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RELATION GRAPHS OF THE SPLIT-SEDENION ALGEBRA

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The paper introduces the Cayley–Dickson split-sedenion algebra. Exact expressions for the annihilators and orthogonalizers of its zero divisors are obtained, and these results are applied in describing relation graphs of the split-sedenions in terms of their diameters and cliques. Bibliography: 10 titles.

1. Introduction

Let $\mathbb{F}$ be an arbitrary field and let $(\mathcal{A}, +, \cdot)$ be an algebra with identity $1_{\mathcal{A}}$ over the field $\mathbb{F}$. $\mathcal{A}$ is assumed to be neither commutative nor associative. Given $a, b \in \mathcal{A}$, we say that

- a and b *commute* if $ab = ba$;
- a and b *anticommute* if $ab + ba = 0$;
- a and b *are orthogonal* if $ab = ba = 0$;
- a is a *left zero divisor* if $a \neq 0$ and there exists a nonzero $x \in \mathcal{A}$ such that $ax = 0$;
- a is a *right zero divisor* if $a \neq 0$ and there exists a nonzero $x \in \mathcal{A}$ such that $xa = 0$;
- a is a *two-sided zero divisor* if it is both a left and a right zero divisor;
- a is a *zero divisor* if it is a left or a right zero divisor.

Definition 1.1.

- The center *of an algebra $\mathcal{A}$ is the set* $C_{\mathcal{A}} = \{a \in \mathcal{A} \mid ab = ba \text{ for all } b \in \mathcal{A}\}$.
- *$Z(\mathcal{A})$ is the set of zero divisors of $\mathcal{A}$.*
- *$Z_{LR}(\mathcal{A})$ is the set of two-sided zero divisors of $\mathcal{A}$.*

Definition 1.2. *Let a be an arbitrary element of an algebra $\mathcal{A}$.*

- The centralizer *of a is* $C_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = ba\}$, *i.e., the set of all elements in $\mathcal{A}$ that commute with a.*
- The anticentralizer *of a is* $\mathrm{Anc}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab + ba = 0\}$, *i.e., the set of all elements in $\mathcal{A}$ that anticommute with a.*
- The left annihilator *of a is the set* $l.\mathrm{Ann}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ba = 0\}$.
- *Similarly,* the right annihilator *of a is* $r.\mathrm{Ann}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = 0\}$.
- The orthogonalizer *of a is* $O_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = ba = 0\}$, *i.e., the set of all elements in $\mathcal{A}$ that are orthogonal to a.*

Remark 1.3. Let $a \in \mathcal{A}$. It can readily be seen that $C_{\mathcal{A}}$, $C_{\mathcal{A}}(a)$, $\mathrm{Anc}_{\mathcal{A}}(a)$, $l.\mathrm{Ann}_{\mathcal{A}}(a)$, $r.\mathrm{Ann}_{\mathcal{A}}(a)$, and $O_{\mathcal{A}}(a)$ are vector spaces over $\mathbb{F}$.

Now we introduce some relation graphs that will be studied in this paper.

Definition 1.4. *Given an algebra $\mathcal{A}$, we define the following structures:*

- the commutativity graph $\Gamma_C(\mathcal{A})$ *is a graph with vertex set $\mathcal{A} \setminus C_{\mathcal{A}}$, and distinct vertices a and b of $\Gamma_C(\mathcal{A})$ are adjacent if and only if $ab = ba$;*
- the orthogonality graph $\Gamma_O(\mathcal{A})$ *is a graph with vertex set $Z_{LR}(\mathcal{A})$, and distinct vertices a and b of $\Gamma_O(\mathcal{A})$ are adjacent if and only if $ab = ba = 0$;*

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- the directed zero divisor graph $\Gamma_Z(\mathcal{A})$ *is a graph with vertex set* $Z(\mathcal{A})$*, and distinct vertices* a *and* b *of* $\Gamma_Z(\mathcal{A})$ *are connected with an edge directed from* a *to* b *if and only if* $ab = 0$.

We will need the following graph theory definitions.

Definition 1.5. *Let* Γ *be a directed or an undirected graph.*

- Γ *is said to be* connected *if for any ordered pair of vertices* (x, y) *there exists a path leading from* x *to* y.
- The distance $d(x, y) = d_\Gamma(x, y)$ *between two vertices* x *and* y *in* Γ *is the number of edges in a shortest path from* x *to* y*. If no path from* x *to* y *exists, then* $d(x, y) = \infty$.
- The diameter $d(\Gamma)$ *of* Γ *is defined as* $\sup_{x,y\in\Gamma} d(x, y)$.

An undirected graph Γ *also has the following invariants:*

- A connected component *of* Γ *is a maximal connected subgraph of* Γ.
- A clique Q *in* Γ *is a subset of vertices of* Γ *such that every two distinct vertices in* Q *are adjacent.*
- *A clique* Q *is said to be* maximal *if for any clique* $\widetilde{Q}$ *such that* $Q \subset \widetilde{Q}$ *we have* $Q = \widetilde{Q}$.

The study of relation graphs of real Cayley–Dickson algebras was initiated in [7] with anticommutativity graphs. Then in [6] relation graphs of the split-complex numbers, split-quaternions, and split-octonions were described. For convenience, in Theorem 3.6 we recall previous results in order to compare them with those obtained in the present paper for the split-sedenions.

Theorem 4.30 in [6] establishes a relationship between the commutativity and orthogonality graphs of real low-dimensional Cayley–Dickson split-algebras. It uses the notion of doubly alternative elements, which was introduced by Moreno in [9] for the algebras of the main sequence and then extended in [6] to the split-algebras.

In this paper, we study zero divisors of the Cayley–Dickson split-sedenion algebra and classify them in terms of the dimensions of their annihilators and orthogonalizers. Then, based on the results obtained, we describe the orthogonality graph and the zero divisor graph of the split-sedenions in terms of their diameters and cliques.

The split-sedenions follow the split-complex numbers, split-quaternions, and split-octonions in the sequence of real Cayley–Dickson split-algebras. For this reason, the next logical step is to study their relation graphs. Moreover, all elements of the split-sedenions are doubly alternative, and they form the last algebra of the sequence possessing this property. Hence they have a convenient criterion for an element to be a zero divisor and a simple description of annihilators and orthogonalizers, see Corollary 3.3 and Lemma 3.5, respectively.

The, paper is organized as follows: Section 2 is devoted to real Cayley–Dickson algebras. Particularly, we describe the Cayley–Dickson process in detail in Sec. 2.1 and recall some properties of the real Cayley–Dickson algebras in Sec. 2.2. Then we introduce the split-sedenions in Sec. 2.3. In Sec. 3, we survey some properties of doubly alternative zero divisors in real Cayley–Dickson split-algebras. Also this section contains some facts concerning relation graphs of real low-dimensional Cayley–Dickson split-algebras. Section 4 completely describes the orthogonality and zero divisor graphs of the split-sedenions.

2. An overview of real Cayley–Dickson algebras

2.1. Construction of Cayley–Dickson algebras. In this section, based on [8,10], we recall the classical method for constructing nonassociative algebras, the so-called Cayley–Dickson algebras.

Definition 2.1.

- *Let* $(\mathcal{A},+,\cdot)$ *be an algebra over a field* $\mathbb{F}$. An involution $a\mapsto\bar a$ *on* $\mathcal{A}$ *is an endomorphism of the vector space* $\mathcal{A}$ *such that for all* $a,b\in\mathcal{A}$ *we have* $\bar{\bar a}=a$ *and* $\overline{ab}=\bar b\bar a$.
- *Now let* $\mathcal{A}$ *have an identity* $1_{\mathcal{A}}$. *The involution* $a\mapsto\bar a$ *on* $\mathcal{A}$ *is said to be* regular *if for any* $a\in\mathcal{A}$, $a+\bar a=t(a)1_{\mathcal{A}}$ *and* $a\bar a=\bar a a=n(a)1_{\mathcal{A}}$, *where* $t(a),n(a)\in\mathbb{F}$. *Here,* $t(a)$ *is called* the trace *of* a, *and* $n(a)$ *is called* the norm *of* a.

Henceforth, we assume that $\mathcal{A}$ is an algebra over a field $\mathbb{F}$ with a regular involution $a\mapsto\bar a$.

Definition 2.2 ([10])**.** The algebra $\mathcal{A}\{\gamma\}$ *produced by the Cayley–Dickson process applied to* $\mathcal{A}$ *with a parameter* $\gamma\in\mathbb{F}$, $\gamma\neq0$, *is defined as the set of ordered pairs of elements of* $\mathcal{A}$ *with operations*

$$\begin{aligned}\alpha(a,b)&=(\alpha a,\alpha b);\\(a,b)+(c,d)&=(a+c,b+d);\\(a,b)(c,d)&=(ac+\gamma\bar d b,da+b\bar c)\end{aligned}$$

and the involution

$$(\overline{a,b})=(\bar a,-b),\quad a,b,c,d\in\mathcal{A},\quad \alpha\in\mathbb{F}.$$

Proposition 2.3 ([10, p. 435])**.** *Properties of* $\mathcal{A}$ *and* $\mathcal{A}\{\gamma\}$ *are interrelated as follows:*

- $\mathcal{A}\{\gamma\}$ *is an algebra over* $\mathbb{F}$ *with the identity* $1_{\mathcal{A}\{\gamma\}}=(1_{\mathcal{A}},0)$ *and a regular involution.*
- *Let* $\mathcal{A}$ *be an* n*-dimensional algebra and let* $\{e_m\}_{m=1,\dots,n}$ *be a basis in* $\mathcal{A}$. *Then* $\mathcal{A}\{\gamma\}$ *is a* $2n$*-dimensional algebra, and* $\{(e_m,0),(0,e_m)\}_{m=1,\dots,n}$ *is a basis in* $\mathcal{A}\{\gamma\}$.
- *Let* $a,b\in\mathcal{A}$, $(a,b)\in\mathcal{A}\{\gamma\}$. *Then*

$$\begin{aligned}t((a,b))&=t(a),\\n((a,b))&=n(a)-\gamma n(b).\end{aligned}$$

Henceforth, we assume that $\mathbb{F}=\mathbb{R}$ and identify $\mathbb{R}1_{\mathcal{A}}$ with $\mathbb{R}$. Consider the following definitions, which are analogous to those for complex numbers.

Definition 2.4.

- The real part *of an element* $a\in\mathcal{A}$ *is* $\mathfrak{Re}(a)=\frac{a+\bar a}{2}$; the imaginary part *of* a *is* $\mathfrak{Im}(a)=\frac{a-\bar a}{2}$, and the norm *of* a *is* $n(a)=a\bar a=\bar a a$.
- *An element* $a\in\mathcal{A}$ *is said to be* pure *if* $\mathfrak{Re}(a)=0$.
- *An element* $(a,b)\in\mathcal{A}\{\gamma\}$ *is said to be* doubly pure *if* $\mathfrak{Re}(a)=\mathfrak{Re}(b)=0$.

Observe that $\mathfrak{Re}(a),n(a)\in\mathbb{R}1_{\mathcal{A}}=\mathbb{R}$ because the involution on $\mathcal{A}$ is regular. Clearly, the notion of norm introduced above agrees with Definition 2.1.

Definition 2.5. *For every integer* $n\geq0$ *and nonzero real numbers* $\gamma_0,\dots,\gamma_{n-1}$, *the real Cayley–Dickson algebra* $\mathcal{A}_n=\mathcal{A}_n\{\gamma_0,\dots,\gamma_{n-1}\}$ *is inductively defined as follows:*

(1) $\mathcal{A}_0=\mathbb{R}$, *and* $e_0^{(0)}=1$ *is its only basis element;*
(2) *if* $\mathcal{A}_n\{\gamma_0,\dots,\gamma_{n-1}\}$ *has already been constructed, then*

$$\mathcal{A}_{n+1}\{\gamma_0,\dots,\gamma_n\}=(\mathcal{A}_n\{\gamma_0,\dots,\gamma_{n-1}\})\{\gamma_n\}.$$

Its basis elements are $e_0^{(n+1)},\dots,e_{2^{n+1}-1}^{(n+1)}$, *where*

$$e_m^{(n+1)}=\begin{cases}(e_m^{(n)},0), & 0\leq m\leq 2^n-1,\\(0,e_{m-2^n}^{(n)}), & 2^n\leq m\leq 2^{n+1}-1.\end{cases}$$

For every integer $n \geq 0$, the structure $\mathcal{A}_n$ in Definition 2.5 is a 2^n-dimensional algebra over $\mathbb{R}$ with the identity $e_0^{(n)}$ and a regular involution, see [6, Lemma 3.14]. We will denote $1 = e_0^{(n)}$ and $r = re_0^{(n)}$ for $r \in \mathbb{R}$.

2.2. Some properties of real Cayley–Dickson algebras. In the sequel, we assume that $\mathcal{A}$ is an arbitrary algebra over a field $\mathbb{F}$ and $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is an arbitrary real Cayley–Dickson algebra. By [8, Exercise 2.5.1], the algebra $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is isomorphic to

$$\mathcal{A}_n\{\operatorname{sgn}(\gamma_0), \dots, \operatorname{sgn}(\gamma_{n-1})\},$$

whence it is sufficient to consider $\gamma_k \in \{\pm 1\}$, $k = 0, \dots, n-1$, only.

Notation 2.6. For $m = 0, \dots, 2^n - 1$, we define

$$\delta_m^{(n)} = \prod_{l=0}^{n-1} (-\gamma_l)^{c_{m,l}},$$

where the exponents $c_{m,l} \in \{0, 1\}$ are the uniquely-defined coefficients of the binary representation

$$m = \sum_{l=0}^{n-1} c_{m,l} 2^l,$$

see [7, Proposition 3.18].

Lemma 2.7 ([6, Lemma 3.16])**.** *Let* $a = a_0 + a_1 e_1^{(n)} + \dots + a_{2^n-1} e_{2^n-1}^{(n)} \in \mathcal{A}_n$. *Then*

$$\bar{a} = a_0 - a_1 e_1^{(n)} - \dots - a_{2^n-1} e_{2^n-1}^{(n)};$$
$$\mathfrak{Re}(a) = a_0;$$
$$\mathfrak{Im}(a) = a_1 e_1^{(n)} + \dots + a_{2^n-1} e_{2^n-1}^{(n)};$$
$$n(a) = \sum_{m=0}^{2^n-1} \delta_m^{(n)} a_m^2.$$

Here, we consider conjugation in the sense of Definition 2.2*, whereas the norm and real and imaginary parts are understood in the sense of Definition* 2.4*.*

Notation 2.8. Given $a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}$ and $b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{A}_n$, define

$$\langle a, b \rangle = \sum_{m=0}^{2^n-1} \delta_m^{(n)} a_m b_m.$$

Proposition 2.9 ([6, Propositions 3.18 and 3.19])**.** *The form* $\langle a, b \rangle$ *is a real-valued symmetric bilinear form, which is associated with the quadratic form* $n(a)$*. Thus,* $\langle a, a \rangle = n(a)$ *and* $2\langle a, b \rangle = a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a$ *for all* $a, b \in \mathcal{A}_n$*.*

The following lemma describes the anticentralizer of an arbitrary nonzero pure element of $\mathcal{A}_n$.

Lemma 2.10 ([7, Lemma 5.8])**.** *Let* $a \in \mathcal{A}_n$*,* $\mathfrak{Re}(a) = 0$*,* $a \neq 0$*. Then*

$$\operatorname{Anc}_{\mathcal{A}_n}(a) = \{b \in \mathcal{A}_n \mid \mathfrak{Re}(b) = 0 \text{ and } \langle a, b \rangle = 0\}.$$

Now we proceed to some concepts related to associativity. The associator of $a, b, c \in \mathcal{A}$ is defined as the element $[a, b, c] = (ab)c - a(bc)$. From the definition of an algebra over a field it follows that the associator is a trilinear function of its arguments.

Definition 2.11 ([8, Definition 2.1.1]).

- *An algebra* $\mathcal{A}$ *is said to be* flexible *if the equality* $(ab)a = a(ba)$ *holds for all* $a, b \in \mathcal{A}$.
- *An element* $a \in \mathcal{A}$ *is said to be* alternative *if the equalities* $a(ax) = a^2x$ *and* $(xa)a = xa^2$ *hold for all* $x \in \mathcal{A}$.
- *An algebra* $\mathcal{A}$ *is said to be* alternative *if all its elements are alternative.*
- *Let* $\mathcal{A}$ *have a regular involution. Then* $\mathcal{A}$ *is called* a composition algebra *if the equality* $n(ab) = n(a)n(b)$ *holds for all* $a, b \in \mathcal{A}$.

Proposition 2.12 ([8, Exercise 2.1.1]). *If* $\mathcal{A}$ *is alternative, then the associator in* $\mathcal{A}$ *is skew-symmetric, that is, it changes sign if an argument transposition is performed.*

Lemma 2.13 ([1, p. 9]). *Let* $\mathcal{A}$ *be an alternative algebra with a regular involution. Then* $\mathcal{A}$ *is a composition algebra.*

Lemma 2.14 ([10, p. 436, p. 438, Theorem 1]).

- $\mathcal{A}_n$ *is commutative if and only if* $n \leq 1$.
- $\mathcal{A}_n$ *is associative if and only if* $n \leq 2$.
- $\mathcal{A}_n$ *is alternative if and only if* $n \leq 3$.
- $\mathcal{A}_n$ *is flexible for all* $n \in \mathbb{N} \cup \{0\}$ *and all* $\gamma_0, \ldots, \gamma_{n-1} \in \mathbb{R} \setminus \{0\}$.

Notation 2.15. Let $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathcal{A}_n$. Denote

$$\operatorname{Lin}(a_1, \ldots, a_m) = \mathbb{R}a_1 + \cdots + \mathbb{R}a_n,$$
$$\operatorname{Lin}^*(a_1, \ldots, a_m) = \operatorname{Lin}(a_1, \ldots, a_m) \setminus \{0\}.$$

2.3. Examples of real Cayley–Dickson algebras

Definition 2.16.

- *An algebra* $\mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\}$ *is said to be an algebra of* the main sequence *if* $\gamma_k = -1$ *for all* $k = 0, \ldots, n-1$. *We denote this algebra by* $\mathcal{M}_n$.
- *An algebra* $\mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\}$ *is called* a Cayley–Dickson split-algebra *if* $\gamma_k = -1$ *for all* $k = 0, \ldots, n-2$ *and* $\gamma_{n-1} = 1$. *We denote it by* $\mathcal{H}_n$ *because the norm on* $\mathcal{H}_n$ *is hyperbolic.*

Proposition 2.17 ([6, Proposition 3.31]).

- *Let* $a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}, b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{M}_n$. *Then* $\langle a, b\rangle = \sum_{m=0}^{2^n-1} a_m b_m$ *is the Euclidean inner product. Particularly,* $n(a) = \sum_{m=0}^{2^n-1} a_m^2$, *whence* $n(a) = 0$ *if and only if* $a = 0$.
- *Let* $a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}, b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{H}_n$. *Then*

$$\langle a, b\rangle = \sum_{m=0}^{2^{n-1}-1} a_m b_m - \sum_{m=2^{n-1}}^{2^n-1} a_m b_m.$$

Example 2.18.

- The complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), octonions ($\mathbb{O}$), and sedenions ($\mathbb{S}$) are algebras of the main sequence for $n = 1$, 2, 3, and 4, respectively. We refer the reader to [1] for the definitions of $\mathbb{H}$ and $\mathbb{O}$ and to [4] for that of $\mathbb{S}$.
- The split-complex numbers ($\widehat{\mathbb{C}}$), split-quaternions (coquaternions; $\widehat{\mathbb{H}}$), and split-octonions (hyperbolic octonions; $\widehat{\mathbb{O}}$) are examples of real low-dimensional split-algebras, all of them being defined in [2]. Yet another example is provided by the split-sedenions ($\widehat{\mathbb{S}}$), which have the same dimension as the sedenions.

Exact definitions and some basic properties of the algebras mentioned above are given below.

Definition 2.19.

- *The algebra of* the split-complex numbers *is* $\widehat{\mathbb{C}} = \mathcal{H}_1$*;*
- *the algebra of* the split-quaternions *is* $\widehat{\mathbb{H}} = \mathcal{H}_2$*;*
- *the algebra of* the split-octonions *is* $\widehat{\mathbb{O}} = \mathcal{H}_3$*;*
- *the algebra of* the split-sedenions *is* $\widehat{\mathbb{S}} = \mathcal{H}_4$*.*

Proposition 2.20.

- $\widehat{\mathbb{C}}$ *is both commutative and associative;*
- $\widehat{\mathbb{H}}$ *is noncommutative and associative;*
- $\widehat{\mathbb{O}}$ *is noncommutative, nonassociative, but alternative;*
- $\widehat{\mathbb{S}}$ *is noncommutative, nonassociative, and nonalternative.*

Proof. The assertions immediately follow from Lemma 2.14. □

Definition 2.21 ([1, p. 6])**.** *The algebra of* octonions $\mathbb{O}$ *is an eight-dimensional algebra over* $\mathbb{R}$*, and its basis elements are* $1, e_1, \dots, e_7$*. The involution in* $\mathbb{O}$ *is given by the formula* $\overline{a_0 + a_1e_1 + \dots + a_7e_7} = a_0 - a_1e_1 - \dots - a_7e_7$*, and multiplication is defined in Table 1.*

$\times$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 1. Multiplication table of the unit octonions.

Proposition 2.22 ([9, p. 1])**.** *The algebras* $\mathbb{O}$ *and* $\mathcal{M}_3$ *are isomorphic,* $\mathbb{O} \cong \mathcal{M}_3$*.*

This proposition implies that $\mathbb{O}$ is a noncommutative, nonassociative, but alternative algebra over $\mathbb{R}$ with the identity 1 and without zero divisors. The corollary below provides a convenient representation for the split-sedenions.

Corollary 2.23. *The algebras* $\widehat{\mathbb{S}}$ *and* $\mathbb{O}\{1\}$ *are isomorphic,* $\widehat{\mathbb{S}} \cong \mathbb{O}\{1\}$*.*

Proof. Indeed, by the definition of $\widehat{\mathbb{S}}$ and Proposition 2.22, we have

$$\widehat{\mathbb{S}} = \mathcal{H}_4 = \mathcal{M}_3\{1\} \cong \mathbb{O}\{1\}.$$

□

3. Doubly alternative zero divisors and low-dimensional split-algebras

By [6, Corollary 4.6], in the case of real Cayley–Dickson algebras, all zero divisors prove to be two-sided, that is, $Z(\mathcal{A}_n) = Z_{LR}(\mathcal{A}_n)$. Now consider the zero divisors $(a, b) \in \mathcal{A}_n$ such that both a and b are alternative elements in $\mathcal{A}_{n-1}$.

Definition 3.1. *The set of* doubly alternative elements *of* $\mathcal{A}_n$ *is defined as*

$$DA(\mathcal{A}_n) = \{(a, b) \in \mathcal{A}_n \mid \text{ both } a \text{ and } b \text{ are alternative elements in } \mathcal{A}_{n-1}\}.$$

Clearly, this definition makes sense for $n \geq 1$ only, which is a natural constraint. All elements of $\mathcal{A}_n$ are doubly alternative if and only if $n \leq 4$, see [6, Proposition 4.10]. Particularly, all elements of the split-complex numbers, split-quaternions, split-octonions, and split-sedenions are doubly alternative. Note that doubly alternative elements are not necessarily alternative, see [6, Lemma 4.16].

Lemma 3.2 ([6, Lemma 4.11]). *Let $(a, b) \in DA(\mathcal{H}_n) \setminus \{0\}$. Then $(a, b) \in Z(\mathcal{H}_n)$ if and only if $n((a, b)) = n(a) - n(b) = 0$.*

Corollary 3.3 ([6, Corollary 4.12]). *If $1 \leq n \leq 4$, then $Z(\mathcal{H}_n) = \{x \in \mathcal{H}_n \setminus \{0\} \mid n(x) = 0\}$.*

However, in general, the set of elements with zero norm is strictly smaller than the set of zero divisors, see [6, Proposition 4.13].

Consider an element $x \in Z(\mathcal{A}_n)$ such that $\mathfrak{Re}(x) \neq 0$ and x is not an isolated vertex in $\Gamma_O(\mathcal{A}_n)$. By [6, Proposition 4.19], we have $n(x) = 0$, and the connected component of $\Gamma_O(\mathcal{A}_n)$ that contains x is the complete bipartite graph, its parts being $\mathrm{Lin}^*(x)$ and $\mathrm{Lin}^*(\bar{x})$. Hence it is natural to introduce the following definition.

Definition 3.4.
- $Z_{\mathfrak{Im}}(\mathcal{A}_n) = \{x \in Z(\mathcal{A}_n) \mid \mathfrak{Re}(x) = 0\}$ *is the set of all zero divisors with zero real part.*
- $\Gamma_O^{\mathfrak{Im}}(\mathcal{A}_n)$ *is the subgraph of* $\Gamma_O(\mathcal{A}_n)$ *on the vertex set* $Z_{\mathfrak{Im}}(\mathcal{A}_n)$.

Lemma 3.5 ([6, Lemma 4.18, Lemma 4.21]).

- *Let $(a, b) \in DA(\mathcal{H}_n) \cap Z(\mathcal{H}_n)$. Then*

$$l.\,\mathrm{Ann}_{\mathcal{H}_n}((a,b)) = \left\{ \left(c, -\frac{(bc)a}{n(a)}\right) \;\middle|\; [a,c,b] = 0 \right\},$$
$$r.\,\mathrm{Ann}_{\mathcal{H}_n}((a,b)) = \left\{ \left(c, -\frac{(b\bar{c})\bar{a}}{n(a)}\right) \;\middle|\; [a,c,b] = 0 \right\}.$$

- *Let $(a, b) \in DA(\mathcal{H}_n) \cap Z_{\mathfrak{Im}}(\mathcal{H}_n)$. Then*

$$O_{\mathcal{H}_n}((a,b)) = \left\{ \left(c, -\frac{(bc)a}{n(a)}\right) \;\middle|\; \mathfrak{Re}(c) = 0, [a,c,b] = 0 \right\}.$$

The next theorem describes relation graphs of the split-quaternions and split-octonions in terms of their connected components and diameters. The counterparts for the split-sedenions are provided by Theorems 4.20 and 4.32.

Theorem 3.6 ([6, Theorems 5.6, 5.9, 5.30, and 5.32]). *The relation graphs of the split-quaternions can be characterized as follows:*

- *Every connected component of $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{H}})$ is a complete graph on the vertex set $\mathrm{Lin}^*(a)$, where $n(a) = 0$, $\mathfrak{Re}(a) = 0$;*
- *$\Gamma_Z(\widehat{\mathbb{H}})$ is connected, and its diameter equals 2.*

As to the split-octonions, their relation graphs have the following properties:

- *$\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{O}})$ is connected, and its diameter equals 3;*
- *$\Gamma_Z(\widehat{\mathbb{O}})$ is connected, and its diameter equals 2.*

Note also that Theorem 4.30 in [6] establishes a relationship between $\Gamma_O(\mathcal{H}_n)$ and $\Gamma_C(\mathcal{H}_n)$ for $2 \leq n \leq 4$; for this reason, in the present paper, we focus on $\Gamma_O(\widehat{\mathbb{S}})$ and $\Gamma_Z(\widehat{\mathbb{S}})$.

4.1. Main properties

Lemma 4.1 ([6, Lemma 5.13])**.** *Let $n \le 3$, $a, b \in \mathcal{A}_n$. Then the set* $\mathrm{Lin}(1, a, b, ab)$ *is closed under multiplication and conjugation.*

Lemma 4.2 ([3, Remark 4.7])**.** *Let $\{1, a, b\} \subset \mathbb{O}$ be an orthonormal system with respect to the inner product $\langle\cdot,\cdot\rangle$. Then there exists an isomorphism $\phi : \mathrm{Lin}(1, a, b, ab) \to \mathbb{H}$ such that $\phi(a) = i$, $\phi(b) = j$, and $\phi(ab) = k$. Particularly, $\{1, a, b, ab\}$ is an orthonormal system as well. We denote $\mathbb{H}\langle a, b\rangle = \mathrm{Lin}(1, a, b, ab)$.*

The next proposition establishes the associativity condition for an arbitrary triple of octonions.

Proposition 4.3. *Let $a, b \in \mathbb{O}$. Then the equation $[a, c, b] = 0$ for $c \in \mathbb{O}$ has the following solutions:*
(1) *if $1, a, b$ are linearly independent, then $[a, c, b] = 0$ if and only if $c \in \mathrm{Lin}(1, a, b, ab)$;*
(2) *otherwise the equality $[a, c, b] = 0$ holds for all $c \in \mathbb{O}$.*

Proof. Let $a' = \mathfrak{Im}(a)$. By b' we denote the orthogonal projection of b on $\mathrm{Lin}(1, a)^{\perp}$ with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle$. Since 1 is the identity of $\mathbb{O}$ and $\mathbb{O}$ is flexible, we have $[a, c, b] = [a', c, b']$. Now consider two cases:
(1) If $1, a, b$ are linearly independent, then $a' \neq 0$ and $b' \neq 0$. By Lemma 8.5 in [3], $[a', c, b'] = 0$ if and only if $c \in \mathrm{Lin}(1, a', b', a'b') = \mathrm{Lin}(1, a, b, ab)$. Moreover, Lemma 4.2 implies that $\dim(\mathrm{Lin}(1, a, b, ab)) = \dim(\mathrm{Lin}(1, a', b', a'b')) = 4$.
(2) Otherwise we have $a' = 0$ or $b' = 0$, whence the equality $[a', c, b'] = 0$ holds trivially for all $c \in \mathbb{O}$. □

In view of Proposition 4.3, it is natural to introduce the following subset of $\widehat{\mathbb{S}}$.

Notation 4.4. $LD(\widehat{\mathbb{S}}) = \{(a, b) \in \widehat{\mathbb{S}} \mid 1, a, b \text{ are linearly dependent}\}$.

Corollary 4.5. *Let $(a, b) \in Z(\widehat{\mathbb{S}})$.*
(1) *If $(a, b) \notin LD(\widehat{\mathbb{S}})$, then $\dim(l.\mathrm{Ann}_{\widehat{\mathbb{S}}}((a, b))) = \dim(r.\mathrm{Ann}_{\widehat{\mathbb{S}}}((a, b))) = 4$;*
(2) *If $(a, b) \in LD(\widehat{\mathbb{S}})$, then $\dim(l.\mathrm{Ann}_{\widehat{\mathbb{S}}}((a, b))) = \dim(r.\mathrm{Ann}_{\widehat{\mathbb{S}}}((a, b))) = 8$.*

Proof. The assertions immediately follow from Lemma 3.5 and Proposition 4.3. □

Corollary 4.6. *Let $(a, b) \in Z_{\mathfrak{Im}}(\widehat{\mathbb{S}})$.*
(1) *If $(a, b) \notin LD(\widehat{\mathbb{S}})$, then $\dim(O_{\widehat{\mathbb{S}}}((a, b))) = 3$;*
(2) *If $(a, b) \in LD(\widehat{\mathbb{S}})$, then $\dim(O_{\widehat{\mathbb{S}}}((a, b))) = 7$.*

Proof. Both assertions immediately follow from Lemma 3.5 and Proposition 4.3. □

4.2. Lower bounds for the diameters of $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ and $\Gamma_Z(\widehat{\mathbb{S}})$. In this section, we construct pairs of elements that are the most distant from each other in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ and $\Gamma_Z(\widehat{\mathbb{S}})$.

Lemma 4.7. *Let $(a, b) \in Z(\widehat{\mathbb{S}})$, $(a, b) \notin LD(\widehat{\mathbb{S}})$. Also let P be a path in $\Gamma_Z(\widehat{\mathbb{S}})$ that starts or ends at (a, b) and contains no inner elements from $LD(\widehat{\mathbb{S}})$. Then all vertices of P belong to* $\mathrm{Lin}(1, a, b, ab) \times \mathrm{Lin}(1, a, b, ab)$*, that is, the Cartesian square of* $\mathrm{Lin}(1, a, b, ab)$.

Proof. Assume, without loss of generality, that P starts at (a, b). Then P is of the form

$$P_n : (a_0, b_0) \longrightarrow (a_1, b_1) \longrightarrow \cdots \longrightarrow (a_n, b_n),$$

where $(a_0, b_0) = (a, b)$, and n is the length of P. We give a proof by induction on n.
• If $n = 0$, then the only element of P is $(a, b) \in \mathrm{Lin}(1, a, b, ab) \times \mathrm{Lin}(1, a, b, ab)$.
• Assume that the assertion has already been proved for $n = k$. Show that it also holds for $n = k + 1$. Indeed, let $P = P_{k+1}$ and assume that its inner elements satisfy $(a_1, b_1), \dots, (a_k, b_k) \notin LD(\widehat{\mathbb{S}})$. Then P_k also starts at (a, b) and contains no inner elements from $LD(\widehat{\mathbb{S}})$. Therefore, by the induction hypothesis, $(a_0, b_0), \dots, (a_k, b_k) \in \mathrm{Lin}(1, a, b, ab) \times \mathrm{Lin}(1, a, b, ab)$. Hence it remains to show that $(a_{k+1}, b_{k+1}) \in \mathrm{Lin}(1, a, b, ab) \times \mathrm{Lin}(1, a, b, ab)$.

We have $(a_k, b_k) \in \mathrm{Lin}(1, a, b, ab) \times \mathrm{Lin}(1, a, b, ab)$ and $(a_k, b_k) \notin LD(\widehat{\mathbb{S}})$. By Lemma 3.5 and Proposition 4.3, from $(a_{k+1}, b_{k+1}) \in r.\,\mathrm{Ann}_{\widehat{\mathbb{S}}}((a_k, b_k))$ it follows that $a_{k+1} \in \mathrm{Lin}(1, a_k, b_k, a_k b_k)$ and $b_{k+1} = -\frac{(b_k \overline{a_{k+1}})\overline{a_k}}{n(a_k)}$. Then, by Lemma 4.1, $b_{k+1} \in \mathrm{Lin}(1, a_k, b_k, a_k b_k)$. By using Lemma 4.1 once again, we obtain that $\mathrm{Lin}(1, a_k, b_k, a_k b_k) \subseteq \mathrm{Lin}(1, a, b, ab)$, implying that $(a_{k+1}, b_{k+1}) \in \mathrm{Lin}(1, a, b, ab) \times \mathrm{Lin}(1, a, b, ab)$. □

Below, we will use the following notation. Its properties are described in Propositions 4.9, 4.11, and 4.12.

Notation 4.8. Let $\{1, a, b\} \subset \mathbb{O}$ be an orthonormal system with respect to the inner product $\langle \cdot, \cdot \rangle$. Then $E_{a,b} = (\sqrt{2}a, 1 + b)$.

Proposition 4.9. *$E_{a,b} \in Z(\widehat{\mathbb{S}})$, and we have*

$$O_{\widehat{\mathbb{S}}}(E_{a,b}) = \mathrm{Lin}\Big((\sqrt{2}a, 1 + b), (\sqrt{2}b, a + ab), (\sqrt{2}ab, 1 - b)\Big);$$
$$l.\,\mathrm{Ann}_{\widehat{\mathbb{S}}}(E_{a,b}) = O_{\widehat{\mathbb{S}}}(E_{a,b}) \oplus \mathrm{Lin}\Big((-\sqrt{2}, a - ab)\Big);$$
$$r.\,\mathrm{Ann}_{\widehat{\mathbb{S}}}(E_{a,b}) = O_{\widehat{\mathbb{S}}}(E_{a,b}) \oplus \mathrm{Lin}\Big((\sqrt{2}, a - ab)\Big).$$

Proof. Corollary 3.3 implies that $E_{a,b} \in Z(\widehat{\mathbb{S}})$ because $n(E_{a,b}) = n(\sqrt{2}a) - n(1 + b) = 2n(a) - (n(1) + n(b)) = 2 - (1 + 1) = 0$. By Lemma 4.2, a and b form a quaternionic subalgebra $\mathbb{H}\langle a, b \rangle \subset \mathbb{O}$. Then the exact expressions for the orthogonalizer and annihilators of $E_{a,b}$ can be obtained from Lemma 3.5 by taking into account the fact that $E_{a,b} \notin LD(\widehat{\mathbb{S}})$. □

Notation 4.10. Denote

$$F_{a,b} = (a + ab, \sqrt{2});$$
$$G_{a,b}^{\alpha,\beta} = \Big(-\sqrt{2}\beta^3 + \alpha(\alpha^2 + \beta^2)a + \sqrt{2}\alpha^2\beta b + \alpha(\alpha^2 - \beta^2)ab,$$
$$\sqrt{2}\alpha^3 + \beta(\alpha^2 + \beta^2)a + \sqrt{2}\alpha\beta^2 b + \beta(\alpha^2 - \beta^2)ab \Big);$$
$$H_{a,b}^{\alpha,\beta} = \Big(\sqrt{2}\beta^3 + \alpha(\alpha^2 + \beta^2)a + \sqrt{2}\alpha^2\beta b + \alpha(\alpha^2 - \beta^2)ab,$$
$$\sqrt{2}\alpha^3 + \beta(\alpha^2 + \beta^2)a + \sqrt{2}\alpha\beta^2 b + \beta(\alpha^2 - \beta^2)ab \Big).$$

Proposition 4.11. *The following equalities hold:*

$$O_{\widehat{\mathbb{S}}}(E_{a,b}) \cap LD(\widehat{\mathbb{S}}) = \mathrm{Lin}(F_{a,b});$$
$$l.\,\mathrm{Ann}_{\widehat{\mathbb{S}}}(E_{a,b}) \cap LD(\widehat{\mathbb{S}}) = \Big\{ G_{a,b}^{\alpha,\beta} \;\big|\; (\alpha, \beta) \in \mathbb{R}^2 \Big\};$$
$$r.\,\mathrm{Ann}_{\widehat{\mathbb{S}}}(E_{a,b}) \cap LD(\widehat{\mathbb{S}}) = \Big\{ H_{a,b}^{\alpha,\beta} \;\big|\; (\alpha, \beta) \in \mathbb{R}^2 \Big\}.$$

Proof. We will only prove the equality for $r.\operatorname{Ann}_{\widehat{\mathbb{S}}}(E_{a,b})$ because the proof for $l.\operatorname{Ann}_{\widehat{\mathbb{S}}}(E_{a,b})$ is entirely similar, whereas the equality for $O_{\widehat{\mathbb{S}}}(E_{a,b})$ can be obtained from the previous two by setting $\beta = 0$ and $\alpha = 1$.

By Proposition 4.9, an arbitrary element of $r.\operatorname{Ann}_{\widehat{\mathbb{S}}}(E_{a,b})$ is of the form

$$\begin{aligned} A &= \kappa_1(\sqrt{2}, a - ab) + \kappa_2(\sqrt{2}a, 1 + b) + \kappa_3(\sqrt{2}b, a + ab) + \kappa_4(\sqrt{2}ab, 1 - b) \\ &= \Big(\sqrt{2}(\kappa_1 + \kappa_2 a + \kappa_3 b + \kappa_4 ab), (\kappa_2 + \kappa_4) + (\kappa_3 + \kappa_1)a + (\kappa_2 - \kappa_4)b + (\kappa_3 - \kappa_1)ab\Big). \end{aligned}$$

From Lemma 4.2 it follows that $1, a, b, ab$ are linearly independent. Then, by the definition of $LD(\widehat{\mathbb{S}})$, $A \in LD(\widehat{\mathbb{S}})$ if and only if

$$\operatorname{rank}\begin{pmatrix} \kappa_2 & \kappa_3 & \kappa_4 \\ \kappa_3 + \kappa_1 & \kappa_2 - \kappa_4 & \kappa_3 - \kappa_1 \end{pmatrix} \le 1.$$

This inequality is equivalent to the system

$$\begin{cases} \Delta_{1,2} = \kappa_2(\kappa_2 - \kappa_4) - \kappa_3(\kappa_3 + \kappa_1) = 0; \\ \Delta_{3,2} = \kappa_4(\kappa_2 - \kappa_4) - \kappa_3(\kappa_3 - \kappa_1) = 0; \\ \Delta_{1,3} = \kappa_2(\kappa_3 - \kappa_1) - \kappa_4(\kappa_3 + \kappa_1) = 0 \end{cases}$$

to be solved.

Let $\kappa_1 = \beta^3$ and let $\kappa_2 + \kappa_4 = \sqrt{2}\alpha^3$. We have $\Delta_{1,2} - \Delta_{3,2} = (\kappa_2 - \kappa_4)^2 - 2\kappa_1\kappa_3 = 0$. Consider the following two cases:

- If $\kappa_1 = 0$, that is, $\beta = 0$, then $\kappa_2 = \kappa_4 = \frac{\alpha^3}{\sqrt{2}}$. Thus, $\Delta_{1,2} = 0$ implies $\kappa_3 = 0$, whence $A = \alpha^3(a + ab, \sqrt{2}) = H^{\alpha,0}_{a,b}$.
- Otherwise $\kappa_1 \neq 0$, whence $\kappa_3 = \frac{(\kappa_2-\kappa_4)^2}{2\kappa_1}$. We also have $\Delta_{1,2} + \Delta_{3,2} = (\kappa_2 + \kappa_4)(\kappa_2 - \kappa_4) - 2\kappa_3^2 = 0$. Denote $x = \kappa_2 - \kappa_4$. Then $\kappa_3 = \frac{x^2}{2\kappa_1}$, implying that $\Delta_{1,2} + \Delta_{3,2} = (\kappa_2 + \kappa_4)x - \frac{x^4}{2\kappa_1^2} = 0$. Here, two cases are possible.
  - ◇ If $x = 0$, then $\kappa_3 = 0$, whence $\Delta_{1,3} = -(\kappa_2 + \kappa_4)\kappa_1 = 0$. Since $\kappa_1 \neq 0$, we infer that $\kappa_2 + \kappa_4 = 0$, that is, $\alpha = 0$ and $\kappa_2 = \kappa_4 = 0$. Then $A = \beta^3(\sqrt{2}, a - ab) = H^{0,\beta}_{a,b}$.
  - ◇ If $x \neq 0$, then $x^3 = 2\kappa_1^2(\kappa_2 + \kappa_4) = (\sqrt{2}\alpha\beta^2)^3$, implying that $x = \sqrt{2}\alpha\beta^2$. Then $\kappa_3 = \frac{x^2}{2\kappa_1} = \alpha^2\beta$. Moreover, $\kappa_2 = \frac{\sqrt{2}\alpha^3 + x}{2} = \frac{\alpha(\alpha^2+\beta^2)}{\sqrt{2}}$ and $\kappa_4 = \frac{\sqrt{2}\alpha^3 - x}{2} = \frac{\alpha(\alpha^2-\beta^2)}{\sqrt{2}}$. Thus, $A = H^{\alpha,\beta}_{a,b}$. □

Proposition 4.12. *The following equality holds:*

$$O_{\widehat{\mathbb{S}}}(F_{a,b}) = \left\{ \left(c, -\frac{c(a + ab)}{\sqrt{2}}\right) \;\middle|\; \mathfrak{Re}(c) = 0 \right\}.$$

Proof. This follows from Lemma 3.5 because $F_{a,b} \in LD(\widehat{\mathbb{S}})$. □

Now we use this construction in Lemma 4.14 in order to obtain a lower bound for the diameter of $\Gamma_Z(\widehat{\mathbb{S}})$.

Proposition 4.13. *Let $a = a' = e_1$, $b = e_2$, $b' = e_4$. Then a, b satisfy the assumptions of Lemma 4.2 and $\mathbb{H}\langle a, b\rangle = \operatorname{Lin}(1, e_1, e_2, e_3)$. Similarly, a', b' satisfy the assumptions of Lemma 4.2 and $\mathbb{H}\langle a', b'\rangle = \operatorname{Lin}(1, e_1, e_4, e_5)$. Finally, $\mathbb{H}\langle a, b\rangle \cap \mathbb{H}\langle a', b'\rangle = \operatorname{Lin}(1, e_1)$.*

Proof. The assertions are verified straightforwardly because $ab = e_3$ and $a'b' = e_5$. □

Lemma 4.14. *Let a, b, a', b' be as defined in Proposition 4.13. Then $d_{\Gamma_Z(\widehat{\mathbb{S}})}(E_{a,b}, E_{a',b'}) \ge 4$.*

Proof. Since $b' \notin \mathbb{H}\langle a,b\rangle$, from Lemma 4.7 it follows that any path connecting $E_{a,b}$ and $E_{a',b'}$ contains at least one element $A \in (\mathbb{H}\langle a,b\rangle \times \mathbb{H}\langle a,b\rangle) \cap LD(\widehat{\mathbb{S}})$. Similarly, $b \notin \mathbb{H}\langle a',b'\rangle$, whence any path connecting $E_{a,b}$ and $E_{a',b'}$ contains at least one element $A' \in (\mathbb{H}\langle a',b'\rangle \times \mathbb{H}\langle a',b'\rangle) \cap LD(\widehat{\mathbb{S}})$.

Assume that $d_{\Gamma_Z(\widehat{\mathbb{S}})}(E_{a,b}, E_{a',b'}) \leq 3$. By Proposition 4.13, $\mathbb{H}\langle a,b\rangle \cap \mathbb{H}\langle a',b'\rangle = \mathrm{Lin}(1, e_1)$, implying that

$$(\mathbb{H}\langle a,b\rangle \times \mathbb{H}\langle a,b\rangle) \cap (\mathbb{H}\langle a',b'\rangle \times \mathbb{H}\langle a',b'\rangle) = \mathrm{Lin}(1,e_1) \times \mathrm{Lin}(1,e_1) = X.$$

However, from Proposition 4.9 it follows that $r.\mathrm{Ann}_{\widehat{\mathbb{S}}}(E_{a,b}) \cap X = l.\mathrm{Ann}_{\widehat{\mathbb{S}}}(E_{a',b'}) \cap X = \{0\}$. Thus, $A \neq A'$, and the path is of the form $E_{a,b} \longrightarrow A \longrightarrow A' \longrightarrow E_{a',b'}$. By Proposition 4.11, we have $A = H^{\alpha,\beta}_{a,b}$ and $A' = G^{\gamma,\delta}_{a',b'}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Now consider AA' as a pair of octonions. Then the condition $AA' = 0$ implies that the first component of AA' vanishes:

$$\begin{aligned}
&\Big(\sqrt{2}\beta^3 + \alpha(\alpha^2+\beta^2)e_1 + \sqrt{2}\alpha^2\beta e_2 + \alpha(\alpha^2-\beta^2)e_3\Big)\\
&\qquad \times \Big(-\sqrt{2}\delta^3 + \gamma(\gamma^2+\delta^2)e_1 + \sqrt{2}\gamma^2\delta e_4 + \gamma(\gamma^2-\delta^2)e_5\Big)\\
&+\Big(\sqrt{2}\gamma^3 - \delta(\gamma^2+\delta^2)e_1 - \sqrt{2}\gamma\delta^2 e_4 - \delta(\gamma^2-\delta^2)e_5\Big)\\
&\qquad \times \Big(\sqrt{2}\alpha^3 + \beta(\alpha^2+\beta^2)e_1 + \sqrt{2}\alpha\beta^2 e_2 + \beta(\alpha^2-\beta^2)e_3\Big) = 0.
\end{aligned}$$

If $\alpha = 0$, then we may set $\beta = 1$ because $A \neq 0$. Hence this equation reads as

$$\begin{aligned}
&\sqrt{2}\Big(-\sqrt{2}\delta^3 + \gamma(\gamma^2+\delta^2)e_1 + \sqrt{2}\gamma^2\delta e_4 + \gamma(\gamma^2-\delta^2)e_5\Big)\\
&\qquad + \Big(\sqrt{2}\gamma^3 - \delta(\gamma^2+\delta^2)e_1 - \sqrt{2}\gamma\delta^2 e_4 - \delta(\gamma^2-\delta^2)e_5\Big)(e_1 - e_3) = 0,
\end{aligned}$$

where the coefficient at $e_7 = e_3e_4$ equals $-\sqrt{2}\gamma\delta^2$. Thus, either $\gamma = 0$ or $\delta = 0$.

(1) If $\gamma = 0$, then we may set $\delta = 1$, but $-2 - (e_1 - e_5)(e_1 - e_3) \neq 0$.
(2) If $\delta = 0$, then we may set $\gamma = 1$, but $\sqrt{2}(e_1 + e_5) + \sqrt{2}(e_1 - e_3) \neq 0$.

In both cases, we have obtained contradictions. Thus, $\alpha \neq 0$.

Similarly, $\beta \neq 0$, $\gamma \neq 0$, and $\delta \neq 0$. Consequently, we may divide our equation by $\alpha^3\delta^3$ and write it in the new variables $\kappa = \dfrac{\beta}{\alpha}$ and $\lambda = \dfrac{\gamma}{\delta}$ as

$$\begin{aligned}
&\Big(\sqrt{2}\kappa^3 + (1+\kappa^2)e_1 + \sqrt{2}\kappa e_2 + (1-\kappa^2)e_3\Big)\\
&\qquad \times \Big(-\sqrt{2} + \lambda(\lambda^2+1)e_1 + \sqrt{2}\lambda^2 e_4 + \lambda(\lambda^2-1)e_5\Big)\\
&+\Big(\sqrt{2}\lambda^3 - (\lambda^2+1)e_1 - \sqrt{2}\lambda e_4 - (\lambda^2-1)e_5\Big)\\
&\qquad \times \Big(\sqrt{2} + \kappa(1+\kappa^2)e_1 + \sqrt{2}\kappa^2 e_2 + \kappa(1-\kappa^2)e_3\Big) = 0.
\end{aligned}$$

The coefficient at $1 = 1^2 = -e_1^2$ equals $K_0 = (\kappa - \lambda)((\kappa^2-1)(\lambda^2-1) - 2\kappa\lambda)$, whereas the coefficient at $e_6 = e_2e_4 = e_5e_3$ equals $K_6 = (\kappa+\lambda)((\kappa^2-1)(\lambda^2-1) + 2\kappa\lambda)$. Since $K_0 = K_6 = 0$, one of the following conditions is fulfilled:

(1) If $\kappa = \lambda$, then $K_6 = 2\lambda((\lambda^2-1)^2 + 2\lambda^2) = 0$, implying that $\kappa = \lambda = 0$.
(2) If $\kappa = -\lambda$, then $K_0 = -2\lambda((\lambda^2-1)^2 + 2\lambda^2) = 0$, implying that $\kappa = \lambda = 0$.
(3) Otherwise we have $(\kappa^2-1)(\lambda^2-1) - 2\kappa\lambda = (\kappa^2-1)(\lambda^2-1) + 2\kappa\lambda = 0$, implying that $\kappa\lambda = 0$, whence either $\kappa = 0$ or $\lambda = 0$.

However, $\kappa = 0$ implies $\beta = 0$, and $\lambda = 0$ implies $\gamma = 0$. Then, as has been shown above, we obtain a contradiction.

Thus, $d_{\Gamma_Z(\widehat{\mathbb{S}})}((a,b),(a',b')) \geq 4$. $\square$

The same argument applies to the lower bound of the diameter of $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ in Lemma 4.17.

Proposition 4.15. *Let $a = e_1$, $b = e_2$, $a' = \frac{e_1+e_4}{\sqrt{2}}$, $b' = \frac{e_2+e_5}{\sqrt{2}}$. Then a, b satisfy the assumptions of Lemma 4.2, and $\mathbb{H}\langle a, b\rangle = \mathrm{Lin}(1, e_1, e_2, e_3)$. Similarly, a', b' satisfy the assumption of Lemma 4.2, and $\mathbb{H}\langle a', b'\rangle = \mathrm{Lin}(1, e_1+e_4, e_2+e_5, e_1+e_3-e_4-e_6)$. Finally, $\mathbb{H}\langle a,b\rangle \cap \mathbb{H}\langle a',b'\rangle = \mathbb{R}$.*

Proof. The assertions are verified straightforwardly using the relations $ab = e_3$ and $a'b' = \frac{e_1+e_3-e_4-e_6}{2}$. $\square$

Proposition 4.16. *Let a, b, a', b' be as defined in Proposition 4.15. Then $O_{\widehat{\mathbb{S}}}(F_{a,b}) \cap (\mathbb{H}\langle a',b'\rangle \times \mathbb{H}\langle a',b'\rangle) = \{0\}$.*

Proof. By Proposition 4.12,

$$O_{\widehat{\mathbb{S}}}(F_{a,b}) = \left\{ \left(c, -\frac{c(a+ab)}{\sqrt{2}}\right) \,\middle|\, \mathfrak{Re}(c) = 0 \right\}.$$

Suppose there exists a certain $c \in \mathbb{H}\langle a',b'\rangle$, $c \neq 0$, such that $d = -\frac{c(a+ab)}{\sqrt{2}} \in \mathbb{H}\langle a',b'\rangle$. Since $\mathbb{O}$ is alternative, we have $a + ab = -\frac{\sqrt{2}}{n(c)}\bar{c}d$. Then from Lemma 4.1 it follows that $\bar{c}d \in \mathbb{H}\langle a',b'\rangle$, whence $a + ab \in \mathbb{H}\langle a',b'\rangle$, a contradiction. $\square$

Lemma 4.17. *Let a, b, a', b' be as defined in Proposition 4.15. Then*

$$d_{\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})}(E_{a,b}, E_{a',b'}) \geq 5.$$

Proof. Any path in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ is a path in $\Gamma_Z(\widehat{\mathbb{S}})$ as well, whence Lemma 4.7 is applicable. Since $b' \notin \mathbb{H}\langle a,b\rangle$, any path connecting $E_{a,b}$ and $E_{a',b'}$ at least contains one element $A \in (\mathbb{H}\langle a,b\rangle \times \mathbb{H}\langle a,b\rangle) \cap LD(\widehat{\mathbb{S}})$. Similarly, $b \notin \mathbb{H}\langle a',b'\rangle$, implying that any path connecting $E_{a,b}$ and $E_{a',b'}$ at least contains one element $A' \in (\mathbb{H}\langle a',b'\rangle \times \mathbb{H}\langle a',b'\rangle) \cap LD(\widehat{\mathbb{S}})$.

Suppose $d_{\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})}(E_{a,b}, E_{a',b'}) \leq 4$. By Proposition 4.15, $\mathbb{H}\langle a,b\rangle \cap \mathbb{H}\langle a',b'\rangle = \mathbb{R}$; therefore, $(\mathbb{H}\langle a,b\rangle \times \mathbb{H}\langle a,b\rangle) \cap (\mathbb{H}\langle a',b'\rangle \times \mathbb{H}\langle a',b'\rangle) = \mathbb{R} \times \mathbb{R}$. Since $\mathfrak{Re}(A) = \mathfrak{Re}(A') = 0$, $A \neq 0$, and $A' \neq 0$, we infer that $A \neq A'$. Then, without loss of generality, we may assume that $E_{a,b}$ is immediately followed by A. By Proposition 4.11, we may take $A = F_{a,b}$. Proposition 4.16 implies that $O_{\widehat{\mathbb{S}}}(F_{a,b}) \cap (\mathbb{H}\langle a',b'\rangle \times \mathbb{H}\langle a',b'\rangle) = \{0\}$. Thus, $d_{\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})}(E_{a,b}, E_{a',b'}) = 4$, and the path is of the form $E_{a,b} \longleftrightarrow A \longleftrightarrow C \longleftrightarrow A' \longleftrightarrow E_{a',b'}$ for some $C \in Z(\widehat{\mathbb{S}})$. By Proposition 4.11, we may set $A' = F_{a',b'}$.

Let $C = (c,d)$. By Proposition 4.12, $C \in O_{\widehat{\mathbb{S}}}(F_{a,b})$ implies $d = -\frac{c(a+ab)}{\sqrt{2}}$. Similarly, $C \in O_{\widehat{\mathbb{S}}}(F_{a',b'})$ implies $d = -\frac{c(a'+a'b')}{\sqrt{2}}$. Then $c(a+ab) = c(a'+a'b')$, whence $c(a+ab-a'-a'b') = 0$. However, $a + ab - a' - a'b' \neq 0$, and $\mathbb{O}$ has no zero divisors. Therefore, $c = 0$ and $C = 0$, a contradiction.

Thus, $d_{\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})}((a,b),(a',b')) \geq 5$. $\square$

4.3. The zero divisor graph of the split-sedenions

Lemma 4.18 ([5, Lemma 5.1])**.** *Let $n \geq 1$, $a \in \mathcal{M}_n$. Then there exists $b \in \mathcal{M}_n$ such that $b^2 = a$.*

Lemma 4.19. *The diameter of* $\Gamma_Z(\widehat{\mathbb{S}})$ *is at most* 4.

Proof. Let $(a,b),(a',b') \in Z(\widehat{\mathbb{S}})$. By Corollary 3.3, we have $n((a,b)) = n((a',b')) = 0$, and, without loss of generality, we may assume that $n(a) = n(b) = n(a') = n(b') = 1$. Now we look for a path of length 4 from (a,b) to (a',b') in the form

$$(a,b) \longrightarrow (1,c) \longrightarrow (x,y) \longrightarrow (1,d) \longrightarrow (a',b').$$

- Consider $c = -b\bar{a}$, $d = -b'a'$. Since $\mathbb{O}$ is a composition algebra, we have $n(c) = n(d) = 1$. By Lemma 3.5, $(a,b)(1,c) = 0$ and $(1,d)(a',b') = 0$.
- We are going to find $(x,y) \in Z(\widehat{\mathbb{S}})$ such that $(1,c)(x,y) = (x,y)(1,d) = 0$. By Lemma 3.5, this condition is equivalent to $y = -c\bar{x} = -dx$. From Lemma 4.18 it follows that there exists $x \in \mathbb{O}$ such that $x^2 = \bar{d}c$. Since $\mathbb{O}$ is a composition algebra and $1 = n(d)n(c) = n(\bar{d})n(c) = n(\bar{d}c) = n(x^2) = (n(x))^2$, we conclude that $n(x) = 1$. Then, by the alternativity of $\mathbb{O}$, $x^2 = \bar{d}c$ implies that $(dx)x = dx^2 = d(\bar{d}c) = (d\bar{d})c = n(d)c = c$. Thus, $dx = (dx)n(x) = (dx)(x\bar{x}) = ((dx)x)\bar{x} = c\bar{x}$, as desired. Now we may set $y = -c\bar{x} = -dx$ and complete the proof. □

Theorem 4.20. *The diameter of* $\Gamma_Z(\widehat{\mathbb{S}})$ *equals* 4.

Proof. This follows directly from Lemmas 4.19 and 4.14. □

4.4. The orthogonality graph of the split-sedenions. The results 4.21–4.23 below play a key role in constructing shortest paths in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$.

Lemma 4.21. *Let* $(a,b) \in Z(\widehat{\mathbb{S}})$, $\mathfrak{Re}(a) = \mathfrak{Re}(b) = 0$, *that is,* (a,b) *is doubly pure. Then* $(a+b,a+b),(a-b,-(a-b)) \in O_{\widehat{\mathbb{S}}}((a,b))$.

Proof. First we show that $(a,b),(b,a) \in O_{\widehat{\mathbb{S}}}((a,b))$. Indeed, $(a,b) = -\overline{(a,b)} \in O_{\widehat{\mathbb{S}}}((a,b))$ because $n((a,b)) = n(a) - n(b) = 0$. Also we have $-\frac{(bb)a}{n(a)} = \frac{n(b)a}{n(a)} = a$, and from Lemma 3.5 for $c = b$ it follows that $(b,a) \in O_{\widehat{\mathbb{S}}}((a,b))$.

Thus, $(a+b,a+b) = (a,b) + (b,a) \in O_{\widehat{\mathbb{S}}}((a,b))$ and $(a-b,-(a-b)) = (a,b) - (b,a) \in O_{\widehat{\mathbb{S}}}((a,b))$. □

Lemma 4.22. *Let* $a \in \mathbb{O}$, $n(a) = 1$, $\mathfrak{Re}(a) = 0$. *Then, for any* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha^2 + \beta^2 = 1$, *we have*

$$O_{\widehat{\mathbb{S}}}((a,\alpha+\beta a)) = \mathbb{R}(a,\alpha+\beta a) \oplus \{(b,(\alpha a-\beta)b) \mid b \in \mathrm{Anc}_{\mathbb{O}}(a)\}.$$

Proof. Let $c \in \mathbb{O}$, $\mathfrak{Re}(c) = 0$. Then c has a unique representation in the form $c = ka+b$ for some $k \in \mathbb{R}$ and $b \in \mathbb{O}$ with $\langle a,b\rangle = 0$. Since $\mathfrak{Re}(b) = \mathfrak{Re}(c-ka) = 0$, from Lemma 2.10 it follows that $b \in \mathrm{Anc}_{\mathbb{O}}(a)$. Thus, the assertion of the lemma immediately follows from Lemma 3.5 because $-((\alpha+\beta a)a)a = -(\alpha+\beta a)(aa) = (\alpha+\beta a)$, $-((\alpha+\beta a)b)a = -(\alpha+\beta a)(ba) = (\alpha+\beta a)(ab) = ((\alpha+\beta a)a)b = (\alpha a-\beta)b$. □

Corollary 4.23. *Let* $a \in \mathbb{O}$, $a \neq 0$, $\mathfrak{Re}(a) = 0$. *Then*

$$O_{\widehat{\mathbb{S}}}((a,a)) = \mathbb{R}(a,a) \oplus \{(b,-b) \mid b \in \mathrm{Anc}_{\mathbb{O}}(a)\},$$
$$O_{\widehat{\mathbb{S}}}((a,-a)) = \mathbb{R}(a,-a) \oplus \{(b,b) \mid b \in \mathrm{Anc}_{\mathbb{O}}(a)\}.$$

Proof. Set $\alpha = 0$, $\beta = \pm 1$ in Lemma 4.22. □

Our next purpose is to obtain an upper bound for the diameter of $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$, and the desired result is presented in Lemma 4.31. The proof of Lemma 4.31 splits into the following steps:

(1) In Lemma 4.25, for Lemma 4.21 to be applicable, we find doubly pure neighbors of an arbitrary zero divisor.

(2) Corollary 4.28 establishes the existence of elements of the form $(a, \pm a)$ at a distance at most 2 from an arbitrary zero divisor.
(3) Then, in Lemma 4.31, these elements are connected by a path using Corollary 4.23.

Proposition 4.24. *Let $(a,b) \in Z(\widehat{\mathbb{S}})$, $\mathfrak{Re}(a) = 0$, $n(a) = n(b) = 1$. Then $b = b_0 + b_1 a + b_2 c$ for some $c \in \mathrm{Lin}(1,a)^{\perp}$, $n(c) = 1$, and $b_0, b_1, b_2 \in \mathbb{R}$, $b_0^2 + b_1^2 + b_2^2 = 1$. Consider $d = ac$. Then a and c generate the quaternionic subalgebra $\mathrm{Lin}(1,a,c,d) = \mathbb{H}\langle a,c\rangle \subset \mathbb{O}$. Particularly, $\{1,a,c,d\}$ is an orthonormal system with respect to the inner product $\langle\cdot,\cdot\rangle$.*

Proof. The assertions immediately follow from Lemma 4.2. □

Lemma 4.25. *Let a,b,c,d be as in Proposition 4.24. Then*
(1) *$(da, bd) = (c, b_2 a - b_1 c + b_0 d)$ and $(db, ad) = -(b_2 a - b_1 c - b_0 d, c)$ are doubly pure;*
(2) *$(da, bd), (db, ad) \in O_{\widehat{\mathbb{S}}}((a,b))$.*

Proof. (1) From Proposition 4.24 it follows that $\mathfrak{Re}(c) = \mathfrak{Re}(d) = 0$, $da = -ad = c$, $cd = -dc = a$. Then

$$db = d(b_0 + b_1 a + b_2 c) = b_0 d + b_1 c - b_2 a,$$
$$bd = (b_0 + b_1 a + b_2 c)d = b_0 d - b_1 c + b_2 a.$$

Since a,c,d are pure, we conclude that $(da, bd) = (c, b_2 a - b_1 c + b_0 d)$ and $(db, ad) = -(b_2 a - b_1 c - b_0 d, c)$ are doubly pure.
(2) Now we use Lemma 3.5 and show that $(da, bd), (db, ad) \in O_{\widehat{\mathbb{S}}}((a,b))$ as follows:
- $\mathfrak{Re}(da) = \mathfrak{Re}(db) = 0$;
- $\mathbb{H}\langle a,c\rangle$ is associative, and $a,b,c,d,db \in \mathbb{H}\langle a,c\rangle$. Therefore, $[a,da,b] = [a,db,b] = 0$.
- Finally,

$$-(b(da))a = -((bd)a)a = -(bd)(aa) = bd \cdot n(a) = bd,$$
$$-(b(db))a = (b(\overline{db}))a = (b(\bar{b}\bar{d}))a = -(b\bar{b})(da) = n(b) \cdot ad = ad. \qquad \square$$

Below, we will use the following notation.

Notation 4.26. Let a,b,c,d be as in Proposition 4.24 and let $K = (k_1,k_2) \in \mathbb{R}^2_* = \mathbb{R}^2 \backslash \{(0,0)\}$. Denote

$$f_K = (k_1 - k_2)((1-b_1)c + b_2 a) + (k_1 + k_2) b_0 d \in \mathbb{O},$$
$$g_K = (k_1 + k_2)((1+b_1)c - b_2 a) - (k_1 - k_2) b_0 d \in \mathbb{O}.$$

Lemma 4.27.
(1) *$f_K = 0$ if and only if at least one of the following conditions is fulfilled:*
- *$k_1 = k_2 = 0$;*
- *$k_1 - k_2 = b_0 = 0$;*
- *$b_0 = b_2 = 1 - b_1 = 0$, that is, $b = a$.*

(2) *$g_K = 0$ if and only if at least one of the following conditions is fulfilled:*
- *$k_1 = k_2 = 0$;*
- *$k_1 + k_2 = b_0 = 0$;*
- *$b_0 = b_2 = 1 + b_1 = 0$, that is, $b = -a$.*

Proof. (1) From Proposition 4.24 it follows that a,c,d are linearly independent. Then $f_K = 0$ if and only if $(k_1 - k_2)(1 - b_1) = (k_1 - k_2) b_2 = (k_1 + k_2) b_0 = 0$. Consider two cases.
- If $1 - b_1 \neq 0$, then $k_1 - k_2 = 0$. Moreover, $(k_1 + k_2) b_0 = 0$, whence either $k_1 + k_2 = 0$ or $b_0 = 0$. Equivalently, we have either $k_1 = k_2 = 0$ or $k_1 - k_2 = b_0 = 0$.
- Now let $1 - b_1 = 0$, that is, $b_1 = 1$. Then $b_0^2 + b_1^2 + b_2^2 = 1$ implies that $b_0 = b_2 = 0$.

- The case where $g_K = 0$ is treated similarly. $\square$

Corollary 4.28.
(1) *If* $f_K \neq 0$, *then* $d_{\Gamma_O(\widehat{\mathbb{S}})}\big((a,b),(f_K,f_K)\big) \leq 2$.
(2) *If* $g_K \neq 0$, *then* $d_{\Gamma_O(\widehat{\mathbb{S}})}\big((a,b),(g_K,-g_K)\big) \leq 2$.

Proof. Let

$$\begin{aligned} h_K = \left(h_K^{(1)}, h_K^{(2)}\right) &= k_1(da, bd) + k_2(db, ad) \\ &= k_1(c, b_2a - b_1c + b_0d) - k_2(b_2a - b_1c - b_0d, c) \\ &= ((k_1 + k_2b_1)c - k_2b_2a + k_2b_0d, -(k_2 + k_1b_1)c + k_1b_2a + k_1b_0d)\,. \end{aligned}$$

By Lemma 4.25, $h_K \in O_{\widehat{\mathbb{S}}}((a,b))$ and $\mathfrak{Re}\left(h_K^{(1)}\right) = \mathfrak{Re}\left(h_K^{(2)}\right) = 0$. Note that

$$\begin{aligned} h_K^{(1)} + h_K^{(2)} &= ((k_1 + k_2b_1)c - k_2b_2a + k_2b_0d) + (-(k_2 + k_1b_1)c + k_1b_2a + k_1b_0d) \\ &= (k_1 - k_2)((1 - b_1)c + b_2a) + (k_1 + k_2)b_0d = f_K \end{aligned}$$

and

$$\begin{aligned} h_K^{(1)} - h_K^{(2)} &= ((k_1 + k_2b_1)c - k_2b_2a + k_2b_0d) - (-(k_2 + k_1b_1)c + k_1b_2a + k_1b_0d) \\ &= (k_1 + k_2)((1 + b_1)c - b_2a) - (k_1 - k_2)b_0d = g_K. \end{aligned}$$

Then $h_K = 0$ implies $f_K = g_K = 0$.

(1) If $f_K \neq 0$, then $h_K \neq 0$. By Lemma 4.21, $(f_K, f_K) = \left(h_K^{(1)} + h_K^{(2)}, h_K^{(1)} + h_K^{(2)}\right) \in O_{\widehat{\mathbb{S}}}(h_K)$, whence $d_{\Gamma_O(\widehat{\mathbb{S}})}\big((a,b),(f_K,f_K)\big) \leq 2$.
(2) If $g_K \neq 0$, then $h_K \neq 0$. By Lemma 4.21, $(g_K, -g_K) = \left(h_K^{(1)} - h_K^{(2)}, -(h_K^{(1)} - h_K^{(2)})\right) \in O_{\widehat{\mathbb{S}}}(h_K)$, whence $d_{\Gamma_O(\widehat{\mathbb{S}})}\big((a,b),(g_K,-g_K)\big) \leq 2$. $\square$

Lemma 4.29. *If* $f_K \neq 0$, *then*

$$\mathrm{Anc}_{\mathbb{O}}(f_K) = \mathrm{Lin}(1, a, c, d)^{\perp} \oplus \mathrm{Lin}\left((k_1 + k_2)b_0c - (k_1 - k_2)(1 - b_1)d, (1 - b_1)a - b_2c\right).$$

Proof. Let A' denote the right-hand side of the relation to be proved. By Lemma 4.27, $f_K \neq 0$ implies $1 - b_1 \neq 0$ and either $k_1 - k_2 \neq 0$ or $(k_1 + k_2)b_0 \neq 0$. Since a, c, d are linearly independent, it follows that $\dim(A') = 6$. Moreover, by Proposition 4.24, $1, a, c, d$ form an orthonormal system, whence $A' \subset \mathfrak{Im}(\mathbb{O})$ and $A' \subset \mathrm{Lin}(f_K)^{\perp}$. Then from Lemma 2.10 it follows that $A' \subset \mathrm{Anc}_{\mathbb{O}}(f_K)$. By Lemma 2.10, we also have $\dim(\mathrm{Anc}_{\mathbb{O}}(f_K)) = \dim(A') = 6$, whence $\mathrm{Anc}_{\mathbb{O}}(f_K) = A'$. $\square$

Lemma 4.30. *Let* $b_0 \neq 0$. *Then* $f_K \neq 0$ *for all* $K \in \mathbb{R}^2_*$, *and* $\bigcup_{K \in \mathbb{R}^2_*} \mathrm{Anc}_{\mathbb{O}}(f_K) = \mathfrak{Im}(\mathbb{O})$.

Proof. By Lemma 4.27, we have $f_K \neq 0$ for all $K \in \mathbb{R}^2_*$. Then we use the representation of $\mathrm{Anc}_{\mathbb{O}}(f_K)$ from Lemma 4.29. It is sufficient to show that

$$\bigcup_{K \in \mathbb{R}^2_*} \mathrm{Lin}\left((k_1 + k_2)b_0c - (k_1 - k_2)(1 - b_1)d, (1 - b_1)a - b_2c\right) = \mathrm{Lin}(a, c, d).$$

Note that $b_0 \neq 0$ implies $1 - b_1 \neq 0$. Obviously, we have

$$\begin{aligned} \bigcup_{K \in \mathbb{R}^2_*} &\mathrm{Lin}((k_1 + k_2)b_0c - (k_1 - k_2)(1 - b_1)d, (1 - b_1)a - b_2c) \\ &= \mathrm{Lin}\left(b_0c, (1 - b_1)d, (1 - b_1)a - b_2c\right) \\ &= \mathrm{Lin}\left(c, d, (1 - b_1)a - b_2c\right) = \mathrm{Lin}\left(c, d, (1 - b_1)a\right) = \mathrm{Lin}\left(a, c, d\right). \end{aligned}$$

This completes the proof. $\square$

Lemma 4.31. *Let* $(a,b),(a',b')\in Z(\widehat{\mathbb{S}})$, $\mathfrak{Re}(a)=\mathfrak{Re}(a')=0$. *Then* $d_{\Gamma_O(\widehat{\mathbb{S}})}((a,b),(a',b'))\leq 5$.

Proof. Without loss of generality, assume that $n(a)=n(a')=1$.

If $b'=-a'$, then set $z=a'$. Otherwise, by Lemma 4.27, there exists $K'\in\mathbb{R}^2_*$ such that $g'_{K'}\neq 0$, where $g'_{K'}$ can be obtained from g_K by replacing every variable with the primed one. Then we set $z=g'_{K'}$. We have $z\neq 0$ and $\mathfrak{Re}(z)=0$. From Corollary 4.28 we obtain that $d_{\Gamma_O(\widehat{\mathbb{S}})}((a',b'),(z,-z))\leq 2$. Now consider the following three cases:

(1) If $b_0\neq 0$, then from Lemma 4.30 it follows that there exists $K\in\mathbb{R}^2_*$ such that $z\in\operatorname{Anc}_{\mathbb{O}}(f_K)$, $f_K\neq 0$. By Corollary 4.28, $d_{\Gamma_O(\widehat{\mathbb{S}})}\big((a,b),(f_K,f_K)\big)\leq 2$, and Corollary 4.23 implies that (f_K,f_K) and $(z,-z)$ are orthogonal. Thus, $d_{\Gamma_O(\widehat{\mathbb{S}})}((a,b),(a',b'))\leq 5$.

(2) If $b_0=0$ and $b\neq a$, then, by Lemma 4.21, $(a-b,-(a-b))\in O_{\widehat{\mathbb{S}}}((a,b))$. Let $y\in\operatorname{Lin}(1,a-b,z)^{\perp}$, $y\neq 0$. Then, by Corollary 4.23, in $\Gamma_O(\widehat{\mathbb{S}})$ there is a path of length 3 of the form

$$(a,b)\longleftrightarrow(a-b,-(a-b))\longleftrightarrow(y,y)\longleftrightarrow(z,-z).$$

Thus, $d_{\Gamma_O(\widehat{\mathbb{S}})}((a,b),(a',b'))\leq 5$.

(3) If $b=a$, then let $x\in\operatorname{Lin}(1,a)^{\perp}$, $x\neq 0$, and let $y\in\operatorname{Lin}(1,x,z)^{\perp}$, $y\neq 0$. Then, by Corollary 4.23, in $\Gamma_O(\widehat{\mathbb{S}})$ there exists the following path of length 3:

$$(a,a)\longleftrightarrow(x,-x)\longleftrightarrow(y,y)\longleftrightarrow(z,-z).$$

Thus, $d_{\Gamma_O(\widehat{\mathbb{S}})}((a,b),(a',b'))\leq 5$. $\square$

Theorem 4.32. *The graph* $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ *is connected, and its diameter equals* 5.

Proof. This immediately follows from Lemmas 4.31 and 4.17. $\square$

The theorem below describes the maximal cliques in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$.

Theorem 4.33. *The maximal cliques in* $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ *are of the form* $\operatorname{Lin}^*(a,b)$, *where* a *and* b *are orthogonal and linearly independent.*

Proof. Let Q be a maximal clique in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$. Consider an arbitrary element $A\in Q$. By Lemma 3.5, we have $\dim(O_{\widehat{\mathbb{S}}}(A))\in\{3,7\}$. Hence the inclusion $Q\subset\operatorname{Lin}^*(A)$ fails, and there exists $B\in Q$ such that A and B are linearly independent. Obviously, $A\in Q_{\widehat{\mathbb{S}}}(A)$ and $B\in Q_{\widehat{\mathbb{S}}}(B)$. Thus, $\operatorname{Lin}^*(A,B)\subset Q$.

Let $A,B,C\in Z_{\mathfrak{Im}}(\widehat{\mathbb{S}})$ be linearly independent. Then A,B,C do not form a 3-cycle in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$. Suppose the contrary. Then two cases are possible.

(1) At least one of the elements A,B,C, say, A has a 7-dimensional orthogonalizer. Then $A=(a,\alpha+\beta a)$ for some $a\in\mathbb{O}$, $\mathfrak{Re}(a)=0$, $\alpha,\beta\in\mathbb{R}$, $\alpha^2+\beta^2=1$. Assume, without loss of generality, that $n(a)=1$. By Lemma 4.22,

$$O_{\widehat{\mathbb{S}}}((a,\alpha+\beta a))=\mathbb{R}(a,\alpha+\beta a)\oplus\big\{(b,(\alpha a-\beta)b)\;\big|\;b\in\operatorname{Anc}_{\mathbb{O}}(a)\big\}.$$

Then $B=(b,(\alpha a-\beta)b)+\gamma A$ and $C=(c,(\alpha a-\beta)c)+\delta A$ for some $b,c\in\operatorname{Anc}_{\mathbb{O}}(a)\setminus\{0\}$, $\gamma,\delta\in\mathbb{R}$. Then $A,B-\gamma A,C-\delta A$ form a 3-cycle in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$, and we may assume, without loss of generality, that $\gamma=\delta=0$, that is, $B=(b,(\alpha a-\beta)b)$ and $C=(c,(\alpha a-\beta)c)$. By Lemma 3.5, $C\in O_{\widehat{\mathbb{S}}}(B)$ implies $[b,c,(\alpha a-\beta)b]=0$. Since $\mathbb{O}$ is flexible, we have $\alpha[b,c,ab]=[b,c,(\alpha a-\beta)b]+\beta[b,c,b]=0$. Consider two cases.

(a) If $\alpha = 0$, then $B = (b, -\beta b)$, $C = (c, -\beta c)$, $\beta = \pm 1$. By Lemma 4.22,

$$O_{\widehat{\mathbb{S}}}((b, -\beta b)) = \mathbb{R}(b, -\beta b) \oplus \left\{ (d, \beta d) \;\middle|\; d \in \mathrm{Anc}_{\mathbb{O}}(b) \right\},$$

whence $C \in O_{\widehat{\mathbb{S}}}(B)$ implies $C \in \mathbb{R}B$, a contradiction.

(b) Now let $[b, c, ab] = 0$. Assume, without loss of generality, that $n(b) = 1$. By Lemma 4.2, a and b generate the quaternionic subalgebra $\mathbb{H}\langle a, b\rangle \subset \mathbb{O}$. Hence $1, b, ab$ are linearly independent. From Proposition 4.3 it follows that $c \in \mathrm{Lin}(1, b, ab, b(ab)) = \mathrm{Lin}(1, a, b, ab)$. Since $c \in \mathrm{Anc}_{\mathbb{O}}(a)$, Lemma 2.10 implies that $c \in \mathrm{Lin}(1, a)^{\perp}$, whence $c \in \mathrm{Lin}(b, ab)$. As A, B, C are linearly independent, b, c are linearly independent. Let $c = \kappa b + ab$. Then $A, B, C - \kappa B$ also form a 3-cycle in $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$. Therefore, we may assume, without loss of generality, that $\kappa = 0$, that is, $C = (ab, (\alpha a - \beta)(ab)) = (ab, -(\alpha + \beta a)b)$. Then the second component of BC equals

$$\begin{aligned}(-(\alpha + \beta a)b)b + ((\alpha a - \beta)b)(\overline{ab}) &= (-(\alpha + \beta a)b)b + ((\alpha a - \beta)b)(ba) \\ &= n(b)((\alpha + \beta a) - (\alpha a - \beta)a) = 2n(b)(\alpha + \beta a) \neq 0.\end{aligned}$$

Thus, $C \notin O_{\widehat{\mathbb{S}}}(B)$, a contradiction.

(2) The orthogonalizers of A, B, C are 3-dimensional. If there exists $D \in \mathrm{Lin}^*(A, B, C)$ such that D has a 7-dimensional orthogonalizer we obtain a contradiction by the previous item.
Otherwise for any $D \in \mathrm{Lin}^*(A, B, C)$ we have $\mathrm{Lin}(A, B, C) \subset O_{\widehat{\mathbb{S}}}(D)$, and

$$\dim(\mathrm{Lin}(A, B, C)) = \dim(O_{\widehat{\mathbb{S}}}(D)) = 3.$$

Thus, $O_{\widehat{\mathbb{S}}}(D) = \mathrm{Lin}(A, B, C)$. Therefore, the induced subgraph of $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ on the vertex set $\mathrm{Lin}^*(A, B, C)$ is a connected component. But from Theorem 4.32 it follows that $\Gamma_O^{\mathfrak{Im}}(\widehat{\mathbb{S}})$ is connected. However, the inclusion $Z_{\mathfrak{Im}}(\widehat{\mathbb{S}}) \subset \mathrm{Lin}(A, B, C)$ cannot hold by the dimension considerations. Thus, we have a contradiction.

Therefore, $Q = \mathrm{Lin}^*(A, B)$. □

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REFERENCES

1. J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* (N.S.), **39**, 145–205 (2002).
2. L. Bentz and D. Tray, "Subalgebras of the split octonions," *Adv. Appl. Clifford Alg.* **28**, No. 2, 40 (2018).
3. D. K. Biss, D. Dugger, and D. C. Isaksen, "Large annihilators in Cayley–Dickson algebras," *Comm. Algebra*, **36**, No. 2, 632–664 (2008).
4. R. E. Cawagas, "On the structure and zero divisors of the Cayley–Dickson sedenion algebra," *Disc. Math. General Algebra Appl.*, **24**, 251–265 (2004).
5. M. Goldberg and T. J. Laffey, "On the radius in Cayley–Dickson algebras," *Proc. Amer. Math. Soc.*, **143**, No. 11, 4733–4744 (2015).
6. A. E. Guterman and S. A. Zhilina, "Cayley–Dickson split-algebras: Doubly alternative zero divisors and relation graphs," Preprint.
7. A. E. Guterman and S. A. Zhilina, "Relationship graphs of real Cayley–Dickson algebras," *Zap. Nauchn. Semin. POMI*, **472**, 44–75 (2018).
8. K. McCrimmon, *A Taste of Jordan Algebras*, Springer-Verlag, New York (2004).

9. G. Moreno, "The zero divisors of the Cayley–Dickson algebras over the real numbers," *Bol. Soc. Mat. Mex.* (tercera serie), **4**, No. 1, 13–28 (1998).
10. R. D. Schafer, "On the algebras formed by the Cayley–Dickson process," *Amer. J. Math.*, **76**, No. 2, 435–446 (1954).