DOI 10.1007/s10958-020-04911-0
Journal of Mathematical Sciences, Vol. 248, No. 6, August, 2020

THE HARMONIC DECOMPOSITION IN CYCLIC HOMOLOGY

M. Karoubi UDC 512.664.2

Abstract. It has been shown by Cuntz and Quillen that in characteristic 0 the kernel of the square of the "noncommutative Laplacian" on the Hochschild and cyclic complexes contains the relevant homology information. In this note, we show that the same property holds for the plain kernel of this Laplacian, as in differential geometry. Using the same ideas, we define a variant of Hochschild homology and cyclic homology and show that we recover the classical definitions in characteristic 0.

1. Hochschild Homology

Let A be a k-algebra with unit, where k is a commutative ring with unit. The classical Hochschild homology $\mathrm{HH}_n(A)$ may be described as the homology of the following complex:

$$\ldots \xrightarrow{b} \Omega^n(A) \xrightarrow{b} \cdots \xrightarrow{b} \Omega^1(A) \xrightarrow{b} \Omega^0(A) \longrightarrow 0,$$

where $\Omega^*(A) = \bigoplus_n \Omega^n(A)$ is the algebra of noncommutative differential forms on A [2,3]. The b operator from $\Omega^n(A)$ to $\Omega^{n-1}(A)$ is defined by the following formula in terms of these differential forms:

$$b(\omega.dx) = (-1)^{n-1}(\omega.x - x.\omega).$$

In [3], we introduced an operator

$$\boldsymbol{\kappa}\colon \Omega^*(A) \to \Omega^*(A)$$

defined by $\boldsymbol{\kappa}(\omega.dx) = (-1)^{n-1}dx.\omega$ on elements of degree $n > 0$ and $\boldsymbol{\kappa} = 1$ on elements of degree 0. As pointed out in the previous references, the operator $\boldsymbol{\kappa}$ is related to the two differentials d and b by the following identity:

$$db + bd = 1 - \boldsymbol{\kappa}.$$

In particular, $\boldsymbol{\kappa}$ commutes with d and b. The operator $1 - \boldsymbol{\kappa}$ is called the noncommutative Laplacian [2], b playing the role of the adjoint of d.

Now we define a variant of Hochschild homology as the homology of the quotient complex

$$\ldots \xrightarrow{b} \Omega^n(A)/(1-\boldsymbol{\kappa}) \xrightarrow{b} \cdots \xrightarrow{b} \Omega^1(A)/(1-\boldsymbol{\kappa}) \xrightarrow{b} \Omega^0(A)/(1-\boldsymbol{\kappa}) \longrightarrow 0.$$

We denote this new homology by ${}_{\boldsymbol{\kappa}}HH_*(A)$. There is a canonical homomorphism between Hochschild homology and its variant:

$$\varphi_n\colon \mathrm{HH}_n(A) \longrightarrow {}_{\boldsymbol{\kappa}}\mathrm{HH}_n(A).$$

Theorem 1.1. *The map φ_n above is an isomorphism for $n = 0$ and $n = 1$. Moreover, if n is invertible in k, then φ_n is an isomorphism.*

Proof. Since the theorem is clear for $n = 0$ and 1, we may assume $n > 1$. As in [3, p. 30], we consider the quotient $\bar{\Omega}^n(A)$ of $\Omega^n(A)$ by the k-submodule $\operatorname{Im} b + \operatorname{Im}(1-\boldsymbol{\kappa})$. The injectivity of φ_n is equivalent to the injectivity of the composition

$$\mathrm{HH}_n(A) \hookrightarrow \Omega^n(A)/\operatorname{Im} b \longrightarrow \Omega^n(A)/[\operatorname{Im} b + \operatorname{Im}(1-\boldsymbol{\kappa})] = \bar{\Omega}^n(A).$$

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 21, No. 6, pp. 165–170, 2016.

The argument in [3] is to remark that $\boldsymbol{\kappa}$ acts as a cyclic group of order n on the quotient group $\Omega^n(A)/\operatorname{Im} b$. Since $\boldsymbol{\kappa}$ is homotopic to 1 in the b-complex, $\mathrm{HH}_n(A)$ is included in the invariant part of $\Omega^n(A)/\operatorname{Im} b$ by the action of $\boldsymbol{\kappa}$. Since n is invertible, this invariant part is isomorphic by the quotient map to the coinvariant part $\bar{\Omega}^n(A)$.

To prove the surjectivity, since b commutes with $\boldsymbol{\kappa}$, we remark that $\boldsymbol{\kappa}$ also acts as a cyclic group of order n on the image of $\Omega^n(A)/\operatorname{Im} b$ in $\Omega^{n-1}(A)$ by the operator b according to [3, p. 30] again. Now let ω be an element of $\Omega^n(A)/\operatorname{Im} b$ such that $b(\omega) = (1-\boldsymbol{\kappa})(\theta)$. We choose rational polynomials $f(\boldsymbol{\kappa})$ and $g(\boldsymbol{\kappa})$ with denominators $1/n$ such that

$$f(\boldsymbol{\kappa})(1-\boldsymbol{\kappa}) + g(\boldsymbol{\kappa})(1+\boldsymbol{\kappa}+\cdots+\boldsymbol{\kappa}^{n-1}) = 1$$

on each of the two previous groups $\Omega^n(A)/\operatorname{Im} b$ and $b\big(\Omega^n(A)\big) \subset \Omega^{n-1}(A)$. More precisely, we may choose

$$f(\boldsymbol{\kappa}) = -\frac{1}{n}(1+2\boldsymbol{\kappa}+3\boldsymbol{\kappa}^2+\cdots+n\boldsymbol{\kappa}^{n-1}), \quad g(\boldsymbol{\kappa}) = \frac{1}{n},$$

which are related by the identity

$$-\frac{1}{n}(1-\boldsymbol{\kappa})(1+2\boldsymbol{\kappa}+3\boldsymbol{\kappa}^2+\cdots+n\boldsymbol{\kappa}^{n-1}) + \frac{1}{n}(1+\boldsymbol{\kappa}+\cdots+\boldsymbol{\kappa}^{n-1}) = 1.$$

Therefore, we have

$$\omega = f(\boldsymbol{\kappa})(1-\boldsymbol{\kappa})(\omega) + g(\boldsymbol{\kappa})(1+\boldsymbol{\kappa}+\cdots+\boldsymbol{\kappa}^{n-1})(\omega)$$

and

$$b(\omega) = b(1-\boldsymbol{\kappa})f(\boldsymbol{\kappa})(\omega) + g(\boldsymbol{\kappa})(1+\boldsymbol{\kappa}+\cdots+\boldsymbol{\kappa}^{n-1})(1-\boldsymbol{\kappa})(\theta) = b(1-\boldsymbol{\kappa})f(\boldsymbol{\kappa})(\omega).$$

If we put $\omega' = \omega - (1-\boldsymbol{\kappa})f(\boldsymbol{\kappa})(\omega)$, we have $b(\omega') = 0$. This shows that φ_n is surjective. □

Remark. According to the harmonic decomposition of noncommutative differential forms proved by Cuntz and Quillen [2], the quotient map between the complexes $\Omega^*(A)$ and $\Omega^*(A)/(1-\boldsymbol{\kappa})^2$ induces a quasi-isomorphism (for the b differential) if $\mathbb{Q} \subset k$. Equivalently, the subcomplex $\operatorname{Ker}(1-\boldsymbol{\kappa})^2$ of "harmonic forms" is quasi-isomorphic to the complex $\Omega^*(A)$. The previous theorem shows that the subcomplex $\operatorname{Ker}(1-\boldsymbol{\kappa})$ enjoys the same property. This result is closer to its analog in differential geometry: as pointed out in [2], the operator $1-\boldsymbol{\kappa}$ is a substitute for the Laplacian in this algebraic setting. On the other hand, there is no obvious projection of $\Omega^*(A)$ onto $\operatorname{Ker}(1-\boldsymbol{\kappa})$ as it is the case with $\operatorname{Ker}(1-\boldsymbol{\kappa})^2$.

Remark. In general, Hochschild homology and its variant do not coincide if we do not assume that $\mathbb{Q} \subset k$. As an example, let A be the $\mathbb{Z}$-algebra $\mathbb{Z}[x]/x^2$. Then $\mathrm{HH}_2(A) \cong \mathbb{Z}$ generated by the differential form $x\,dx\,dx$. On the other hand, ${}_{\boldsymbol{\kappa}}\mathrm{HH}_2(A) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ generated by $x\,dx\,dx$ and $dx\,dx$.

2. Cyclic Homology

Since $db + bd = 1 - \boldsymbol{\kappa}$, we may define a mixed complex in the sense of Kassel [4] with the two differentials b and d given on the quotient module $\Omega^*(A)/(1-\boldsymbol{\kappa})$:

$$\begin{array}{ccc} \Omega^*(A)/(1-\boldsymbol{\kappa}) & \xrightarrow{d} & \Omega^*(A)/(1-\boldsymbol{\kappa}) \\ {\scriptstyle b}\big\downarrow & & {\scriptstyle b}\big\downarrow \\ \Omega^*(A)/(1-\boldsymbol{\kappa}) & \xrightarrow{d} & \Omega^*(A)/(1-\boldsymbol{\kappa}) \end{array} .$$

We define our variant ${}_{\boldsymbol{\kappa}}\mathrm{HC}_*(A)$ of cyclic homology as the homology of this mixed complex. Our first task is to compare this definition with the classical one. We recall that Connes' operator

$$B\colon \Omega^{n-1}(A) \to \Omega^n(A)$$

is deduced from the operators d, b, and $\boldsymbol{\kappa} = 1 - db - bd$ by the formula

$$B = (1+\boldsymbol{\kappa}+\cdots+\boldsymbol{\kappa}^{n-1}).d.$$

On the quotient module $\Omega^*(A)/(1-\boldsymbol{\kappa})$, it is reduced to $N = nd$ in degree $n-1$.

We also recall that the cyclic complex $\mathrm{CC}_*(A)$ is defined as

$$\mathrm{CC}_n(A) = \Omega^n(A) \oplus \Omega^{n-2}(A) \oplus \dots .$$

The total differential $D\colon \mathrm{CC}_n(A) \to \mathrm{CC}_{n-1}(A)$ is defined as the following matrix:

$$D = \begin{pmatrix} b & B & 0 & 0 & \dots \\ 0 & b & B & 0 & \\ 0 & 0 & b & B & \dots \\ 0 & 0 & 0 & b & \\ \vdots & & \vdots & & \ddots \end{pmatrix}.$$

If we take the quotient by $\mathrm{Im}(1-\boldsymbol{\kappa})$, the operator B reduces to the operator N above and D may be written $\bar{D}$ on the quotient with

$$\bar{D} = \begin{pmatrix} b & (n-1)d & 0 & 0 & \dots \\ 0 & b & (n-3)d & 0 & \\ 0 & 0 & b & (n-5)d & \dots \\ 0 & 0 & 0 & b & \\ \vdots & & \vdots & & \ddots \end{pmatrix}.$$

Now we consider the automorphism of $\mathrm{CC}_n(A)$ defined by

$$(x_n, x_{n-2}, \dots) \mapsto (\alpha_n x_n, \alpha_{n-2} x_{n-2}, \alpha_{n-4} x_{n-4}, \dots),$$

where $\alpha_n = (n-1).(n-3)\dots$. Modulo this automorphism, the differential $\bar{D}$ may be written as

$$D' = \begin{pmatrix} b & d & 0 & 0 & \dots \\ 0 & b & d & 0 & \\ 0 & 0 & b & d & \dots \\ 0 & 0 & 0 & b & \\ \vdots & & \vdots & & \ddots \end{pmatrix}.$$

Therefore, if we assume $(n+1)!$ invertible in k, we get the same homology for $\bar{D}$ and D'. Moreover, according to the theorem above, the canonical map $\varphi_m\colon \mathrm{HH}_m(A) \longrightarrow {}_{\boldsymbol{\kappa}}\mathrm{HH}_m(A)$ is an isomorphism for $m \le n+1$. Thanks to the five lemma, we deduce the following theorem.

Theorem 2.1. *Let us assume $(n+1)!$ is invertible in k. Then the previous zigzag maps induce an isomorphism*

$$\mathrm{HC}_n(A) \cong {}_{\boldsymbol{\kappa}}\mathrm{HC}_n(A).$$

Now we consider the reduced de Rham complex

$$\tilde{\Omega}^*(A) = \Omega^*(A)/k$$

and the quotient complex $\tilde{\Omega}^*(A)/(1-\boldsymbol{\kappa})$. We claim that Connes' property, as formalized in [2], is valid for this quotient. More precisely, we have the following proposition.

Proposition 2.2. *Let us assume $(n+1)!$ is invertible in k. Then the cohomology (for the differential d) of degree n of the quotient complex $\tilde{\Omega}^*(A)/(1-\boldsymbol{\kappa})$ is trivial. In particular, if $\mathbb{Q} \subset k$, the mixed complex $(\tilde{\Omega}^*(A)/(1-\boldsymbol{\kappa}), b, d)$ satisfies Connes's property.*

Proof. We already know that the noncommutative reduced de Rham complex $\tilde{\Omega}^*(A)$ is acyclic [3]. Therefore, it is enough to prove that if $d(\omega) \in \mathrm{Im}(1-\boldsymbol{\kappa})$, there exists ω' such that $d(\omega') = 0$ and $\omega' - \omega \in \mathrm{Im}(1-\boldsymbol{\kappa}) + \mathrm{Im}\, d$. For this, we apply exactly the same argument as in Theorem 1.1, since the operator $\boldsymbol{\kappa}$ acts as a cyclic group of order $n+1$ on $\tilde{\Omega}^n(A)/\mathrm{Im}(d)$. □

The following theorem is in [2, 5]: our proof is slightly different, with the differential d instead of Connes' differential B. We note that the operator $\boldsymbol{k}$ also acts as a cyclic group of order $n+1$ on the other quotient $\tilde{\Omega}^n(A)/\operatorname{Im}(d) \cong (A/k.1)^{\otimes(n+1)}$, up to the sign $(-1)^{n+1}$.

Theorem 2.3. *Let us assume $(n+1)!$ is invertible in k. Then the reduced cyclic homology in degree n is the homology of the reduced cyclic complex of Connes*

$$\dots \xrightarrow{b} \bar{A}^{\otimes 3}/(1-\boldsymbol{k}) \xrightarrow{b} \bar{A}^{\otimes 2}/(1-\boldsymbol{k}) \xrightarrow{b} \bar{A} \longrightarrow 0,$$

where $\bar{A} = A/k.1$ and b is the usual Hochschild differential.

We know that the usual Hochschild homology and cyclic homology are Morita invariant. The same property is true for their variants. The key point is that an inner automorphism of an algebra A sends a Hochschild cycle to a homologous one, since we are dealing with commutators $[\omega, a]$, where ω is a differential form and a is an algebra element. The Morita invariance of our variant of Hochschild homology implies the Morita invariance of its variant in cyclic homology.

REFERENCES

1. A. Connes, "Noncommutative differential geometry," *Publ. Math. IHES*, **62**, 257–360 (1985).
2. J. Cuntz and D. Quillen, "Operators on noncommutative differential forms and cyclic homology," in: *Geometry, Topology and Physics*, International Press, Cambridge (1995), pp. 77–111.
3. M. Karoubi, *Homologie cyclique et K-théorie*, Soc. Math. de France (1987), Astérisque, Vol. 149.
4. C. Kassel, "Cyclic homology, comodules and mixed complexes," *J. Algebra*, **107**, 195–216 (1987).
5. J.-L. Loday and D. Quillen, "Cyclic homology and the Lie algebra homology of matrices," *Comment. Math. Helv.*, **59**, 565–591 (1984).

Max Karoubi
Université Paris 7 — Mathématiques,
Institut de Mathématiques de Jussieu / Paris Rive Gauche, 75013 Paris, France
E-mail: max.karoubi@gmail.com