)} &= -K_n\left(\varepsilon_0^{(n)}, \Gamma_n, \Theta\right) \alpha_n (\Theta - \Theta_0), \quad 1 \le n \le N,
\end{aligned} \tag{7}$$

$\mathbf{A}_n = \left(a_{ij}^{(n)}\right)$ are symmetric 6×6-matrices whose nonzero components according to (5) are determined by relations (7) from [20]. The operation of transposition is denoted by the asterisk.

We assume that relations (5) [and, hence, relations (6) together with (7)] satisfy sufficient conditions for the convergence of the method of successive approximations (see [9, p. 199]) similar to the method of variable elasticity parameters [12]. In what follows, according to this assumption, relations (5) and (6) are regarded as linearized and, therefore, the components of the vector $\mathbf{p}_n$ [see (7)] in each iteration can be treated as known initial (or, more precisely, temperature) stresses. Formally, the linearized matrix equalities (6) completely coincide with relations (8) from [19] and, therefore, by repeating the arguments from [19, 20] in each iteration, we deduce effective linear governing equations for the layered composite medium with regular structure, which can be represented in the matrix form as follows:

$$\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon} + \mathbf{p}, \tag{8}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the six-component vector columns of mean stresses and strains whose structures are similar to (7); $\mathbf{A} = (a_{ij})$ is a known 6×6-matrix, which can be interpreted (in each iteration) as the matrix of effective stiffnesses of the layered composition; $\mathbf{p} = \{p_i\}$ is a known, in the current iteration, six-component vector column of mean thermal stresses whose components are as follows:

$$p_1 = p_2 \neq 0, \quad p_3 \neq 0, \quad \text{and} \quad p_j \equiv 0, \quad j = 4, 5, 6.$$

The components of the matrix $\mathbf{A}$ and the vector $\mathbf{p}$ in (8) are determined by the matrix equalities (10) from [19]. If the mean strains $\boldsymbol{\varepsilon}$ are known from the solution of the corresponding boundary-value problem for a composite layered body (in particular, for a plate) whose mechanical behavior is described by the governing equation (8), then, by using matrix relations (13) and (14) from [19], we can also find the strains $\boldsymbol{\varepsilon}_n$ in nth component of the layered composition [see (6) and (7)]. Thus, by the method of successive approximations, we can refine the values of the coefficients g_n and K_n in (5) [or, equivalently, the values of components of the matrices $\mathbf{A}_n$ and vectors $\mathbf{p}_n$ in (6) with regard for (7)] and, by using the scheme described above (and in [19, 20]), construct the next approximation to the solution. Then the outlined iterative process is continued until we guarantee its convergence with the required accuracy.

We use the following static hypothesis traditional for thin-walled structural elements: $\sigma_{33}(\mathbf{r}) \approx 0$ [1–3, 7, 11, 13, 21, 29, 34, 35, 39]. Thus, we can transform the system of six algebraic equalities (8) to the following form (by eliminating the strain ε_{33}):

$$\sigma_{ii} = b_{i1}\varepsilon_{11} + b_{i2}\varepsilon_{22} + p_{ii}, \quad \sigma_{12} = b_{33}\varepsilon_{12}, \quad \sigma_{i3} = b_{jj}\varepsilon_{i3},$$
$$j = i + 3, \quad i = 1, 2, \tag{9}$$

where

$$b_{ij} \equiv a_{ij} - \frac{a_{i3}a_{3j}}{a_{33}}, \quad p_{ii} \equiv p_i - \frac{a_{i3}p_3}{a_{33}}, \quad i, j = 1, 2,$$

$$\varepsilon_{33} = -\frac{1}{a_{33}}(a_{31}\varepsilon_{11} + a_{32}\varepsilon_{22} + p_3),$$

$$b_{11} = b_{22}, \quad b_{21} = b_{12}, \quad b_{44} = b_{55}, \tag{10}$$

$$b_{33} \equiv a_{66}, \quad b_{44} \equiv a_{55}, \quad b_{55} \equiv a_{44},$$

a_{ij} are the components of the matrix $\mathbf{A}$ in (8), which has the same block-diagonal structure as the matrices $\mathbf{A}_n$ in (6) [see (5)]; p_i are the nonzero components of the vector $\mathbf{p}$ in (8); σ_{ij} and ε_{ij} are the components of mean stresses and strains in the composition [the components of the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in (8)]. According to (10), the quantities b_{ij} and p_{ii} in (9) are assumed to be known from the solution of the analyzed problem in the previous iteration of the method of successive approximations.

By using relations (9), in view of (1), (4), and (10), we can find all internal force factors acting in the plate for the current iteration:

$$M_{jj}^{(\ell)}(\mathbf{x}) = \sum_{i=1}^{2}\left(a_{ji}^{(\ell)}\,\partial_i u_i^0 - b_{ji}^{(\ell)}\,\partial_i^2 u_3^0 + 2\sum_{k=0}^{K} c_{ji}^{(\ell,k)}\,\partial_i \varepsilon_{i3}^{(k)} + 2d_{ji}^{(\ell)}\,\partial_i \varepsilon_{i3}^{(+)} - 2e_{ji}^{(\ell)}\,\partial_i \varepsilon_{i3}^{(-)} \right) + P_{jj}^{(\ell)}(\mathbf{x}),$$

$$M_{12}^{(\ell)}(\mathbf{x}) = \frac{a_{33}^{(\ell)}}{2}\left(\partial_1 u_2^0 + \partial_2 u_1^0\right) - b_{33}^{(\ell)}\,\partial_1\partial_2 u_3^0 + \sum_{k=0}^{K} c_{33}^{(\ell,k)}\left(\partial_1 \varepsilon_{23}^{(k)} + \partial_2 \varepsilon_{13}^{(k)}\right) + d_{33}^{(\ell)}\left(\partial_1 \varepsilon_{23}^{(+)} + \partial_2 \varepsilon_{13}^{(+)}\right) - e_{33}^{(\ell)}\left(\partial_1 \varepsilon_{23}^{(-)} + \partial_2 \varepsilon_{13}^{(-)}\right),$$

$$M_{j3}^{(\ell)}(\mathbf{x}) = \sum_{k=0}^{K} g_{j+3,j+3}^{(\ell,k)}\varepsilon_{j3}^{(k)} + h_{j+3,j+3}^{(\ell)}\varepsilon_{j3}^{(+)} - f_{j+3,j+3}^{(\ell)}\varepsilon_{j3}^{(-)}, \tag{11}$$

$$j = 1, 2, \quad \ell = 0, 1, \ldots, \frac{2}{3}K + 1, \quad \mathbf{x} \in G,$$

where

$$P_{jj}^{(\ell)}(\mathbf{x}) = \int_{-h}^{h} p_{jj}(\mathbf{r}) x_3^{\ell}\, dx_3, \qquad M_{ji}^{(\ell)}(\mathbf{x}) \equiv \int_{-h}^{h} \sigma_{ji}(\mathbf{r}) x_3^{\ell}\, dx_3,$$

$$a_{ji}^{(\ell)}(\mathbf{x}) = \int_{-h}^{h} b_{ji}(\mathbf{r}) x_3^{\ell}\, dx_3, \qquad b_{ji}^{(\ell)}(\mathbf{x}) = \int_{-h}^{h} b_{ji}(\mathbf{r}) x_3^{\ell+1}\, dx_3,$$

$$d_{ji}^{(\ell)}(\mathbf{x}) = \beta \int_{-h}^{h} b_{ji}(\mathbf{r}) \frac{x_3^{\ell+1}}{2h}\left(\frac{x_3}{2} + h\right) dx_3,$$

$$e_{ji}^{(\ell)}(\mathbf{x}) = \beta \int_{-h}^{h} b_{ji}(\mathbf{r}) \frac{x_3^{\ell+1}}{2h}\left(\frac{x_3}{2} - h\right) dx_3,$$

$$c_{ji}^{(\ell,k)}(\mathbf{x}) = \int_{-h}^{h} b_{ji}(\mathbf{r}) \frac{x_3^{\ell+k+1}}{h^2}\left(\frac{h^2}{k+1} - \frac{\beta x_3^2}{k+3}\right) dx_3,$$

$$g_{ii}^{(\ell,k)}(\mathbf{x}) = \int_{-h}^{h} b_{ii}(\mathbf{r}) \frac{x_3^{\ell+k}}{h^2}\left(h^2 - \beta x_3^2\right) dx_3, \tag{12}$$

$$h_{ii}^{(\ell)}(\mathbf{x}) = \beta \int_{-h}^{h} b_{ii}(\mathbf{r}) \frac{x_3^{\ell}}{2h}(x_3 + h)\, dx_3,$$

$$f_{ii}^{(\ell)}(\mathbf{x}) = \beta \int_{-h}^{h} b_{ii}(\mathbf{r}) \frac{x_3^{\ell}}{2h}(x_3 - h)\, dx_3,$$

$$c_{ji}^{(\ell,k)} = c_{ji}^{(k,\ell)} = c_{ij}^{(\ell,k)} = c_{ij}^{(k,\ell)}, \qquad g_{ii}^{(\ell,k)} = g_{ii}^{(k,\ell)},$$

$$a_{ji}^{(\ell)} = a_{ij}^{(\ell)}, \quad b_{ji}^{(\ell)} = b_{ij}^{(\ell)}, \quad d_{ji}^{(\ell)} = d_{ij}^{(\ell)}, \quad e_{ji}^{(\ell)} = e_{ij}^{(\ell)}, \quad b_{ji}^{(\ell)} = a_{ji}^{(\ell+1)},$$

$$0 \le \ell \le \frac{2}{3}K + 1, \qquad 0 \le k \le K.$$

Here and in what follows, we assume that the number K is a multiple of three. According to (12), $M_{ij}^{(0)} \equiv F_{ij}$ and $M_{ij}^{(1)} \equiv M_{ij}$ are, respectively, the membrane forces and bending and torsional moments acting in the plate; $M_{i3}^{(0)} \equiv F_{i3}$, $i, j = 1, 2$, are the transverse forces, and the other force factors are mathematical moments of higher orders. The quantities $P_{jj}^{(\ell)}$ in (11) can be regarded as initial (more precisely, temperature) force factors.

By using the differential balance equations for an element of continuum [2, 7, 8, 12]

$$\partial_1 \sigma_{i1} + \partial_2 \sigma_{i2} + \partial_3 \sigma_{i3} = -X_i(\mathbf{r}), \qquad i = 1, 2, 3, \quad \mathbf{x} \in G, \quad |x_3| \le h, \tag{13}$$

where X_i are components of the bulk load acting upon the material of the layered composition, we can write all necessary balance equations for the plate satisfied by the force factors (11). For this purpose, we multiply (13) by x_3^{ℓ} and integrate the result over the thickness of the plate. Thus, in view of (12), we find

$$\partial_1 M_{i1}^{(\ell)} + \partial_2 M_{i2}^{(\ell)} - \ell M_{i3}^{(\ell-1)} = -X_i^{(\ell)} - h^{\ell}\left[\sigma_{i3}^{(+)} - (-1)^{\ell}\sigma_{i3}^{(-)}\right],$$

$$0 \le \ell \le \frac{2}{3}K + 1, \quad i = 1, 2,$$

$$\partial_1 M_{13}^{(\ell)} + \partial_2 M_{23}^{(\ell)} = -X_3^{(\ell)} + \ell \int_{-h}^{h} x_3^{\ell-1}\sigma_{33}\, dx_3 - h^{\ell}\left[\sigma_{33}^{(+)} - (-1)^{\ell}\sigma_{33}^{(-)}\right], \tag{14}$$

$$0 \le \ell \le \frac{2}{3}K,$$

where

$$X_i^{(\ell)}(\mathbf{x}) = \int_{-h}^{h} X_i(\mathbf{r}) x_3^{\ell}\, dx_3, \qquad \sigma_{i3}^{(\pm)}(\mathbf{x}) = \sigma_{i3}(\mathbf{x}, \pm h), \quad i = 1, 2, 3. \tag{15}$$

In deducing relations (14), we have used the formulas of integration by parts.

Equations (14) do not contain the moments of the highest orders $M_{13}^{(\ell)}$ and $M_{23}^{(\ell)}$ with $\ell = \frac{2}{3}K + 1$, and, therefore, it is not necessary to compute them by using relations (11). The stresses $\sigma_{i3}^{(\pm)}$ [see (15)] are known from the static boundary conditions imposed on the front surfaces of the plate $x_3 = \pm h$. Further, the stress $\sigma_{33}(\mathbf{r})$ in the last relation in (14) is regarded as known from the solution obtained in the previous iteration of the method of successive approximations, just as $P_{jj}^{(\ell)}$ and all coefficients on the right-hand sides of relations (11). Therefore, the integral in this relation can be transferred to the right.

In view of the fact that the thickness of the plate is much smaller than its characteristic size in plan, we can approximate the stress σ_{33} by Hermitian polynomials in the variable x_3. As the initial approximation, we can take a cubic polynomial constructed according to the known values of $\sigma_{33}^{(\pm)}(\mathbf{x})$ and

$$\partial_3 \sigma_{33}\big|_{x_3 = \pm h} = -X_3(\mathbf{x}, \pm h) - \partial_1 \sigma_{13}^{(\pm)}(\mathbf{x}) - \partial_2 \sigma_{23}^{(\pm)}(\mathbf{x});$$

see equality (13) for $i = 3$. In particular, for

$$X_3 \equiv 0 \quad \text{and} \quad \sigma_{i3}^{(\pm)} \equiv 0, \quad i = 1, 2,$$

we obtain

$$\sigma_{33}(\mathbf{r}) = -\frac{\sigma_{33}^{(+)}(\mathbf{x}) - \sigma_{33}^{(-)}(\mathbf{x})}{4h^3} x_3(x_3^2 - 3h^2) + \frac{\sigma_{33}^{(+)}(\mathbf{x}) + \sigma_{33}^{(-)}(\mathbf{x})}{2}, \quad \mathbf{x} \in G, \quad |x_3| \le h. \tag{16}$$

The subsequent refinement of the stress $\sigma_{33}(\mathbf{r})$ is possible on the basis of Hermitian polynomials of higher orders, by using not only the values of

$$\sigma_{33}^{(\pm)}(\mathbf{x}) \quad \text{and} \quad \partial_3 \sigma_{33}\big|_{x_3 = \pm h},$$

as in (16), but also the additional values of $\partial_3 \sigma_{33}$ known from the solution obtained in the previous iteration at the inner points of the plate $x_3 \in (-h, h)$. These additional values are also obtained from relation (13) ($i = 3$) if we replace the stresses σ_{13} and σ_{23} by their previous approximations computed according to formulas (9) with regard for (10). The greater the number of the applied intermediate values of $\partial_3 \sigma_{33}$ across the thickness of the plate, the more accurate the approximation of the stress $\sigma_{33}(\mathbf{r})$ by Hermitian polynomials of high orders.

Substituting the obtained approximation of the function $\sigma_{33}(\mathbf{r})$ in equality (14) and, in particular, substituting relation (16) in (14), for the current iteration, we obtain, in view of (15), the required system of differential equations that should be satisfied by the force factors (11). Note that the right-hand sides of these equations are known. This system should be also supplemented with the following four static boundary conditions on the front surfaces of the plate [see (9) with regard for (1)]:

$$b_{j+3,j+3}\left(\frac{h^2-\beta x_3^2}{h^2}\sum_{k=0}^{K} x_3^k \varepsilon_{j3}^{(k)} + \beta\frac{x_3+h}{2h}\varepsilon_{j3}^{(+)} - \beta\frac{x_3-h}{2h}\varepsilon_{j3}^{(-)}\right)\Bigg|_{x_3=\pm h}$$

$$= \sigma_{j3}^{(\pm)}(\mathbf{x}), \qquad \mathbf{x}\in G, \text{v} \quad j=1,2, \tag{17}$$

where right-hand sides and coefficients b_{ii}, $i=j+3$, are known for the current iteration.

In the case of application of the Reddy theory or its refinements $(\beta=1,\ K\geq 0)$, we can unambiguously determine the approximations to the functions $\varepsilon_{j3}^{(\pm)}(\mathbf{x})$, $j=1,2$, in the current iteration from Eqs. (17). However, if we use the Reissner $(\beta=0,\ K=0)$ or classical $(\beta=0,\ \varepsilon_{j3}^{(k)}\equiv 0)$ theories, then it is impossible to satisfy the static boundary conditions (17) in the general case.

Recall that the temperature stresses p_{ii} in (9) with (10) were obtained under the assumption that $\sigma_{33}\approx 0$. It is possible to refine the quantities p_{ii} in relations (9) by using the following expressions instead of (10):

$$p_{ii} \equiv p_i - \frac{a_{i3}(p_3-\sigma_{33})}{a_{33}}, \quad i=1,2,$$

where σ_{33} is already known as a result of approximation by Hermitian polynomials [e.g., from (16)]. In the first iteration, as earlier, it is necessary to use the condition $\sigma_{33}\approx 0$.

Substituting (11) in equalities (14), we arrive at the balance equations written in the resolving form:

$$\sum_{i=1}^{2}\partial_j\left(a_{ji}^{(\ell)}\partial_i u_i^0 - b_{ji}^{(\ell)}\partial_i^2 u_3^0 + 2\sum_{k=0}^{K} c_{ji}^{(\ell,k)}\partial_i\varepsilon_{i3}^{(k)} + 2d_{ji}^{(\ell)}\partial_i\varepsilon_{i3}^{(+)} - 2e_{ji}^{(\ell)}\partial_i\varepsilon_{i3}^{(-)}\right)$$

$$+ \partial_{3-j}\Bigg[\frac{a_{33}^{(\ell)}}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) - b_{33}^{(\ell)}\partial_1\partial_2 u_3^0$$

$$+ \sum_{k=0}^{K} c_{33}^{(\ell,k)}\left(\partial_1\varepsilon_{23}^{(k)} + \partial_2\varepsilon_{13}^{(k)}\right)$$

$$+ d_{33}^{(\ell)}(\partial_1\varepsilon_{23}^{(+)} + \partial_2\varepsilon_{13}^{(+)}) - e_{33}^{(\ell)}(\partial_1\varepsilon_{23}^{(-)} + \partial_2\varepsilon_{13}^{(-)})\Bigg]$$

$$- \ell\left(\sum_{k=0}^{K} g_{j+3,j+3}^{(\ell-1,k)}\varepsilon_{j3}^{(k)} + h_{j+3,j+3}^{(\ell-1)}\varepsilon_{j3}^{(+)} - f_{j+3,j+3}^{(\ell-1)}\varepsilon_{j3}^{(-)}\right)$$

$$= -X_j^{(\ell)} - h^\ell \left[\sigma_{j3}^{(+)} - (-1)^\ell \sigma_{j3}^{(-)}\right] - \partial_j P_{jj}^{(\ell)},$$

$$j = 1, 2, \quad 0 \le \ell \le \frac{2}{3} K + 1,$$

$$\sum_{j=1}^{2} \partial_j \left(\sum_{k=0}^{K} g_{j+3,j+3}^{(\ell,k)} \varepsilon_{j3}^{(k)} + h_{j+3,j+3}^{(\ell)} \varepsilon_{j3}^{(+)} - f_{j+3,j+3}^{(\ell)} \varepsilon_{j3}^{(-)} \right) \tag{18}$$

$$= -X_3^{(\ell)} + \ell \int_{-h}^{h} x_3^{\ell-1} \sigma_{33}\, dx_3 - h^\ell \left[\sigma_{33}^{(+)} - (-1)^\ell \sigma_{33}^{(-)}\right],$$

$$0 \le \ell \le \frac{2}{3} K, \quad \mathbf{x} \in G.$$

Here, it is necessary to take into account relations (15). If K is a multiple of three, then we arrive at a closed system of $2K+9$ equations (17) and (18) containing $2K+9$ unknown kinematic variables

$$u_i^0, \ u_3^0, \ \varepsilon_{i3}^{(\pm)}, \ \varepsilon_{i3}^{(k)}, \ i = 1, 2, \ 0 \le k \le K,$$

that depend only on two coordinates x_1 and x_2.

For the unambiguous integration of this system of equations, it is necessary to formulate the corresponding boundary conditions. On one part of the end surface of plate (denoted by G_σ), we impose the following static boundary conditions:

$$\begin{aligned} \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + 2\sigma_{12} n_1 n_2 &= \sigma_{nn}, \\ (\sigma_{22} - \sigma_{11}) n_1 n_2 + \sigma_{12}\left(n_1^2 - n_2^2\right) &= \sigma_{n\tau}, \\ \sigma_{13} n_1 + \sigma_{23} n_2 &= \sigma_{n3}, \\ n_1 = \cos\gamma, \quad n_2 = \sin\gamma, &\quad \mathbf{r} \in G_\sigma. \end{aligned} \tag{19}$$

On the other part of the end surface (denoted by G_u), we impose the following kinematic boundary conditions [see (2) and (3)]:

$$u_3^0(\mathbf{x}) = u_{30}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_u, \tag{20}$$

$$u_i(\mathbf{r}) = u_{i0}(\mathbf{r}), \quad \mathbf{r} \in G_u, \quad i = 1, 2, \tag{21}$$

where σ_{nn}, $\sigma_{n\tau}$, and σ_{n3} are, respectively, the normal, tangential (in the plane $x_3 = \text{const}$), and transverse (in the direction x_3) external surface loads specified on G_σ; Γ_u is the projection of the part of end surface G_u onto the reference plane $x_3 = 0$ (i.e., Γ_u is a part of the contour Γ bounding the domain G occupied by the plate in plan); $u_{30}(\mathbf{x})$ is the deflection given on Γ_u; u_{i0} are the displacements in the tangential directions x_i, $i = 1, 2$, specified on the end surface G_u, and γ is the angle specifying the direction of the outer normal to the contour Γ (or, equivalently, to the end surface $G_\sigma \cup G_u$) and measured from the direction x_1.

To get static boundary conditions represented in terms of the force factors [see (11)], we multiply (19) by x_3^ℓ with subsequent integration of the result over the thickness of the plate. Thus, by using (12), we find

$$\begin{aligned} M_{11}^{(\ell)} n_1^2 + M_{22}^{(\ell)} n_2^2 + 2M_{12}^{(\ell)} n_1 n_2 &= M_{nn}^{(\ell)}, \\ \left(M_{22}^{(\ell)} - M_{11}^{(\ell)}\right) n_1 n_2 + M_{12}^{(\ell)}\left(n_1^2 - n_2^2\right) &= M_{n\tau}^{(\ell)}, \qquad 0 \le \ell \le \frac{2}{3}K + 1, \\ M_{13}^{(\ell)} n_1 + M_{23}^{(\ell)} n_2 &= M_{n3}^{(\ell)}, \qquad 0 \le \ell \le \frac{2}{3}K, \quad \mathbf{x} \in \Gamma_\sigma, \end{aligned} \tag{22}$$

where

$$\begin{aligned} M_{nn}^{(\ell)} &\equiv \int_{-h}^{h} \sigma_{nn} x_3^\ell \, dx_3, \\ M_{n\tau}^{(\ell)} &\equiv \int_{-h}^{h} \sigma_{n\tau} x_3^\ell \, dx_3, \\ M_{n3}^{(\ell)} &\equiv \int_{-h}^{h} \sigma_{n3} x_3^\ell \, dx_3, \end{aligned} \tag{23}$$

Γ_σ is the projection of the part of end surface G_σ onto the reference plane $x_3 = 0$,

$$\Gamma = \Gamma_\sigma \cup \Gamma_u;$$

$M_{nn}^{(\ell)}$, $M_{n\tau}^{(\ell)}$, and $M_{n3}^{(\ell)}$ are force factors given on the contour Γ_σ, and, according to (23), $M_{nn}^{(0)}$ and $M_{n\tau}^{(0)}$ are given normal and tangential (to Γ_σ) membrane forces, $M_{n3}^{(0)}$ is a given transverse force, and $M_{nn}^{(1)}$ and $M_{n\tau}^{(1)}$ are, respectively, bending and torsional moments given on Γ_σ [the other quantities on the right-hand sides of equalities (22) are given mathematical moments of higher orders].

Since we use only partial sums of the power series in (3) $(K < \infty)$, it is impossible to satisfy the kinematic boundary conditions (21) at any point $\mathbf{r}$ of the end surface of the plate G_u for any dependence of the functions u_{i0} on the transverse coordinate x_3. Therefore, by analogy with the static boundary conditions [see (19), (22), and (23)], we satisfy the kinematic conditions (21) in the integral sense on the corresponding end face of the plate, i.e., integrate (21) over the thickness of the plate with weights x_3^ℓ. Hence, by virtue of (3), we obtain

$$\frac{h^{\ell+1}}{\ell+1}\left(1-(-1)^{\ell+1}\right)u_i^0-\frac{h^{\ell+2}}{\ell+2}\left(1-(-1)^{\ell+2}\right)\partial_i u_3^0$$
$$+\ 2\sum_{k=0}^{K}h^{k+\ell+2}\left(1-(-1)^{\ell+k+2}\right)$$
$$\times\left(\frac{1}{(k+1)(\ell+k+2)}-\frac{\beta}{(k+3)(\ell+k+4)}\right)\varepsilon_{i3}^{(k)}$$
$$+\ \beta h^{\ell+2}\left[\frac{1}{2(\ell+3)}\left(1-(-1)^{\ell+3}\right)+\frac{1}{\ell+2}\left(1-(-1)^{\ell+2}\right)\right]\varepsilon_{i3}^{(+)}$$
$$-\ \beta h^{\ell+2}\left[\frac{1}{2(\ell+3)}\left(1-(-1)^{\ell+3}\right)-\frac{1}{\ell+2}\left(1-(-1)^{\ell+2}\right)\right]\varepsilon_{i3}^{(-)}$$
$$=\ u_{i0}^{(\ell)}(\mathbf{x}),\qquad \mathbf{x}\in\Gamma_u,\quad i=1,2,\quad 0\le\ell\le K+1, \tag{24}$$

where

$$u_{i0}^{(\ell)}(\mathbf{x})\ \equiv\ \int_{-h}^{h}u_{i0}(\mathbf{r})x_3^\ell\,dx_3,$$
$$\mathbf{x}\in\Gamma_u,\quad \mathbf{r}\in G_u,\quad i=1,2,\quad 0\le\ell\le K+1. \tag{25}$$

Substituting relations (11) in Eqs. (22), we obtain static boundary conditions at the end face of the plate in the resolving form:

$$\sum_{j=1}^{2}n_j^2\ \sum_{i=1}^{2}\Bigg(a_{ji}^{(\ell)}\,\partial_i u_i^0-b_{ji}^{(\ell)}\,\partial_i^2 u_3^0$$
$$+\ 2\ \sum_{k=0}^{K}c_{ji}^{(\ell,k)}\,\partial_i\varepsilon_{i3}^{(k)}+2d_{ji}^{(\ell)}\,\partial_i\varepsilon_{i3}^{(+)}-2e_{ji}^{(\ell)}\,\partial_i\varepsilon_{i3}^{(-)}\Bigg)$$
$$+\ n_1n_2\Big[a_{33}^{(\ell)}\left(\partial_1u_2^0+\partial_2u_1^0\right)-2b_{33}^{(\ell)}\,\partial_1\partial_2u_3^0$$
$$+\ 2\ \sum_{k=0}^{K}c_{33}^{(\ell,k)}\left(\partial_1\varepsilon_{23}^{(k)}+\partial_2\varepsilon_{13}^{(k)}\right)$$

$$+\ 2d_{33}^{(\ell)}\big(\partial_1\varepsilon_{23}^{(+)}+\partial_2\varepsilon_{13}^{(+)}\big)-2e_{33}^{(\ell)}\big(\partial_1\varepsilon_{23}^{(-)}+\partial_2\varepsilon_{13}^{(-)}\big)\Big]$$

$$=\ M_{nn}^{(\ell)}-\sum_{j=1}^{2}n_j^2P_{jj}^{(\ell)},$$

$$n_1n_2\sum_{j=1}^{2}(-1)^j\sum_{i=1}^{2}\Bigg(a_{ji}^{(\ell)}\partial_iu_i^0-b_{ji}^{(\ell)}\partial_i^2u_3^0$$

$$+\ 2\sum_{k=0}^{K}c_{ji}^{(\ell,k)}\partial_i\varepsilon_{i3}^{(k)}+2d_{ji}^{(\ell)}\partial_i\varepsilon_{i3}^{(+)}-2e_{ji}^{(\ell)}\partial_i\varepsilon_{i3}^{(-)}\Bigg)$$

$$+\ \frac{1}{2}\big(n_1^2-n_2^2\big)\Big[a_{33}^{(\ell)}\big(\partial_1u_2^0+\partial_2u_1^0\big)-2b_{33}^{(\ell)}\partial_1\partial_2u_3^0$$

$$+\ 2\sum_{k=0}^{K}c_{33}^{(\ell,k)}\big(\partial_1\varepsilon_{23}^{(k)}+\partial_2\varepsilon_{13}^{(k)}\big)+2d_{33}^{(\ell)}$$

$$\times\ \big(\partial_1\varepsilon_{23}^{(+)}+\partial_2\varepsilon_{13}^{(+)}\big)-2e_{33}^{(\ell)}\big(\partial_1\varepsilon_{23}^{(-)}+\partial_2\varepsilon_{13}^{(-)}\big)\Big]$$

$$=\ M_{n\tau}^{(\ell)}\ -\ n_1n_2\sum_{j=1}^{2}(-1)^jP_{jj}^{(\ell)},\qquad 0\le\ell\le\frac{2}{3}K+1, \tag{26}$$

$$\sum_{j=1}^{2}n_j\left(\sum_{k=0}^{K}g_{j+3,j+3}^{(\ell,k)}\varepsilon_{j3}^{(k)}+h_{j+3,j+3}^{(\ell)}\varepsilon_{j3}^{(+)}-f_{j+3,j+3}^{(\ell)}\varepsilon_{j3}^{(-)}\right)\ =\ M_{n3}^{(\ell)},\qquad 0\le\ell\le\frac{2}{3}K,\quad \mathbf{x}\in\Gamma_\sigma,$$

where it is necessary to take into account equalities (23).

Thus, for the unambiguous integration of the system of resolving equations (17), (18) at any point of the contour Γ, it is necessary to impose either the static boundary conditions (26) or the kinematic boundary conditions (20) and (24) with (25). It is also possible to use mixed boundary conditions [a combination of (20), (24), and (26)], e.g., in the case of free support of the end face.

Within the framework of the Reddy theory and its refinements (at $\beta=1$), in the current iteration of the method of successive approximations, it is possible to assume that the functions $\varepsilon_{j3}^{(\pm)}$, $j=1,2$, in Eqs. (18) and equalities (26) are known from the preliminary solution of the system of equations (17). At the same time, in the Reissner and classical theories ($\beta=0$), there are no functions of this kind in relations (18) and (26) [see (12) for $\beta=0$].

On the basis of the well-known formulas of transition from a Cartesian coordinate system to a cylindrical system [6], by using relations presented above, we can deduce a system of resolving equations and the corresponding boundary conditions in a polar coordinate system, which is convenient when the domain G occupied by the plate in plan is either a circle, or a ring, or a sector of circle (or ring).

CONCLUSIONS

The expansions of transverse shear strains formed in layered plates with regular structures in the polynomials of different orders in the transverse coordinate make it possible to construct refined theories of the thermoelastoplastic bending of these structures, which take into account their weakened resistance to transverse shears with different degrees of accuracy and make it possible to satisfy the force boundary conditions for tangential stresses on the front surfaces. In the first approximation, the obtained relations yield the equations of traditional nonclassical Reddy theory whose complexity of realization is the same as for the Reissner theory. However, within the framework of this theory, it is impossible to satisfy boundary conditions for the tangential stresses on the front surfaces of the plate. The application of kinematic relations more accurate than the equations of the Reddy theory leads to a significant complication of the boundary-value problems as compared with the equations of the Reissner and Reddy theories.

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