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SOME BOUNDS FOR INVERSES INVOLVING MATRIX SPARSITY PATTERN

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The paper considers some subclasses of the class of nonsingular $\mathcal{H}$-matrices whose definitions involve matrix sparsity pattern. For matrices A in these subclasses, upper bounds for $\|A^{-1}\|_\infty$ are derived and shown to be sharper than the corresponding bounds ignoring matrix sparsity. Bibliography: 26 titles.

1. Introduction

In the last years, the problem of bounding the l_∞ norm of inverse matrices from above has been intensively studied, see, e.g., [4–7, 11–16, 18–22, 24]. To this end, different approaches have been elaborated and applied to matrices from different classes. In this paper, the technological approach recently suggested in [20, 21] is exploited. This approach is essentially based on the following result, see, e.g., [26, 20], whose proof is provided below for completeness.

Lemma 1.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be a nonsingular matrix. Then there is a vector $x \in \mathbb{C}^n$ such that $\|x\|_\infty = 1$ and*

$$\|A^{-1}\|_\infty^{-1} = \|Ax\|_\infty = \max_{1\leq i\leq n} |(Ax)_i|. \tag{1.1}$$

Proof. Since

$$\|A^{-1}\|_\infty = \sup_{y\neq 0} \frac{\|A^{-1}y\|_\infty}{\|y\|_\infty},$$

we have

$$\|A^{-1}\|_\infty^{-1} = \inf_{z\neq 0} \frac{\|Az\|_\infty}{\|z\|_\infty} = \min_{\|z\|_\infty=1} \|Az\|_\infty = \max_{1\leq i\leq n} |(Ax)_i|,$$

where $x \in \mathbb{C}^n$ and $\|x\|_\infty = 1$. □

In the present paper, specifically, we consider some known and new matrix classes that are subclasses of the class of nonsingular $\mathcal{H}$-matrices and contain the class {SDD} of Strictly Diagonally Dominant matrices. For matrices from the classes considered, new upper bounds for the infinity norm of their inverses are suggested. A common feature of the main results obtained below is that they are stated with account for the sparsity patterns of the matrices in question. It is worth mentioning that the classes obtained in this way contain the corresponding subclasses that result if the sparsity pattern is ignored. Moreover, in application to the latter subclasses the new bounds are in general sharper than the bounds ignoring sparsity considerations.

The paper is organized as follows. Section 2 considers the so-called S-SOB and S-OB matrices, introduced and shown to be nonsingular $\mathcal{H}$-matrices in [17]. Here and in what follows, by S a nonempty proper subset of the index set $\langle n\rangle = \{1, \dots, n\}$ is denoted. The main theorem of Sec. 2 states an upper bound for $\|A^{-1}\|_\infty$ for an S-SOB matrix A. As a corollary, an upper bound for an S-OB matrix A is obtained. Section 3 is devoted to the so-called S-SDDS (S-SDD Sparse) matrices, which are obtained by combining sparsity considerations with the known definition of S-SDD matrices. It is proved that every S-SDDS matrix A is a nonsingular

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$\mathcal{H}$-matrix, and an upper bound for $\|A^{-1}\|_\infty$ is established. As a corollary, the known bound for the norm of the inverse to an S-SDD matrix [22, 11] is obtained. Finally, in Sec. 4, we derive a generalization of the upper bound on $\|A^{-1}\|_\infty$ for an OB (Ostrowski–Brauer) matrix (also referred to as a DSDD (Double SDD) matrix) proposed in [24], also see [20].

It should be mentioned that all the bounds considered in the paper are applicable to SDD matrices and improve the following classical result.

Theorem 1.1 ([1, 25])**.** *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be an SDD matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in\langle n\rangle} \frac{1}{|a_{ii}| - r_i(A)}. \tag{1.2}$$

We conclude this introduction by specifying some notation used throughout the paper. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and a subset $S \subset \langle n\rangle$, $n \geq 2$, we denote

$$r_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n;$$

$$r_i^j(A) = r_i(A) - |a_{ij}|, \quad \text{where} \quad j \neq i, \quad i = 1, \ldots, n,$$

and also

$$r_i^S(A) = \begin{cases} \sum_{\substack{j\in S\\ j\neq i}} |a_{ij}|, & i \in S, \\ \sum_{j\in S} |a_{ij}|, & i \notin S, \end{cases} \qquad i = 1, \ldots, n;$$

$\bar{S} = \langle n\rangle\backslash S$ is the complement of S in $\langle n\rangle$, and $|S|$ is the cardinality of S; $\mathcal{M}(A) = (m_{ij})$, where $m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j \end{cases}$ is the comparison matrix for A.

2. S-SOB and S-OB matrices

Let S be an arbitrary nonempty proper subset of the index set. In accordance with [17], we say that a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, is an S-SOB (S-Sparse Ostrowski–Brauer) matrix if the following conditions are fulfilled:

(i) $|a_{pp}| > r_p^S(A)$ for all $p \in S$;

(ii) $|a_{qq}| > r_q^{\bar{S}}(A)$ for all $q \in \bar{S}$;

(iii) for all $p \in S$ and all $q \in \bar{S}$ such that $a_{pq} \neq 0$,

$$[|a_{pp}| - r_p^S(A)]\, |a_{qq}| > r_p^{\bar{S}}(A)\, r_q(A); \tag{2.1}$$

(iv) for all $p \in S$ and all $q \in \bar{S}$ such that $a_{qp} \neq 0$,

$$[|a_{qq}| - r_q^{\bar{S}}(A)]\, |a_{pp}| > r_q^S(A)\, r_p(A). \tag{2.2}$$

The above definition takes into account the sparsity pattern of the matrix A. If the matrix sparsity is ignored and we require that conditions (2.1) and (2.2) be fulfilled for all $p \in S$ and all $q \in \bar{S}$, then conditions (i) and (ii) become exuberant, and we obtain the following simplified definition, given in [17].

Let S be an arbitrary nonempty proper subset of the index set. In accordance with [17], we say that a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, is an S-OB (S-Ostrowski–Brauer) matrix if conditions (2.1) and (2.2) are fulfilled for all $p \in S$ and all $q \in \bar{S}$.

In [17], it is proved that all the S-SOB and S-OB matrices are nonsingular $\mathcal{H}$-matrices; moreover, the following inclusions hold:

$$\{\text{SDD}\} \subsetneqq \{S\text{-OB}\} \subsetneqq \{S\text{-SOB}\} \subsetneqq \mathcal{H}.$$

In Theorem 2.1 and Corollary 2.1 below, we establish upper bounds for the infinity norm of inverses to S-SOB and S-OB matrices.

Theorem 2.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be an S-SOB matrix, where $S \subset \langle n\rangle$, $1 \leq |S| \leq n-1$. Then*

$$\|A^{-1}\|_\infty \leq \max\left\{ \max_{\substack{i\in S:\\ r_i^{\bar S}(A)=0}} \frac{1}{|a_{ii}| - r_i^S(A)},\ \max_{\substack{j\in \bar S:\\ r_j^{S}(A)=0}} \frac{1}{|a_{jj}| - r_j^{\bar S}(A)},\right.$$

$$\left. \max_{\substack{i\in S, j\in \bar S:\\ a_{ij}\neq 0}} f_{ij}(A,S),\ \max_{\substack{i\in S, j\in \bar S:\\ a_{ji}\neq 0}} f_{ji}(A,\bar S) \right\}. \tag{2.3}$$

Here and in what follows, we use the notation

$$f_{ij}(A,S) = \frac{|a_{jj}| + r_i^{\bar S}(A)}{[|a_{ii}| - r_i^S(A)]|a_{jj}| - r_i^{\bar S}(A)\, r_j(A)}, \tag{2.4}$$

where $i \in S$, $j \in \bar S$.

Proof. In accordance with Lemma 1.1, choose a vector $x = (x_i) \in \mathbb{C}^n$ such that

$$\|x\|_\infty = 1 \quad \text{and} \quad |(Ax)_i| \leq \|A^{-1}\|_\infty^{-1} \quad \text{for all} \quad i = 1,\dots,n. \tag{2.5}$$

Assume that

$$|x_p| = 1 = \|x\|_\infty. \tag{2.6}$$

First consider the case where $p \in S$.

If $r_p^{\bar S}(A) = 0$, then we have

$$(Ax)_p = a_{pp}x_p + \sum_{\substack{j\in S\\ j\neq p}} a_{pj}x_j,$$

implying, in view of (2.6), that

$$|a_{pp}| = |a_{pp}x_p| \leq |(Ax)_p| + \sum_{\substack{j\in S\\ j\neq p}} |a_{pj}|\,|x_j| \leq |(Ax)_p| + r_p^S(A).$$

Then, by (2.5), we have

$$|a_{pp}| - r_p^S(A) \leq \|A^{-1}\|_\infty^{-1}. \tag{2.7}$$

Since, for the S-SOB matrix A, the left-hand side of (2.7) is positive, we obtain

$$\|A^{-1}\|_\infty \leq \frac{1}{|a_{pp}| - r_p^S(A)} \leq \max_{\substack{i\in S:\\ r_i^{\bar S}(A)=0}} \frac{1}{|a_{ii}| - r_i^S(A)}. \tag{2.8}$$

This proves the bound (2.3) in the case under consideration.

Now assume that $r_p^{\bar S}(A) \neq 0$. In this case, there is an index $q \in \bar S$ such that $a_{pq} \neq 0$ and

$$|x_q| = \max_{\substack{j\in \bar S:\\ a_{pj}\neq 0}} |x_j|. \tag{2.9}$$

Using (2.6) and (2.9), we derive

$$|a_{pp}| = |a_{pp}x_p| \leq |(Ax)_p| + r_p^S(A) + r_p^{\bar S}(A)|x_q|,$$

whence

$$|a_{pp}| - |(Ax)_p| - r_p^S(A) \leq r_p^{\bar S}(A)|x_q|. \tag{2.10}$$

If $x_q = 0$, then from inequality (2.10), by using (2.5), we infer

$$|a_{pp}| - r_p^S(A) \le |(Ax)_p| \le \|A^{-1}\|_\infty^{-1}.$$

Using the conditions $a_{qq} \neq 0$ and (iii), we derive

$$\|A^{-1}\|_\infty \le \frac{1}{|a_{pp}| - r_p^S(A)} = \frac{|a_{qq}|}{[|a_{pp}| - r_p^S(A)]|a_{qq}|}$$
$$\le \frac{|a_{qq}| + r_p^{\bar{S}}(A)}{[|a_{pp}| - r_p^S(A)]|a_{qq}| - r_p^{\bar{S}}(A)\, r_q(A)} \le \max_{\substack{i\in S, j\in\bar{S}:\\ a_{ij}\neq 0}} f_{ij}(A,S).$$

Thus, if $x_q = 0$, then the desired bound is established.

Now let $x_q \neq 0$. By considering the qth component of Ax and using (2.6), we obtain

$$|a_{qq}|\, |x_q| \le |(Ax)_q| + r_q(A). \tag{2.11}$$

Now from (2.10) and (2.11) it follows that

$$[|a_{pp}| - r_p^S(A) - |(Ax)_p|]|a_{qq}| \le r_p^{\bar{S}}(A)[|(Ax)_q| + r_q(A)],$$

or

$$[|a_{pp}| - r_p^S(A)]|a_{qq}| - r_p^{\bar{S}}(A)\, r_q(A) \le |(Ax)_p||a_{qq}| + |(Ax)_q| r_p^{\bar{S}}(A).$$

Therefore, in view of (2.5), we have

$$[|a_{pp}| - r_p^S(A)]|a_{qq}| - r_p^{\bar{S}}(A)\, r_q(A) \le \|A^{-1}\|_\infty^{-1}[|a_{qq}| + r_p^{\bar{S}}(A)].$$

Taking into account that for the S-SOB matrix A, the left-hand side of the latter relation is positive, we write

$$\|A^{-1}\|_\infty \le \frac{|a_{qq}| + r_p^{\bar{S}}(A)}{[|a_{pp}| - r_p^S(A)]|a_{qq}| - r_p^{\bar{S}}(A) r_q(A)}, \tag{2.12}$$

which implies that

$$\|A^{-1}\|_\infty \le \max_{\substack{i\in S, j\in\bar{S}:\\ a_{ij}\neq 0}} f_{ij}(A,S).$$

Now, in order to complete the proof, we must consider the case where $p \in \bar{S}$.

If $r_p^S(A) = 0$, then, arguing as above, we derive

$$|a_{pp}| = |a_{pp} x_p| \le |(Ax)_p| + r_p^{\bar{S}}(A)$$

and

$$\|A^{-1}\|_\infty \le \frac{1}{|a_{pp}| - r_p^{\bar{S}}(A)} \le \max_{\substack{j\in\bar{S}:\\ r_j^S(A)=0}} \frac{1}{|a_{jj}| - r_j^{\bar{S}}(A)}.$$

If $r_p^S(A) \neq 0$, then we choose $q \in S$ in such a way that $a_{pq} \neq 0$ and

$$|x_q| = \max_{\substack{j\in S:\\ a_{pj}\neq 0}} |x_j|.$$

If $|x_q| = 0$, then we have

$$|a_{pp}| - r_p^{\bar{S}}(A) \le |(Ax)_p| \le \|A^{-1}\|_\infty^{-1}$$

and

$$\|A^{-1}\|_\infty \le \frac{1}{|a_{pp}| - r_p^{\bar S}(A)} \le \frac{|a_{qq}| + r_p^S(A)}{[|a_{pp}| - r_p^{\bar S}(A)]|a_{qq}| - r_p^S(A) r_q(A)}$$
$$\le \max_{\substack{i\in S, j\in \bar S:\\ a_{ji}\neq 0}} \frac{|a_{ii}| + r_j^S(A)}{[|a_{jj}| - r_j^{\bar S}(A)]|a_{ii}| - r_j^S(A) r_i(A)} = \max_{\substack{i\in S, j\in \bar S:\\ a_{ji}\neq 0}} f_{ji}(A, \bar S).$$

Finally, if $x_q \neq 0$, then, arguing as in proving (2.12), we arrive at the inequality

$$\|A^{-1}\|_\infty \le \frac{|a_{qq}| + r_p^S(A)}{[|a_{pp}| - r_p^{\bar S}(A)]|a_{qq}| - r_p^S(A) r_q(A)}$$
$$\le \max_{\substack{i\in S, j\in \bar S:\\ a_{ji}\neq 0}} \frac{|a_{ii}| + r_j^S(A)}{[|a_{jj}| - r_j^{\bar S}(A)]|a_{ii}| - r_j^S(A) r_i(A)} = \max_{\substack{i\in S, j\in \bar S:\\ a_{ji}\neq 0}} f_{ji}(A, \bar S).$$

This completes the proof of the theorem. □

Now from Theorem 2.1 we derive a corollary. Observe that if $r_i^{\bar S}(A) = 0$, $i \in S$, then for all $j \in \bar S$,

$$f_{ij}(A, S) = \frac{1}{|a_{ii}| - r_i^S(A)}$$

and, similarly, if $r_j^S(A) = 0$, $j \in \bar S$, then for all $i \in S$,

$$f_{ji}(A, \bar S) = \frac{1}{|a_{jj}| - r_j^{\bar S}(A)}.$$

In view of the latter relations, we have

$$\max\left\{ \max_{\substack{i\in S:\\ r_i^{\bar S}(A)\neq 0}} \frac{1}{|a_{ii}| - r_i^S(A)},\ \max_{\substack{i\in S, j\in \bar S:\\ a_{ij}\neq 0}} f_{ij}(A, S) \right\} \le \max_{i\in S, j\in \bar S} f_{ij}(A, S)$$

and also

$$\max\left\{ \max_{\substack{j\in \bar S\\ r_j^{S}(A)\neq 0}} \frac{1}{|a_{jj}| - r_j^{\bar S}(A)},\ \max_{\substack{i\in S, j\in \bar S:\\ a_{ji}\neq 0}} f_{ji}(A, \bar S) \right\} \le \max_{i\in S, j\in \bar S} f_{ji}(A, \bar S).$$

Thus, Theorem 2.1 implies the following somewhat simpler but also less sharp bound, valid for S-OB matrices, for which (contrary to S-SOB matrices) the quantities $f_{ij}(A, S)$ and $f_{ji}(A, \bar S)$ are well defined for all $i \in S$ and $j \in \bar S$.

Corollary 2.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, be an S-OB matrix, where $S \subset \langle n\rangle$, $1 \le |S| \le n-1$. Then*

$$\|A^{-1}\|_\infty \le \max_{i\in S, j\in \bar S} \max\left\{ f_{ij}(A, S),\ f_{ji}(A, \bar S)\right\}. \tag{2.13}$$

It is of importance to mention that for an SDD matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, the bound (2.13) is in general sharper than the classical bound (1.2), i.e.,

$$\max_{i\in S, j\in \bar S} \max\left\{ f_{ij}(A, S),\ f_{ji}(A, \bar S)\right\} \le \max_{i\in\langle n\rangle} \frac{1}{|a_{ii}| - r_i(A)}. \tag{2.14}$$

Indeed, as is readily seen, for all $i \neq j$ and an arbitrary S,

$$f_{ij}(A, S) \le \frac{1}{|a_{ii}| - r_i(A)}$$

whenever

$$|a_{ii}| - r_i(A) \le |a_{jj}| - r_j(A),$$

and, similarly,

$$f_{ij}(A, S) \le \frac{1}{|a_{jj}| - r_j(A)}$$

whenever

$$|a_{jj}| - r_j(A) \le |a_{ii}| - r_i(A),$$

and (2.14) follows.

3. S-SDDS and S-SDD matrices

In this section, we introduce into consideration a new matrix class $\{S\text{-SDDS}\}$, which takes into account the matrix sparsity pattern. Then we show that matrices from this class are nonsingular $\mathcal{H}$-matrices and obtain an upper bound for the infinity norm of their inverses.

Recall that a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, is said to be S-SDD (e.g., see [8, 13]) if the following two conditions are fulfilled:

$$|a_{ii}| > r_i^S(A) \quad \text{for all} \quad i \in S \tag{3.1}$$

and

$$[|a_{ii}| - r_i^S(A)]\,[|a_{jj}| - r_j^{\bar{S}}(A)] > r_i^{\bar{S}}(A)\, r_j^S(A) \quad \text{for all} \quad i \in S \quad \text{and} \quad j \in \bar{S}. \tag{3.2}$$

As is well known, the class of S-SDD matrices is a subclass of the class of nonsingular $\mathcal{H}$-matrices and contains the subclass of SDD matrices.

We extend the class $\{S\text{-SDD}\}$ by introducing the following definition.

A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, is said to be S-SDDS (S-SDD Sparse) if the following conditions are fulfilled:

$$|a_{ii}| > r_i^S(A) \quad \text{for all} \quad i \in S; \tag{3.3}$$

$$|a_{jj}| > r_j^{\bar{S}}(A) \quad \text{for all} \quad j \in \bar{S}, \tag{3.4}$$

and

$$\begin{gathered}[|a_{ii}| - r_i^S(A)]\,[|a_{jj}| - r_j^{\bar{S}}(A)] > r_i^{\bar{S}}(A)\, r_j^S(A) \\ \text{for all } i \in S \text{ and all } j \in \bar{S} \ \text{ such that } \ a_{ij} \ne 0 \ \text{ or } \ a_{ji} \ne 0.\end{gathered} \tag{3.5}$$

Observe that conditions (3.3)–(3.5) being obviously fulfilled for any S-SDD matrix A, we have

$$\{\text{SDD}\} \subseteq \{S\text{-SDD}\} \subseteq \{S\text{-SDDS}\}. \tag{3.6}$$

We start the study of the S-SDDS matrices by establishing the following basic result.

Lemma 3.1. *Let $S \subset \langle n\rangle$, where $n \ge 2$ and $1 \le |S| \le n-1$, let a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be singular, and let*

$$Ax = 0 \tag{3.7}$$

for a nonzero vector $x = (x_i) \in \mathbb{C}^n$. Let

$$|x_p| = \max_{i\in\langle n\rangle} |x_i|. \tag{3.8}$$

If $p \in S$, then either

$$|a_{pp}| - r_p^S(A) \le 0 \tag{3.9}$$

or there exists an index $q \in \bar{S}$ such that $a_{pq} \ne 0$ and

$$[|a_{pp}| - r_p^S(A)]\,[|a_{qq}| - r_q^{\bar{S}}(A)] \le r_p^{\bar{S}}(A)\, r_q^S(A). \tag{3.10}$$

Proof. If $r_p^{\bar{S}}(A) = 0$, then we have

$$0 = (Ax)_p = a_{pp}x_p + \sum_{\substack{j \neq p \\ j \in S}} a_{pj}x_j,$$

implying that

$$|a_{pp}||x_p| \leq r_p^S(A)|x_p|,$$

which proves (3.9).

If $r_p^{\bar{S}}(A) \neq 0$, then we choose $q \in \bar{S}$ such that $a_{pq} \neq 0$ and

$$|x_q| = \max_{\substack{j \in \bar{S} \\ a_{pj} \neq 0}} |x_j|. \tag{3.11}$$

In this case, we have

$$|a_{pp}||x_p| \leq r_p^S(A)|x_p| + r_p^{\bar{S}}(A)|x_q|$$

and

$$[|a_{pp}| - r_p^S(A)]|x_p| \leq r_p^{\bar{S}}(A)|x_q|. \tag{3.12}$$

If $x_q = 0$, then from (3.12) it follows that

$$[|a_{pp}| - r_p^S(A)] \leq 0,$$

whence inequality (3.9) holds.

If $x_q \neq 0$, then we derive

$$|a_{qq}||x_q| \leq r_q^S(A)|x_p| + r_q^{\bar{S}}(A)|x_q|,$$
$$[|a_{qq}| - r_q^{\bar{S}}(A)]|x_q| \leq r_q^S(A)|x_p|. \tag{3.13}$$

From (3.12) and (3.13), taking into account that $x_p \neq 0$ and $x_q \neq 0$, we obtain

$$[|a_{pp}| - r_p^S(A)]\ [|a_{qq}| - r_q^{\bar{S}}(A)] \leq r_p^{\bar{S}}(A)\ r_q^S(A).$$

This completes the proof of the lemma. □

From Lemma 3.1 we immediately obtain the following matrix nonsingularity criterion, which involves the matrix sparsity pattern and depends on the partition $\langle n \rangle = S \cup \bar{S}$.

Theorem 3.1. *Let a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $n \geq 2$, be an S-SDDS matrix for a certain subset $S \subset \langle n \rangle$, where $1 \leq |S| \leq n - 1$. Then A is nonsingular.*

Proof. Suppose A is singular and $Ax = 0$ for a nonzero vector $x = (x_i)$. Let p be defined in accordance with (3.8).

If $p \in S$, then, by Lemma 3.1, either (3.9) holds or $a_{pq} \neq 0$, $q \in \bar{S}$, and (3.10) is valid. If $p \in \bar{S}$, then, by Lemma 3.1, where S is replaced by $\bar{S}$, either $|a_{pp}| - r_p^{\bar{S}}(A) \leq 0$ or, for a certain $q \in S$, we have $a_{pq} \neq 0$ and

$$[|a_{qq}| - r_q^S(A)]\ [|a_{pp}| - r_p^{\bar{S}}(A)] \leq r_q^{\bar{S}}(A)\ r_p^S(A).$$

But, under the assumptions of the theorem, both cases are impossible, whence A is nonsingular. □

The next theorem claims that any S-SDDS matrix actually is a nonsingular $\mathcal{H}$-matrix.

Theorem 3.2. *Let, for a certain subset $S \subset \langle n \rangle$, where $n \geq 2$ and $1 \leq |S| \leq n - 1$, a matrix $A \in \mathbb{C}^{n \times n}$ be an S-SDDS matrix. Then it is a nonsingular $\mathcal{H}$-matrix.*

Proof. Note that conditions (3.3)–(3.5) are fulfilled for A if and only if they are fulfilled for the comparison matrix $\mathcal{M}(A)$. Therefore, by Theorem 3.1, $\mathcal{M}(A)$ is nonsingular, and it remains to show that $\mathcal{M}(A)$ is an $\mathcal{M}$-matrix. To this end, by virtue of [2, Condition D_{15} of Theorem 6.2.3], it is sufficient to prove that the shifted matrix $\mathcal{M}(A)+\varepsilon I_n$ is nonsingular for every $\varepsilon \geq 0$. However, conditions (3.3)–(3.5), which hold for $\mathcal{M}(A)$, hold for $\mathcal{M}(A)+\varepsilon I_n$ a fortiori. Thus, by Theorem 3.1, $\mathcal{M}(A)+\varepsilon I_n$ is nonsingular for all $\varepsilon \geq 0$, whence A is a nonsingular $\mathcal{H}$-matrix. □

In view of (3.6) and Theorem 3.2, we have

$$\{\text{SDD}\} \subseteq \{S\text{-SDD}\} \subseteq \{S\text{-SDDS}\} \subseteq \{\mathcal{H}\},$$

which yields a new proof of the fact that any S-SDD matrix is a nonsingular $\mathcal{H}$-matrix.

Now we are ready to present an upper bound for $\|A^{-1}\|_\infty$ for an S-SDDS matrix A.

Theorem 3.3. *Let, given a subset $S \subset \langle n\rangle$, where $n \geq 2$ and $1 \leq |S| \leq n-1$, $A=(a_{ij}) \in \mathbb{C}^{n\times n}$ be an S-SDDS matrix. Then*

$$\|A^{-1}\|_\infty \leq \max\left\{ \max_{\substack{i\in S:\\ r_i^{\bar S}(A)=0}} \frac{1}{|a_{ii}|-r_i^S(A)},\ \max_{\substack{j\in \bar S:\\ r_j^{S}(A)=0}} \frac{1}{|a_{jj}|-r_j^{\bar S}(A)},\right.$$
$$\left. \max_{\substack{p\in S, q\in \bar S:\\ a_{pq}\neq 0}} f_{pq}^S(A),\ \max_{\substack{p\in S, q\in \bar S:\\ a_{qp}\neq 0}} f_{qp}^{\bar S}(A) \right\}. \tag{3.14}$$

Here, for $i \in S$ and $j \in \bar S$ such that $a_{ij} \neq 0$, we set

$$f_{ij}^S(A) := \frac{|a_{jj}| - r_j^{\bar S}(A) + r_i^{\bar S}(A)}{[|a_{ii}| - r_i^S(A)]\,[|a_{jj}| - r_j^{\bar S}(A)] - r_i^{\bar S}(A)\, r_j^S(A)}. \tag{3.15}$$

Proof. In accordance with Lemma 1.1, assume that a vector $x=(x_i)$ is such that

$$|(Ax)_i| \leq \|A^{-1}\|_\infty^{-1}, \quad i=1,\dots,n, \tag{3.16}$$

and

$$1 = |x_p| = \max_{i\in\langle n\rangle} |x_i|.$$

First consider the case where $p \in S$.

If $r_p^{\bar S}(A)=0$, then we have

$$|a_{pp}| - r_p^S(A) \leq |(Ax)_p| \leq \|A^{-1}\|_\infty^{-1}. \tag{3.17}$$

Since A is an S-SDDS matrix, the leftmost expression in (3.17) is positive, and we derive

$$\|A^{-1}\|_\infty \leq \frac{1}{|a_{pp}| - r_p^S(A)} \leq \max_{\substack{i\in S:\\ r_i^{\bar S}(A)=0}} \frac{1}{|a_{ii}|-r_i^S(A)}. \tag{3.18}$$

If $r_p^{\bar S}(A) \neq 0$, then we choose $q \in \bar S$ in such a way that $a_{pq} \neq 0$ and

$$|x_q| = \max_{\substack{j\in \bar S:\\ a_{pj}\neq 0}} |x_j|.$$

Then, as is readily seen, we have

$$|a_{pp}| - r_p^S(A) - |(Ax)_p| \leq r_p^{\bar S}(A)|x_q| \tag{3.19}$$

and

$$[|a_{qq}| - r_q^{\bar S}(A)]|x_q| \leq |(Ax)_q| + r_q^S(A). \tag{3.20}$$

If $x_q \neq 0$, then from (3.19) and (3.20) it follows that

$$[|a_{pp}| - r_p^S(A) - |(Ax)_p|]\,[|a_{qq}| - r_q^{\bar{S}}(A)] \le r_p^{\bar{S}}(A)\,[|(Ax)_q| + r_q^S(A)],$$

or

$$\begin{aligned}[|a_{pp}| - r_p^S(A)]\,[|a_{qq}| - r_q^{\bar{S}}(A)] - r_p^{\bar{S}}(A)\,r_q^S(A) &\le |(Ax)_p|[|a_{qq}| - r_q^{\bar{S}}(A)] + |(Ax)_q| r_p^{\bar{S}}(A)\\ &\le \|A^{-1}\|_\infty^{-1}[|a_{qq}| - r_q^{\bar{S}}(A) + r_p^{\bar{S}}(A)],\end{aligned}$$

where we have used (3.16). Since the first expression in the latter string of inequalities is positive, we obtain

$$\|A^{-1}\|_\infty \le \frac{|a_{qq}| - r_q^{\bar{S}}(A) + r_p^{\bar{S}}(A)}{[|a_{pp}| - r_p^S(A)]\,[|a_{qq}| - r_q^{\bar{S}}(A)] - r_p^{\bar{S}}(A)\,r_q^S(A)} = f_{pq}^S(A). \tag{3.21}$$

In the case where $x_q = 0$, by (3.19) and (3.16), we have

$$|a_{pp}| - r_p^S(A) \le |(Ax)_p| \le \|A^{-1}\|_\infty^{-1},$$

whence

$$\|A^{-1}\|_\infty \le \frac{1}{|a_{pp}| - r_p^S(A)}. \tag{3.22}$$

Show that for every $q \in \bar{S}$ and, in particular, for q chosen above,

$$\frac{1}{|a_{pp}| - r_p^S(A)} \le f_{pq}^S(A). \tag{3.23}$$

Indeed,

$$\begin{aligned}\frac{1}{|a_{pp}| - r_p^S(A)} &= \frac{|a_{qq}| - r_q^{\bar{S}}(A)}{[|a_{pp}| - r_p^S(A)]\,[|a_{qq}| - r_q^{\bar{S}}(A)]}\\ &\le \frac{|a_{qq}| - r_q^{\bar{S}}(A) + r_p^{\bar{S}}(A)}{[|a_{pp}| - r_p^S(A)]\,[|a_{qq}| - r_q^{\bar{S}}(A)] - r_p^{\bar{S}}(A)\,r_q^S(A)} = f_{pq}^S(A).\end{aligned}$$

Thus, in the case where $p \in S$, by (3.18), (3.21), and (3.22)–(3.23), we have

$$\|A^{-1}\|_\infty \le \max\left\{ \max_{\substack{i\in S:\\ r_i^{\bar{S}}(A)=0}} \frac{1}{|a_{ii}| - r_i^S(A)},\ \max_{\substack{p\in S, q\in\bar{S}:\\ a_{pq}\neq 0}} f_{pq}^S(A) \right\}. \tag{3.24}$$

In the case where $p \in \bar{S}$, the desired result is obtained from (3.24) by interchanging S with $\bar{S}$ and p with q. This completes the proof. □

As is trivial to see, if $r_i^{\bar{S}}(A) = 0$, where $i \in S$, then, for every $j \in \bar{S}$, we have

$$f_{ij}^S(A) = \frac{1}{|a_{ii}| - r_i^S(A)}. \tag{3.25}$$

Similarly, if $r_j^S(A) = 0$, where $j \in \bar{S}$, then, for every $i \in S$, we have

$$f_{ji}^{\bar{S}}(A) = \frac{1}{|a_{jj}| - r_j^{\bar{S}}(A)}. \tag{3.26}$$

By applying Theorem 3.3 to an S-SDD matrix A, taking into account that for such a matrix A the quantities $f_{ij}^S(A)$ and $f_{ji}^{\bar{S}}(A)$ are well defined for all $i \in S$ and all $j \in \bar{S}$, and using (3.25)–(3.26), we obtain the following known bound, originally established in [22] (also see [11]).

Corollary 3.1. *Let $S \subset \langle n \rangle$, $1 \le |S| \le n-1$, $n \ge 2$ and let $A \in \mathbb{C}^{n\times n}$ be an S-SDD matrix. Then*

$$\|A^{-1}\|_\infty \le \max_{i\in S, j\in \bar{S}} \max\left\{ f_{ij}^S(A),\ f_{ji}^{\bar{S}}(A) \right\}, \tag{3.27}$$

where $f_{ij}^S(A)$ is defined in (3.15).

As is well known (see, e.g., [11]), for an SDD matrix A the bound (3.27) generally improves the classical bound (1.2), whence the bound (3.14) improves (1.2) a fortiori.

4. OBS and OB matrices

In this section, we consider matrices $A \in \mathbb{C}^{n\times n}$, $n \ge 2$, that satisfy the following condition:

$$|a_{ii}|\,|a_{jj}| > r_i(A)\, r_j(A) \quad \text{for all} \quad i \ne j. \tag{4.1}$$

As is known since 1937, see [23] and [3], such matrices are nonsingular. Moreover, as is nondifficult to ascertain (applying, e.g., the argument used in the proof of Theorem 3.2), the matrices satisfying (4.1) are nonsingular $\mathcal{H}$-matrices. We call them OB (Ostrowski–Brauer) matrices, and some authors say that such matrices are DSDD (doubly SDD). Relatively recently, in 2008, Pan and Cheng [24] (also see [9]) established the following upper bound for the inverse of a DSDD matrix. (One can guess that in [20] the same bound is obtained.)

Theorem 4.1 ([24])**.** *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, be an OB matrix. Then*

$$\|A^{-1}\|_\infty \le \max_{i\ne j} \frac{|a_{jj}| + r_i(A)}{|a_{ii}|\,|a_{jj}| - r_i(A)\, r_j(A)}. \tag{4.2}$$

In this section, we show that the bound (4.2) can be sharpened and, simultaneously, extended to a larger matrix class. Recall the following result, established in [10].

Theorem 4.2. *Let a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, be irreducible, satisfy the condition*

$$|a_{ii}|\,|a_{jj}| \ge r_i(A)\, r_j(A) \quad \textit{for all} \quad i \ne j \quad \textit{such that} \quad a_{ij} \ne 0, \tag{4.3}$$

and let at least one of the inequalities in (4.3) *be strict. Then A is nonsingular.*

Theorem 4.2 implies the following result.

Corollary 4.1. *Let a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, be free of zero rows and satisfy the condition*

$$|a_{ii}|\,|a_{jj}| > r_i(A)\, r_j(A) \quad \textit{for all} \quad i \ne j \quad \textit{such that} \quad a_{ij} \ne 0. \tag{4.4}$$

Then A is nonsingular.

Proof. If A is an irreducible matrix, then it is nonsingular by Theorem 4.2.

If A is reducible, then each of its irreducible components $A[S]$, where $S \subset \langle n \rangle$, $2 \le |S| \le n-1$, a fortiori satisfies conditions of the type (4.4), whence all of them are nonsingular.

Finally, if $A[i]$, $i \in \langle n \rangle$, is an irreducible component of order 1, then it is nonsingular by virtue of the assumption that A is free of zero rows. □

Arguing as in the proof of Theorem 3.2, one can readily strengthen the above result as follows.

Theorem 4.3. *Let a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, be free of zero rows and satisfy condition* (4.4)*. Then A is a nonsingular $\mathcal{H}$-matrix.*

Note that the class of matrices satisfying the assumptions of Theorem 4.1, i.e., the class of OB (DSDD) matrices is a subclass of the class of matrices satisfying the hypotheses of Theorem 4.3 because from (4.1) it immediately follows that all the diagonal entries of A are nonzero, whence A is free of zero rows.

Observe that in Corollary 4.1 and Theorem 4.3, the condition that A is free of zero rows is equivalent to the condition that all the diagonal entries of A are nonzero.

Matrices $A \in \mathbb{C}^{n\times n}$, $n \geq 2$, that satisfy the hypotheses of Theorem 4.3 will be referred to as OBS (OB Sparse) matrices.

Obviously,

$$\{\text{SDD}\} \subset \{\text{OB}\} \subset \{\text{OBS}\}.$$

The main result of this section is the following theorem.

Theorem 4.4. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be an OBS matrix. Then*

$$\|A^{-1}\|_\infty \leq \max \left\{ \max_{\substack{i\in\langle n\rangle:\\ r_i(A)=0}} |a_{ii}|^{-1},\ \max_{i:\, r_i(A)\neq 0} \max_{\substack{j\neq i:\\ a_{ij}\neq 0}} \frac{|a_{jj}| + r_i(A)}{|a_{ii}|\,|a_{jj}| - r_i(A)\, r_j(A)} \right\}. \tag{4.5}$$

Proof. As in the proof of Theorem 3.3, choose a vector $x \neq 0$ with $\|x\|_\infty = 1$ such that

$$\|Ax\|_\infty \leq \|A^{-1}\|_\infty^{-1} \tag{4.6}$$

and assume that

$$|x_p| = \|x\|_\infty = 1. \tag{4.7}$$

If $r_p(A) = 0$, then we have

$$|(Ax)_p| = |a_{pp}||x_p| = |a_{pp}|,$$

implying, in view of (4.6), that

$$\|A^{-1}\|_\infty \leq |a_{pp}|^{-1}.$$

Thus, in this case,

$$\|A^{-1}\|_\infty \leq \max_{\substack{i\in\langle n\rangle:\\ r_i(A)=0}} |a_{ii}|^{-1},$$

and the bound (4.5) is established.

Now let $r_p(A) \neq 0$ and let $q \neq p$ be such that $a_{pq} \neq 0$ and

$$|x_q| = \max_{\substack{j\neq p:\\ a_{pj}\neq 0}} |x_j|. \tag{4.8}$$

By using (4.7) and (4.8), we readily obtain

$$|a_{pp}| \leq |(Ax)_p| + r_p(A)|x_q| \tag{4.9}$$

and

$$|a_{qq}||x_q| \leq |(Ax)_q| + r_q(A). \tag{4.10}$$

If $x_q = 0$, then, by (4.9) and (4.6), we have

$$|a_{pp}| \leq |(Ax)_p| \leq \|A^{-1}\|_\infty^{-1},$$

whence, for any $j \neq p$,

$$\|A^{-1}\|_\infty \leq \frac{1}{|a_{pp}|} = \frac{|a_{jj}|}{|a_{pp}|\,|a_{jj}|} < \frac{|a_{jj}| + r_p(A)}{|a_{pp}|\,|a_{jj}| - r_p(A)\, r_j(A)} \tag{4.11}$$

and

$$\|A^{-1}\|_\infty \leq \min_{j\neq p} \frac{|a_{jj}| + r_p(A)}{|a_{pp}|\,|a_{jj}| - r_p(A)\, r_j(A)}.$$

Thus, in the case considered, the bound (4.5) is obviously valid.

It remains to consider the situation where $r_p(A) \neq 0$ and $x_q \neq 0$. In this case, from (4.9) and (4.10) we obtain

$$[|a_{pp}| - |(Ax)_p|]|a_{qq}| \leq r_p(A)[|(Ax)_q| + r_q(A)],$$

or

$$|a_{pp}|\, |a_{qq}| - r_p(A)\, r_q(A) \leq |(Ax)_p||a_{qq}| + |(Ax)_q|\, r_p(A).$$

In view of (4.6), it follows that

$$|a_{pp}|\, |a_{qq}| - r_p(A)\, r_q(A) \leq \|A^{-1}\|_\infty^{-1}[|a_{qq}| + r_p(A)],$$

whence

$$\|A^{-1}\|_\infty \leq \max_{\substack{j\neq p:\\ a_{pj}\neq 0}} \frac{|a_{jj}| + r_p(A)}{|a_{pp}|\, |a_{jj}| - r_p(A)\, r_j(A)} \leq \max_{i:\, r_i(A)\neq 0} \max_{\substack{j\neq i:\\ a_{ij}\neq 0}} \frac{|a_{jj}| + r_i(A)}{|a_{ii}|\, |a_{jj}| - r_i(A)\, r_j(A)}.$$

This completes the proof of the theorem. □

As it follows from (4.11), under the assumption that

$$|a_{ii}|\, |a_{jj}| > r_i(A)\, r_j(A) \quad \text{for all} \quad i \neq j,$$

the bound (4.5) obviously does not exceed the bound (4.2). Furthermore, the bound (4.5) holds for a wider class of matrices than the bound (4.2).

We conclude this section by showing that for an SDD matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, the bound (4.2) is at least as good as the classical bound (1.2).

Indeed, assume that A is an SDD matrix. If $r_i(A) \neq 0$, then the inequality

$$\frac{|a_{jj}| + r_i(A)}{|a_{ii}|\, |a_{jj}| - r_i(A)\, r_j(A)} \leq \frac{1}{|a_{ii}| - r_i(A)} \tag{4.12}$$

is equivalent to

$$|a_{ii}| - r_i(A) \leq |a_{jj}| - r_j(A),$$

and if $r_i(A) = 0$, then (4.12) is an equality. On the other hand, the inequality

$$\frac{|a_{jj}| + r_i(A)}{|a_{ii}|\, |a_{jj}| - r_i(A)\, r_j(A)} \leq \frac{1}{|a_{jj}| - r_j(A)}$$

amounts to

$$|a_{jj}| - r_j(A) \leq |a_{ii}| - r_i(A).$$

Thus, we always have

$$\frac{|a_{jj}| + r_i(A)}{|a_{ii}|\, |a_{jj}| - r_i(A)\, r_j(A)} \leq \max\left\{\frac{1}{|a_{ii}| - r_i(A)},\ \frac{1}{|a_{jj}| - r_j(A)}\right\}. \tag{4.13}$$

This shows that for an SDD matrix A the bound (4.2) and, consequently, (4.5) as well generally improve the classical bound (1.2).

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