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# $\pm 1$-MATRICES WITH VANISHING PERMANENT

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*The problem of finding $(-1, 1)$-matrices with vanishing permanent was posed by Edward Wang in 1974. This paper states and proves bounds on the number of negative entries in a matrix with zero permanent and minimal number of negative entries among all matrices of the same equivalence class. Then representatives of every equivalence class of matrices with zero permanent are found for $n \le 5$. Bibliography: 20 titles.*

## 1. Introduction

**Definition 1.1.** Let $A$ be a matrix of order $n$ over the field of reals. The function that maps the matrix into the number $\mathrm{per}\,(A) = \sum\limits_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where $S_n$ is the symmetric group over the set $\{1, 2, \dots, n\}$, is called the permanent function, and the value of the function is called the permanent of the matrix.

The permanent function has applications in some areas of mathematics, such as graph theory, combinatorics, discrete mathematics, as well as in other sciences, such as genetics, economics, and physics, see [19] and [9]. However, in contrast with the determinant function, for the permanent function no numerical algorithm with polynomial complexity is known. Moreover, the problem of computing the permanent has proved to be $\sharp P$-complete, see [14] and [16]. This has given rise to many questions concerning the values of the function; the most widely known list of problems and conjectures in this area is due to H. Minc [14]. Here, the $(0, 1)$-matrices and $(-1, 1)$-matrices are of particular interest. Information on applications and problems concerning $(0, 1)$-matrices can be found in [2], also see [8] and the bibliography therein. The set of $(-1, 1)$-matrices is also widely used (for example, in economics, see [3]) and studied (for example, in connection with the problem of conversion, see [13], and in the theory of Hadamard matrices, see [7]). Also some estimates for the number of matrices with prescribed order and value of the permanent were obtained in [10].

In this paper, we investigate the vanishing problem for the permanents of $(-1, 1)$-matrices. The problem of the existence of a $(-1, 1)$-matrix of order $n$ with zero permanent was proposed and partly solved in [17]. For the complete solution, see [18]. The present paper aims at computing the number of such matrices and describing their structure, which still remains an open question. Yet another related problem, stated in [17], namely, the problem of bounding from above the value of the permanent of a matrix in terms of its rank has recently been solved in [4] by the proof of Kräuter's conjecture [11]. For details concerning other problems from [17], see [12] and [15]. Information on the current state of research in the area can be found in [1, 7, 18] and [20].

Here, we prove bounds on the number of $-1$'s in the minimal with respect to the number of $-1$'s representatives of the *equivalence* classes of matrices with zero permanent. Also we classify, up to *equivalence*, all the matrices of order less than or equal to 5 with zero permanent. We use the terminology and notation introduced in the preceding papers [5] and [6] on the same topic, as well as those from [12]. Since matrix transposition and line permutation do not change the permanent and the matrix structure, we do not distinguish between matrices that can be transformed to the same matrix by a sequence of such operations. We say that matrices

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are *equivalent* if they can be transformed to the same form by some line permutations, line negations (i.e., multiplication of some lines by $-1$) and, if necessary, matrix transposition. Fix the order $n \in \mathbb{N}$ and consider the set $\Omega_n$ of $(-1,1)$-matrices of order $n$. A *negative partial generalized diagonal* of length $m$ is any set of the form $\{a_{i_1\sigma(i_1)}, \dots, a_{i_m\sigma(i_m)}\}$ consisting of $m$ negative entries of a matrix $A \in \Omega_n$, where $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, with each $i_j$ occurring only once, $m < n$, and $\sigma \in S_n$. If $m = n$, then such a set is called *negative generalized diagonal*. By $k_m$, $0 \le m \le n$, we denote the number of negative (partial) generalized diagonals of length $m$. The value of $k_0$ is always set to be 1. Note that $k_1$ is exactly the number of negative entries in the matrix.

The paper is organized as follows. In Sec. 2, we prove bounds for the number of $-1$'s in the minimal with respect to the number of $-1$'s representatives of the equivalence classes of matrices with zero permanent. In Sec. 3, we find representatives of all such classes for $n \le 4$. In Sec. 4, we find representatives of all such classes for $n = 5$.

We conclude this section by recalling some known results.

**Lemma 1.2** (**[5, Lemma 2.3]**). *Let $A \in \Omega_n$. Then*

$$\operatorname{per}(A) = \sum_{j=0}^{n} (-2)^j \cdot k_j \cdot (n-j)!.$$

**Proposition 1.3** (**[12, Lemma 5]**). *Let $A \in \Omega_n$ and let $n = 2^t - 1$ for a positive integer $t$. Then*

$$\operatorname{per}(A) \not\vdots\; 2^{n - \lfloor \log_2 n \rfloor}.$$

**Corollary 1.4** (**[12, Lemma 4]**). *For any $n = 2^t - 1$, where $t \in \mathbb{N}$, no matrix $A \in \Omega_n$ has zero permanent.*

The following proposition is the converse of Corollary 1.4.

**Proposition 1.5** (**[18, Theorem 1]**). *For every $n$ except for $n = 2^t - 1$, $t \in \mathbb{N}$, there exists a matrix $A \in \Omega_n$ with zero permanent.*

**Proposition 1.6** (**[6, Lemma 3.1]**). *Fix $n, t \in \mathbb{N}$, $2^{t-1} \le n < 2^t - 1$, and let $\mathfrak{M} = \mathfrak{M}(n)$ be the set of nonnegative integers $m \le n$ such that in the binary representation of $n - m$ there are exactly $t - 1$ units. Then, for any matrix $A \in \Omega_n$,*

$$\operatorname{per}(A) \;\vdots\; 2^{n-t+2} \quad \textit{if and only if} \quad \sum_{m \in \mathfrak{M}} k_m \;\vdots\; 2.$$

## 2. The general case

In this section, we state and prove several results, which allow us to reduce the number of matrices to be considered in searching for matrices with the smallest number of $-1$'s among those with zero permanent.

**Lemma 2.1.** *An arbitrary matrix from $\Omega_n$ can be transformed into a matrix with no more than $\lfloor \frac{n}{2} \rfloor$ negative entries in every line by multiplying some of its rows and columns by $-1$.*

*Proof.* Indeed, by multiplying a row or a column with strictly more than $\lfloor \frac{n}{2} \rfloor$ negative entries by $-1$, we decrease the number of such entries in the matrix. Thus, since the number of entries is finite, we will ultimately obtain a matrix of the desired form. □

**Lemma 2.2.** *For $n = 2^t$, $t \in \mathbb{N}$, and $A \in \Omega_n$, $\operatorname{per}(A)$ can vanish only if the number of negative entries of $A$ is even.*

*Proof.* By applying Proposition 1.6, we obtain that for $n = 2^t$ the permanent of a $(-1, 1)$-matrix is divisible by $2^{n-t+1}$ if and only if $k_1$ is divisible by 2. Thus, in the case of an odd $k_1$, the permanent cannot vanish. □

**Lemma 2.3.** *For $n = 2^t + 1$, $3 \le t \in \mathbb{N}$, the permanent of a matrix $A \in \Omega_n$ can vanish only if the number $k_2$ of negative partial generalized diagonals of length 2 is even.*

*Proof.* This assertion follows from Proposition 1.6 in the same way as Lemma 2.2. □

**Lemma 2.4.** *If, in a $(-1, 1)$-matrix of order $2k$, there are at least $k$ rows $R_{i_1}, \dots, R_{i_k}$ with at least $k$ negative entries $a_{i_q j_1}, \dots, a_{i_q j_k}$ and at least one negative entry $(i, j)$ distinct from $(i_q, j_m)$ for all $q, m = 1, \dots, k$, then the number of negative entries in the matrix can be decreased by line negation.*

*Proof.* If a matrix row contains $k + 1$ or more negative entries, then we negate this row, and the number of negative entries decreases. Now assume that there are no such rows. Consider the rows $R_{i_1}, \dots, R_{i_k}$ and column $j$. By $l$ denote the number of rows among the chosen $k$ rows with which column $j$ intersects by a negative entry. Negate column $j$.

1. If $k = l$, then originally column $j$ has contained at least $k + 1$ negative entries, whence the number of such entries in the entire matrix is decreased by its negation.

2. If $l < k$, then, upon negation of column $j$, no more than $2k - 2l - 2$ negative entries appear. The total number can even decrease if sufficiently many among the other $k$ rows intersect with column $j$ by negative entries. Then consider the $k - l$ rows among $R_{i_1}, \dots, R_{i_k}$ that originally have intersected with column $j$ by unity. As a result of the negation of column $j$, each of them contains $k + 1$ negative entries. Negate these $k - l$ rows. Then the total number of negative entries will decrease by $2(k - l) = 2k - 2l > 2k - 2l - 1$, i.e., after the sequence of negations considered the total number of negative entries will decrease at least by 2. □

**Corollary 2.5.** *If $n > 2$ is even and a matrix $A \in \Omega_n$ with zero permanent has more than $\frac{n^2}{2} - \frac{n}{2} - 1$ negative entries, then it can be transformed into a matrix with fewer negative entries by line negation.*

*Proof.* If $A$ contains a row with more than $\frac{n}{2}$ negative entries, then, by negating the row, we obtain the desired matrix. If there are no such rows, then there are at least $\frac{n}{2}$ rows with $\frac{n}{2}$ negative entries, and one extra negative entry in some other row; otherwise the total number of negative entries does not exceed the value indicated in the assumption of the corollary. Therefore, the assumptions of Lemma 2.4 are satisfied, and the result of Corollary 2.5 follows from Lemma 2.4. □

## 3. Matrices of order $n \le 4$

It is clear that for $n = 1$ there are no $(-1, 1)$-matrices with permanent 0, and for $n = 2$ half of all $(-1, 1)$-matrices have zero permanent; those are the matrices with odd number of $-1$'s. For $n = 3$, by Proposition 1.4, no such matrices exist.

**Lemma 3.1.** *Any matrix $A \in \Omega_4$ with zero permanent can be transformed by line negation into a matrix with two or four $-1$'s.*

*Proof.* By virtue of Lemma 2.1, we may only consider matrices with no more than eight $-1$'s in total and no more than two of them in a line. By Proposition 1.6, the permanent of a $(-1, 1)$-matrix of order 4 is divisible by 8 if and only if the number $k_1$ of negative entries is even. Consequently, if the number of negative entries is odd, then the permanent cannot vanish. Thus, it remains to consider the cases $k_1 = 6$ and $k_1 = 8$.

1. If the matrix under consideration has eight negative entries, then exactly two of them occur in every line. Thus, by multiplying an arbitrary row by $-1$, we obtain a matrix with eight negative entries that has two columns each of which contains three negative entries. The negation of these two columns yields a matrix with four negative entries.

2. Let the matrix contain six $-1$'s. If there exist a row and a column with two $-1$'s in each of them that intersect by 1, then we can negate these row and column, which will decrease the number of $-1$'s by two. If there are no such rows and columns, then we have a matrix of the form

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

In this case, we can obtain a matrix with four $-1$'s by negating a row with one $-1$ and then negating two columns with three $-1$'s. Note, in addition, that the matrix obtained has a row with three $-1$'s, whence actually one can obtain a matrix with two $-1$'s. This completes the proof of the lemma. □

**Proposition 3.2.** *For $n = 4$, any given $(-1, 1)$-matrix with vanishing permanent is equivalent to one of the following matrices:*

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

*Proof.* By Lemma 3.1, it is sufficient to consider matrices with two or four $-1$'s. By Lemma 2.1, we may assume that no three of the $-1$'s occur in the same line. Thus, up to line permutation and matrix transposition, there are the following eight cases to consider:

$$A_1 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad B_6 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Observe that $A_2$ is equivalent to $B_4$, $B_1$ is equivalent to $B_3$, and $B_2$ is equivalent to $B_5$. Now, applying Lemma 1.2 in the case $n = 4$, we see that for $k_1 = 4$ the permanent is divisible by 16 if and only if the difference $k_2 - k_3$ is odd, i.e., the permanent cannot vanish if the latter difference is even, which is the case for $B_1$. Then, computing the permanents of $A_1$, $A_2$, $B_5$, and $B_6$, we conclude that only $\operatorname{per}(A_2)$ is nonzero. This completes the proof. □

## 4. Matrices of order 5

**Lemma 4.1.** *Any matrix from $\Omega_5$ with vanishing permanent is equivalent to a matrix with no more than seven $-1$'s.*

*Proof.* By Lemma 2.1, it is unnecessary to consider matrices with more than ten $-1$'s in total, as well as matrices with more than two $-1$'s in a line. If the matrix under consideration contains ten or nine $-1$'s, then there exist two columns with two $-1$'s in each of them and a row with two $-1$'s that intersects the latter columns by a unity. The negation of these row and columns yields us a matrix with the desired number of negative entries. In the case of eight $-1$'s, there exist matrices without such intersections. Namely, those are the matrices with two $-1$'s that are unique negative entries in some rows and columns. Up to line permutation and matrix transposition, such matrices are of the following form:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

By computing the permanent, we obtain that it is equal to $-8$, i.e., is nonzero. This completes the proof. □

**Proposition 4.2.** *Any* $(-1,1)$*-matrix of order* 5 *with zero permanent is equivalent to one of the following* 11 *matrices:*

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

*Proof.* By Lemma 4.1, it is sufficient to consider the matrices containing no more than seven $-1$'s with no more than two of them in every line. As is straightforwardly verified, the permanents of matrices with no more than two $-1$'s do not vanish. Thus, we are left with five options $k_1 = 3, 4, 5, 6, 7$.

0. Note that, by Proposition 1.6, the permanent of a $(-1,1)$-matrix of order 5 can vanish only if $k_0 + k_2$ is even, i.e., only if $k_2$ is odd.

1. $k_1 = 7$. If there exists a $4 \times 4$ submatrix containing all the $-1$'s, then there are three rows and three columns with two $-1$'s, one row and one column with one $-1$, and one row

and one column free of negative entries. Therefore, either there is a row with two $-1$'s that intersects a column with one $-1$ by this $-1$ or the matrix is as follows:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In the former case, we first negate that row, which results in that one extra $-1$ and two columns containing three $-1$'s each appear. Then we negate the latter two columns and obtain a matrix with six $-1$'s. In the latter case, the permanent vanishes.

Now assume that all the seven $-1$'s occur in exactly four rows and five columns. As above, if the row with one $-1$ intersects a column with two $-1$'s by $-1$, then we first negate the column and then the two rows that now have three $-1$'s. In this way, the number of $-1$'s is decreased. If there is no such intersection, then there is a $3 \times 4$ submatrix with six $-1$'s. If it contains a $2 \times 2$ negative block, then we negate a row that is not involved in the block and then negate the two columns from the block. This results in a matrix with six $-1$'s. If the $3 \times 4$ submatrix contains no $2 \times 2$ negative block, then the matrix is as follows:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Its permanent equals 8.

Finally, if the seven $-1$'s are distributed among all of the rows and columns, then there are the following three cases, which are not equivalent with respect to line permutation and matrix transposition and to which the above-described procedures of decreasing the number of negative entries are inapplicable:

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

The permanent of the first matrix equals $-16$, whereas the permanents of the second and third ones vanish.

2. $k_1 = 3$. From item 0 and Lemma 2.1 it follows that we must compute the permanents of the following two matrices:

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The first one equals 32, and the second one vanishes.

3. $k_1 = 4$. By Lemma 2.1 and item 0, it is sufficient to compute the permanents of the following two matrices:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Indeed, for all the other matrices with four $-1$'s, either three of them lie in a line or $k_2$ is even. The permanent of the first matrix is zero, and the permanent of the second one equals 16.

4. $k_1 = 5$. If the negative entries occur in all of the five rows and five columns, then they form a diagonal of length 5, whence $k_2 = 10$, and the permanent cannot vanish by item 0.

If the negative entries occur in four rows and five columns, then the matrix is of the form

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Its permanent equals 16.

If the negative entries occur in four rows and four columns, then there are two matrices

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which are not equivalent with respect to line permutation and matrix transposition. The permanents of both matrices equal 8.

If the negative entries occur in three rows and five columns, then the entire matrix has the form

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Its permanent equals 8.

If the negative entries occur in three rows and four columns, then there are the following two cases:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In the first case, the permanent vanishes; in the second case, it equals 16.

Finally, assume that all of the negative entries are located in a $3 \times 3$ submatrix. Again, we have two cases,

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In the first case, the permanent equals 8; in the second case, it equals $-8$.

5. $k_1 = 6$. If the six negative entries occur in a $3 \times 3$ submatrix, then the entire matrix is of the form

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Its permanent equals 16.

If the six negative entries occur in three rows and four columns, then the matrix has one of the following two forms:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In both cases, the permanent equals 8.

If the six negative entries occur in three rows and five columns, then the entire matrix is as follows:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Its permanent vanishes.

If the six negative entries are located in four rows and four columns, then the entire matrix has one of the following five forms:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The third matrix is equivalent, with respect to line negation, to a matrix with five $-1$'s. The permanents of the first, second, and fourth matrices vanish, and the permanent of the fifth one equals 16.

If the six negative entries occur in five rows and four columns, then the matrix has one of the following three forms:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{pmatrix}.$$

The first matrix is equivalent, with respect to line negation, to a matrix with five $-1$'s. The permanents of the second and third matrices are equal to 8.

Finally, if the six negative entries are distributed among all the five rows and five columns, then the matrix has one of the following two forms:

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In the first case, the permanent equals 16; in the second one, it vanishes.

This completes the proof of the proposition. □

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