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# A nonlocal boundary-value problem for a fourth-order mixed-type equation

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(*Presented by I. I. Skrypnik*)

**Abstract.** The criterion of uniqueness of a solution of the problem with periodicity and nonlocal and boundary conditions is established by the spectral analysis for a fourth-order mixed-type equation in a rectangular region. When constructing a solution in the form of the sum of a series, we use the completeness in the space $L_2$, the system of eigenfunctions of the corresponding problem orthogonally conjugate. When proving the convergence of a series, the problem of small denominators arises. Under some conditions imposed on the parameters of the data of the problem and given functions, the stability of the solution is proved.

**Keywords.** Mixed type equations, fourth-order partial differential equations, ill-posed problem, nonlocal problem, small denominators, uniqueness, stability.

## 1. Introduction

The work is devoted to the study of a nonlocal boundary-value problem for a fourth-order partial differential equation of the mixed type.

Let $u(x,t)$ on the region $\Omega = \{(x,t)| \, (-1,\, 1)\times(0,T),\, x\neq 0\}$ satisfy the equation

$$sgn\, x\frac{\partial^4 u}{\partial t^4} - Lu = 0, \tag{1.1}$$

where $Lu \equiv -\frac{\partial}{\partial x}\left(p(x)u_x(x,t)\right) + q(x)u(x,t)$ and $p(x)$, $p'(x)$, $q(x)$ are continuous functions on the segment $[-1,\, 1]$, $q(x)\geqslant 0$, $p(x)>p_0$, and $p_0$ is some positive constant.

**Problem.** Find a function $u(x,t)$ satisfying Eq. (1.1) on the region $\Omega$ and the following conditions: nonlocal

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} + \left.\frac{\partial^i u}{\partial t^i}\right|_{t=T} = \varphi_i(x),\ \ i=0,1,2,3,\ x\in[-1,\, 1], \tag{1.2}$$

boundary

$$u(-1,t)=u(1,t)=0,\ \ t\in[0,\, T], \tag{1.3}$$

and bonding ones

$$\left.\frac{\partial^i u}{\partial x^i}\right|_{x=-0} = \left.\frac{\partial^i u}{\partial x^i}\right|_{x=+0},\ \ i=0,1,\ t\in[0,\, T]. \tag{1.4}$$

Nonlocal problems for mixed-type differential equations with partial derivatives were currently studied very actively, and equations of the first and second orders were mainly considered by a number of researchers, such as A. A. Dezin, G. Infante, T. Jankowski, V. A. Ilyin, M. A. Naimark, E. I. Moiseev, K. B. Sabitov, S. G. Krein, G. I. Laptev, O. A. Repin, I. E. Egorov, A. I. Kozhanov.

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Fourth-order partial differential equations were studied in the works by M. Smirnov [12], T. D. Djuraev and A. K. Sopuev [3], A. I. Kozhanov [9], V. B. Dmitriev [4], K. S. Fayazov and I. O. Khajiev [7], T. K. Yuldashev [14], and others.

Note that nonlocal problems can be ill-posed in the sense of J. Hadamard. In work [13], a parabolic equation with nonlocal temporary variable conditions was investigated. To construct a stable approximate solution of such problems, an approach with a nonlocal condition was used instead of the initial condition. The regularizing properties of such a method were established in the usual class of bounded solutions. In [11], the boundary-value problem for a second-order differential equation with a negative self-adjoint operator was considered in a Hilbert space $H$. In [6], K.S.Fayazov investigated boundary-value problems for a second-order differential equation with self-adjoint operator coefficients in a Hilbert space.

The initial-boundary-value problem for Eq. (1.1) was investigated in [7]. Using the methods of spectral decompositions and energy integrals, the theorems of uniqueness and conditional stability of a solution on a set of correctness were proved. An approximate solution was constructed by the regularization method, and the estimate of an error of the norm of the difference between the exact and approximate solutions was obtained.

As a matter of fact, problem (1.1)–(1.4) is incorrect in the sense of J. Hadamard. Namely, there is no continuous dependence of the solution on the data of the problem. In addition, the problem of “small denominators” (see [10]) arises. To be more exact, the problem has no unique solution for all $T$. The Dirichlet problem for the wave equation, which is also incorrect, was studied in [2] and [1].

This paper presents the conditions on the data of the problem in which this provides the solution of the problem is unique and a conditionally stable set of correctness. We give some facts from [5], which are explained by further presentation of our results.

We will seek a solution of the problem (1.1)–(1.4) $u(x,t)$ in the form of a Fourier series in the eigenfunctions of the following spectral problem: to find the values of $\lambda$ for which the problem

$$\begin{cases} sgn\,(x)\frac{d}{dx}\,(p(x)X'(x)) - sgn\,(x)q(x)X(x) + \lambda X(x) = 0, \\ X(-1) = X(1) = 0, \\ X(-0) = X(+0),\ X'(-0) = X'(+0) \end{cases} \tag{1.5}$$

has a nontrivial solution.

By $\{X_k^+(x)\}_{k=1}^{\infty}$, $\{X_k^-(x)\}_{k=1}^{\infty}$, we denote the eigenfunctions with the corresponding positive $\{\lambda_k^+\}_{k=1}^{\infty}$ and negative $\{\lambda_k^-\}_{k=1}^{\infty}$ eigenvalues, and the numbers $\lambda_k^+$, $-\lambda_k^-$ form nondecreasing sequences.

According to [5], the eigenfunctions of problem (1.5) have the property

$$\left(sgn\,x\,X_k^{\pm}, X_j^{\pm}\right) = \pm\delta_{kj},\ ,\ \left(sgn\,x\,X_k^{+}, X_j^{-}\right) = 0\ \forall k, j \in N,$$

where $\delta_{kj}$ is the Kronecker symbol.

Let $(u,v) = \int\limits_{-1}^{1} uv dx$ be a scalar product in $L_2(-1,1)$, $\|u\| = \left(\int\limits_{-1}^{1} u^2(x,t)dx\right)^{1/2}$, and

$$\|u(x,t)\|_0^2 = \sum_{k=1}^{\infty}\left\{\left|\left(sgnx\ u(x,t),\, X_k^+\right)\right|^2 + \left|\left(sgnx\ u(x,t),\, X_k^-\right)\right|^2\right\}. \tag{1.6}$$

The eigenfunctions of the problem (1.5) form a Riesz basis in $H_0$, and the norm in the space $L_2(-1,\,1)$ defined by equality (1.6) is equivalent to the original one [5].

## 2. Form of a solution

By a generalized solution of the boundary-value problem (1.1)–(1.4), we understand a function $u(x,t) \in W_2^{1,3}(\Omega)$ satisfying conditions

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} + \left.\frac{\partial^j u}{\partial t^j}\right|_{t=T} = \varphi_j(x),\ \ j = 0,1,2,$$

$u(-1,t) = u(1,t)$ and the identity

$$\int_0^T \int_{-1}^1 (sgnx\, u_{ttt} V_t + u_x p V_x + uqV)\, dxdt = \int_{-1}^1 sgnx\, \varphi_3(x)\, V(x,T) dx \qquad (2.1)$$

for any function $V(x,t) \in W_{x,t}^{2,4}(\Omega)$, $\left.\frac{\partial^j V(x,t)}{\partial t^j}\right|_{t=0} + \left.\frac{\partial^j V(x,t)}{\partial t^j}\right|_{t=T} = 0$, $V(-1,t) = V(1,t) = 0$, $j = 0,1,2,3$.

Let the solution of the problem (1.1)–(1.4) exist and have the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k^+(t) X_k^+ + \sum_{k=1}^{\infty} u_k^-(t) X_k^-,$$

where $u_k^{\pm}(t)$ for each $k = 1,2,3,...$ are solutions of the following problems:

$$\begin{cases} \{u_k^+(t)\}_{tttt} - \mu_k^4 u_k^+(t) = 0, \\ \left.\frac{d^j}{dt^j} u_k^+(t)\right|_{t=0} + \left.\frac{d^j}{dt^j} u_k^+(t)\right|_{t=T} = \varphi_{j_k}^+,\ j = 0,1,2,3, \end{cases} \qquad (2.2)$$

$$\begin{cases} \{u_k^-(t)\}_{tttt} + 4\gamma_k^4 u_k^-(t) = 0, \\ \left.\frac{d^j}{dt^j} u_k^-(t)\right|_{t=0} + \left.\frac{d^j}{dt^j} u_k^-(t)\right|_{t=T} = \varphi_{j_k}^-,\ j = 0,1,2,3, \end{cases} \qquad (2.3)$$

where $\varphi_{j_k}^{\pm} = \pm\left(\operatorname{sgn} x\, X_k^{\pm}(x), \varphi_j(x)\right)$, $j = 0,1,2,3$, herewith $\gamma_k^4 = \mu_k^4/4$, $\mu_k^4 = \pm\lambda_k^{\pm}$.

Now we turn to the solution of the problem (2.2). Let $\frac{1}{\mu_k^2}\frac{d^2 u_k^+}{dt^2} = \vartheta_k^+$, $w_k^+ = u_k^+ - \vartheta_k^+$, $v_k^+ = u_k^+ + \vartheta_k^+$. After some transformations, we have

$$\begin{aligned} &\{v_k^+\}_{tt} - \mu_k^2 v_k^+ = 0, \\ &v_k^+(0) + v_k^+(T) = \varphi_{0_k}^+ + \mu_k^{-2}\varphi_{2_k}^+, \\ &\{v_k^+(t)\}_t\big|_{t=0} + \{v_k^+(t)\}_t\big|_{t=T} = \varphi_{1_k}^+ + \mu_k^{-2}\varphi_{3_k}^+ \end{aligned}$$

and

$$\begin{aligned} &\{w_k^+\}_{tt} + \mu_k^2 w_k^+ = 0, \\ &w_k^+(0) + w_k^+(T) = \varphi_{0_k}^+ - \mu_k^{-2}\varphi_{2_k}^+, \\ &\{w_k^+(t)\}_t\big|_{t=0} + \{w_k^+(t)\}_t\big|_{t=T} = \varphi_{1_k}^+ - \mu_k^{-2}\varphi_{3_k}^+. \end{aligned}$$

The solution of the last problems can be represented as

$$v_k^+(t) = \frac{1}{2}\left(F(\mu_k,t)\left(\varphi_{0_k}^+ + \mu_k^{-2}\varphi_{2_k}^+\right) + G(\mu_k,t)\left(\varphi_{1_k}^+ + \mu_k^{-2}\varphi_{3_k}^+\right)\right),$$

$$w_k^+(t) = \frac{1}{2}\left(\overline{F}(\mu_k,t)\left(\varphi_{0_k}^+ - \mu_k^{-2}\varphi_{2_k}^+\right) + \overline{G}(\mu_k,t)\left(\varphi_{1_k}^+ - \mu_k^{-2}\varphi_{3_k}^+\right)\right),$$

where

$$F(\mu_k,t)=\frac{ch\mu_k t+ch\left(\mu_k(T-t)\right)}{1+ch\mu_k T},\overline{F}(\mu_k,t)=\frac{\cos\mu_k t+\cos\left(\mu_k(T-t)\right)}{1+\cos\mu_k T},$$

$$G(\mu_k,t)=\frac{sh\mu_k t+sh\left(\mu_k(t-T)\right)}{1+ch\mu_k T},\overline{G}(\mu_k,t)=\frac{sin\mu_k t+sin\left(\mu_k(t-T)\right)}{1+\cos\mu_k T}.$$

Then

$$\begin{aligned}u_k^+(t)&=\frac{1}{2}\left(v_k^+(t)+w_k^+(t)\right)\\&=\frac{1}{4}\left(\left(F(\mu_k,t)+\overline{F}(\mu_k,t)\right)\varphi_{0_k}^+ +\mu_k^{-1}\left(G(\mu_k,t)+\overline{G}(\mu_k,t)\right)\varphi_{1_k}^+\right.\\&\quad\left.+\mu_k^{-2}\left(F(\mu_k,t)-\overline{F}(\mu_k,t)\right)\varphi_{2_k}^+ +\mu_k^{-3}\left(G(\mu_k,t)-\overline{G}(\mu_k,t)\right)\varphi_{3_k}^+\right).\quad(2.4)\end{aligned}$$

Similarly, for the problem (2.3), we have $\frac{1}{2i\gamma_k^2}\frac{d^2u_k^-}{dt^2}=\vartheta_k^-$, $w_k^-=u_k^- -\vartheta_k^-$, $v_k^-=u_k^- +\vartheta_k^-$

$$\begin{aligned}&\left\{v_k^-\right\}_{tt}-2i\gamma_k^2 v_k^-=0,\\&v_k^-(0)+v_k^-(T)=\varphi_{0_k}^- -0,5i\gamma_k^{-2}\varphi_{2_k}^-,\\&\left\{v_k^-(t)\right\}_t\Big|_{t=0}+\left\{v_k^-(t)\right\}_t\Big|_{t=T}=\varphi_{1_k}^- -0,5i\gamma_k^{-2}\varphi_{3_k}^-,\end{aligned}$$

and

$$\begin{aligned}&\left\{w_k^-\right\}_{tt}+2i\gamma_k^2 w_k^-=0,\\&w_k^-(0)+w_k^-(T)=\varphi_{0_k}^- +0,5i\gamma_k^{-2}\varphi_{2_k}^-,\\&\left\{w_k^-(t)\right\}_t\Big|_{t=0}+\left\{w_k^-(t)\right\}_t\Big|_{t=T}=\varphi_{1_k}^- +0,5i\gamma_k^{-2}\varphi_{3_k}^-.\end{aligned}$$

Then $u_k^-(t)=\frac{1}{2}\left(v_k^-+w_k^-\right)$, where

$$v_k^-(t)=\frac{1}{2}\left(f_{0_k}^- -0,5i\gamma_k^{-2}f_{2_k}^-\right)F(z,t)+\frac{1}{2z}\left(f_{1_k}^- -0,5i\gamma_k^{-2}f_{3_k}^-\right)G(z,t),$$

$$w_k^-(t)=\frac{1}{2}\left(f_{0_k}^- +0,5i\gamma_k^{-2}f_{2_k}^-\right)F(\bar{z},t)+\frac{1}{2\bar{z}}\left(f_{1_k}^- +0,5i\gamma_k^{-2}f_{3_k}^-\right)G(\bar{z},t),$$

where $z=\gamma_k+i\gamma_k$.

After a simplification, we have

$$u_k^-(t)=\frac{P_1(\gamma_k,t)}{4\Delta_k^2}\varphi_{0_k}^- +\frac{P_2(\gamma_k,t)}{8\gamma_k\Delta_k^2}\varphi_{1_k}^- +\frac{P_3(\gamma_k,t)}{8\gamma_k^2\Delta_k^2}\varphi_{2_k}^- +\frac{P_4(\gamma_k,t)}{16\gamma_k^3\Delta_k^2}\varphi_{3_k}^-,\quad(2.5)$$

where $\Delta_k=ch\gamma_k T+\cos\gamma_k T$,

$$\begin{aligned}P_1(\gamma_k,t)&=2ch\gamma_k t\cos\gamma_k t+2ch(\gamma_k(t-T))\cos(\gamma_k(t-T))\\&\quad+ch(\gamma_k(t+T))\cos(\gamma_k(t-T))+ch(\gamma_k(t-T))\cos(\gamma_k(t+T))\\&\quad+ch\gamma_k t\cos(\gamma_k(t-2T))+ch(\gamma_k(t-2T))\cos\gamma_k t,\end{aligned}$$

$$\begin{aligned}P_2(\gamma_k,t)&=2sh\gamma_k t\cos\gamma_k t+2sh(\gamma_k(t-T))\cos(\gamma_k(t-T))\\&\quad+sh(\gamma_k(t+T))\cos(\gamma_k(t-T))+sh(\gamma_k(t-T))\cos(\gamma_k(t+T))\\&\quad+sh\gamma_k t\cos(\gamma_k(t-2T))+sh(\gamma_k(t-2T))\cos\gamma_k t\\&\quad+2ch\gamma_k t\sin\gamma_k t+2ch(\gamma_k(t-T))sin(\gamma_k(t-T))\\&\quad+ch(\gamma_k(t+T))sin(\gamma_k(t-T))+ch(\gamma_k(t-T))sin(\gamma_k(t+T))\\&\quad+ch\gamma_k t\sin(\gamma_k(t-2T))+ch(\gamma_k(t-2T))sin\gamma_k t,\end{aligned}$$

$$P_3(\gamma_k,t) = 2sh\gamma_k t\sin\gamma_k t + 2sh(\gamma_k(t-T))sin(\gamma_k(t-T)) + sh(\gamma_k(t+T))sin(\gamma_k(t-T)) + sh(\gamma_k(t-T))sin(\gamma_k(t+T)) + sh\gamma_k t\sin(\gamma_k(t-2T)) + sh(\gamma_k(t-2T))sin\gamma_k t,$$

$$P_4(\gamma_k,t) = 2ch\gamma_k t\sin\gamma_k t + 2ch(\gamma_k(t-T))sin(\gamma_k(t-T)) + ch(\gamma_k(t+T))sin(\gamma_k(t-T)) + ch(\gamma_k(t-T))sin(\gamma_k(t+T)) + ch\gamma_k t\sin(\gamma_k(t-2T)) + ch(\gamma_k(t-2T))sin\gamma_k t - 2sh\gamma_k t\cos\gamma_k t - 2sh(\gamma_k(t-T))cos(\gamma_k(t-T)) - sh(\gamma_k(t+T))cos(\gamma_k(t-T)) - sh(\gamma_k(t-T))cos(\gamma_k(t+T)) - sh\gamma_k t\cos(\gamma_k(t-2T)) - sh(\gamma_k(t-2T))cos\gamma_k t.$$

## 3. Theorems

**Theorem 3.1.** *For the uniqueness of a solution of the problem* (1.1)–(1.4) *in the class* $W_2^{1,3}(\Omega)$, *it is necessary and sufficient that the equation*

$$\mu_k T = \pi + 2\pi n$$

*has no solutions in integers* $k, n$ $(k, n \in N)$.

*Proof. Necessity.* If, for some positive integer $k$, the expression $1+\cos\mu_k T$ vanishes, then the homogeneous problem (1.1)–(1.4), i.e., $\varphi_j(x) = 0,\ j = 0,1,2,3$, has nontrivial solutions of the form

$$u(x,t) = (sh\mu_k t + ch\mu_k t + \sin\mu_k t + \cos\mu_k t)\, X_k^+(x).$$

The solution of the inhomogeneous problem (1)–(4), if exists, will not be unique.

*Sufficiency.* Let there exist two solutions $u_1(x,t),\ u_2(x,t)$ of problem (1.1)–(1.4) from the space $(L_2(-1,1); C[0,T])$. Then the function $u(x,t) = u_1(x,t) - u_2(x,t)$ is a solution of the homogeneous problem (1.1)–(1.4), where $\varphi_j(x) \equiv 0,\ j = 0,1,2,3$. Hence, we get $u_k^+(t) \equiv 0,\ u_k^-(t) \equiv 0$. Therefore, $u(x,t) \equiv 0\ \forall(x,t) \in \Omega$. The theorem was proved. □

**Theorem 3.2.** *Let* $\varphi_j(x) \in W_2^{4-j+\varepsilon}(-1;1)$, $j = 0,1,2,3$, *and let the conditions of Theorem 4 be satisfied. Then, for* $T \neq \frac{(2n+1)\pi}{\mu_k}$ *(n, k are natural numbers), there exists a unique solution of problem* (1)–(4) *which belongs to the space* $W_2^{1,3}(\Omega)$ *and depends continuously on the functions* $\varphi_j(x)$, $j = 0,1,2,3$ *in the sense that the estimate*

$$\|u(x,t)\|_0^2 \leqslant C_0\|\varphi_0(x)\|^2_{W_2^{4+\varepsilon}} + C_1\|\varphi_1(x)\|^2_{W_2^{3+\varepsilon}} + C_2\|\varphi_2(x)\|^2_{W_2^{2+\varepsilon}} + C_3\|\varphi_3(x)\|^2_{W_2^{1+\varepsilon}}$$

*is valid, where* $C_j$ *are constants,* $j = 0,1,2,3$, $0 < \varepsilon < 1$.

*Proof.* The existence of a solution of problem (1.1)–(1.4) under the condition $T \neq \frac{(2n+1)\pi}{\mu_k}$ is related to the problem of small denominators, since the expression $1+\cos\mu_k T$ in the denominators in formula (2.4), being nonzero, can become arbitrarily small for an infinite set $k \in N$. Note , for an arbitrary integer $k > 0$,

$$\left|\cos\frac{\mu_k T}{2}\right| = \left|\sin\left(\frac{\mu_k T - \pi}{2} - n\pi\right)\right| = \left|\sin\left(\left(\frac{\mu_k T - \pi}{2\pi} - n\right)\pi\right)\right| > 2\left|\left(\frac{\mu_k T - \pi}{2\pi} - n\right)\right| = 2k\left|\left(\frac{\mu_k T - \pi}{2\pi k} - \frac{n}{k}\right)\right|,$$

where $n$ is a nonnegative integer satisfying the inequality

$$\left|\left(\frac{\mu_k T-\pi}{2\pi}-n\right)\right|<\frac{1}{2}.$$

Moreover, in deriving the upper inequality, we take into account that, for all $x \in (0,\ \pi/2)$, the inequality $\sin x > 2x/\pi$ is satisfied. According to [10, Ch. 1] for almost all (in the sense of Lebesgue measure) numbers $T>0$, the inequality

$$\left|\frac{\mu_k T-\pi}{2\pi k}-\frac{n}{k}\right|<\frac{1}{k^{2+\varepsilon/2}},\ 0<\varepsilon<1,$$

$k>0$, $n>0$, it has at most finitely many integer solutions.

It is also known from number theory that, for each $\frac{\mu_k T-\pi}{2\pi k}$, there are constants $\delta_1>0$ and $\varepsilon>0$ for which the inequality

$$\left|\frac{\mu_k T-\pi}{2\pi k}-\frac{n}{k}\right|>\frac{\delta_1}{k^{2+\varepsilon/2}},\ 0<\varepsilon<1,$$

holds for all (except a finite number) pairs of integers $n$ and $k$, $k\neq 0$. Hence, we have

$$1+\cos\mu_k T = 2\cos^2\frac{\mu_k T}{2}>\frac{\delta^2}{k^{2+\varepsilon}}, \tag{3.1}$$

where $\delta^2=8\delta_1^2$. From equality (2.4), using the inequality $(a+b+c+d)^2\leqslant 4(a^2+b^2+c^2+d^2)$, the monotonically increasing function $ch\mu_k T$, and (3.1), we obtain

$$\begin{aligned}\left\{u_k^+\right\}^2 &\leqslant \left(2+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{0_k}^+\mu_k^{4+\varepsilon}\right)^2+\left(\tfrac{2}{\mu_k}+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{1_k}^+\mu_k^{3+\varepsilon}\right)^2\\ &+\left(\tfrac{2}{\mu_k^2}+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{2_k}^+\mu_k^{2+\varepsilon}\right)^2+\left(\tfrac{2}{\mu_k^3}+\tfrac{2}{\delta^2}\right)^2\left(\varphi_{3_k}^+\mu_k^{1+\varepsilon}\right)^2.\end{aligned}$$

We proceed to estimate the function $u_k^-(t)$. It follows from representation (2.5) that the expression $\Delta_k = ch\gamma_k T+\cos\gamma_k T$ is in the denominator. This expression increases monotonically, and $\Delta_k>2$ for all $\gamma_k$, $T$. Therefore, in (2.5), there is no problem of small denominators.

We consider one of the terms in (2.5),

$$\frac{ch\left(\mu_k(t+T)\right)\cos\left(\mu_k(t-T)\right)}{\left(ch\left(\mu_k T\right)+\cos\left(\mu_k T\right)\right)^2}\leqslant\frac{ch\left(2\mu_k T\right)}{\left(ch\left(\mu_k T\right)+\cos\left(\mu_k T\right)\right)^2}<m,$$

where $m$ is a bounded constant. This estimate is true for any $\gamma_k$, $T$. The remaining terms are also bounded for all $\gamma_k$, $T$. Considering these facts, we estimate the function $u_k^-(t)$

$$\left\{u_k^-(t)\right\}^2\leqslant 4m^2\left\{\varphi_{0_k}^-\right\}^2+32\frac{m^2}{\gamma_k}\left\{\varphi_{1_k}^-\right\}^2+8\frac{m^2}{\gamma_k^2}\left\{\varphi_{2_k}^-\right\}^2+16\frac{m^2}{\gamma_k^3}\left\{\varphi_{3_k}^-\right\}^2.$$

Combining $u_k^+(t)$ and $u_k^-(t)$, we have

$$\begin{aligned}\left\{u_k^+(t)\right\}^2+\left\{u_k^-(t)\right\}^2 &\leqslant C_0\left(\mu_k^{4+\varepsilon}\right)^2\left(\left\{\varphi_{0_k}^+\right\}^2+\left\{\varphi_{0_k}^-\right\}^2\right)\\ &+C_1\left(\mu_k^{3+\varepsilon}\right)^2\left(\left\{\varphi_{1_k}^+\right\}^2+\left\{\varphi_{1_k}^-\right\}^2\right)+C_2\left(\mu_k^{2+\varepsilon}\right)^2\left(\left\{\varphi_{2_k}^+\right\}^2+\left\{\varphi_{2_k}^-\right\}^2\right)\\ &+C_3\left(\mu_k^{1+\varepsilon}\right)^2\left(\left\{\varphi_{3_k}^+\right\}^2+\left\{\varphi_{3_k}^-\right\}^2\right),\end{aligned}$$

where

$$C_0 = \max\left\{2(1+\delta^{-2}), 4m^2\right\},\ C_1 = \max\left\{2(\mu_1^{-1}+\delta^{-2}), 32m^2\gamma_1^{-1}\right\},$$
$$C_2 = \max\left\{2(\mu_1^{-2}+\delta^{-2}), 8m^2\gamma_1^{-2}\right\},\ C_3 = \max\left\{2(\mu_1^{-3}+\delta^{-2}), 16m^2\gamma_1^{-3}\right\}.$$

Adding the last inequalities in $k$, we obtain

$$\sum_{k=1}^{\infty}\left\{u_k^+(t)\right\}^2+\left\{u_k^-(t)\right\}^2 \leqslant C_0\left\|\varphi_0(x)\right\|^2_{W_2^{4+\varepsilon}}$$
$$+C_1\left\|\varphi_1(x)\right\|^2_{W_2^{3+\varepsilon}}+C_2\left\|\varphi_2(x)\right\|^2_{W_2^{2+\varepsilon}}+C_3\left\|\varphi_3(x)\right\|^2_{W_2^{1+\varepsilon}},$$

and this yields the required inequality. □

## 4. Numerical calculations

For the numerical solution of problem (1.1)–(1.4), we take the initial data as follows:

$$p(x)=1,\ q(x)=0,\ \varphi_0(x)=x^2-1,\ \varphi_j(x)=0,\ j=1,2,3.$$

Then $\pm\lambda_k^{\pm}$ are solutions of the equation $tg\sqrt{\pm\lambda_k^{\pm}}+th\sqrt{\pm\lambda_k^{\pm}}=0$. If we denote $\alpha=\sqrt{\pm\lambda_k^{\pm}}$, solutions of the equation $tg\,\alpha+th\,\alpha=0$ can be easily found by the Newton method. When $\varepsilon=10^{-15}$, we calculate $\alpha_1\approx 2.36502037243135$, $\alpha_2\approx 5.49780391900084$, $\alpha_3\approx 8.63937982869974$, $\alpha_4\approx 11.7809724510202$, $\alpha_k\approx -\frac{\pi}{4}+\pi k$, $k>4$, $k\in N$. Then $\mu_k=\sqrt{\alpha_k}$, $\gamma_k=\sqrt{\alpha_k/2}$, $k=1,2\ldots$ .

Let the solution of problem (1.1)–(1.4) exist and have the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k^+(t)X_k^+ + \sum_{k=1}^{\infty}u_k^-(t)X_k^-, \tag{4.1}$$

where

$$u_k^+(t)=\frac{1}{4}\left(\frac{ch\mu_k t+ch\left(\mu_k(T-t)\right)}{1+ch\mu_k T}+\frac{\cos\mu_k t+\cos\left(\mu_k(T-t)\right)}{1+\cos\mu_k T}\right)\varphi^+_{0_k},$$

$$u_k^-(t)=\frac{1}{4\Delta_k^2}\left(2ch\gamma_k t\cos\gamma_k t+2ch(\gamma_k(t-T))\cos(\gamma_k(t-T))\right.$$
$$+ch(\gamma_k(t+T))\cos(\gamma_k(t-T))+ch(\gamma_k(t-T))\cos(\gamma_k(t+T))$$
$$\left.+ch\gamma_k t\cos(\gamma_k(t-2T))+ch(\gamma_k(t-2T))\cos\gamma_k t\right)\varphi^-_{0_k},$$

$$X_k^+(x)=\begin{cases}\frac{\sin\alpha_k(x-1)}{\cos\alpha_k},\ 0<x\leqslant 1,\\ \frac{sh\alpha_k(x+1)}{ch\alpha_k},\ -1\leqslant x<0,\end{cases}$$

$$X_k^-(x)=\begin{cases}\frac{sh\alpha_k(x-1)}{ch\alpha_k},\ 0<x\leqslant 1,\\ \frac{\sin\alpha_k(x+1)}{\cos\alpha_k},\ -1\leqslant x<0.\end{cases}$$

Let the equation $\mu_k T=\pi+2\pi n$ have no exact solutions for $k,n\in N$. The numerical solution is presented in Table 4.1 (case $T=2.5$).

Let the equation $\mu_k T = \pi + 2\pi n$ have a solution for $k, n \in N$. The numerical solution is given in Table 4.2 by approximate values close to solution of the indicated equation because problem has no unique solution. As the parameters in calculations, we take $n = 1$, $k = 5$, and $T = 2.4397728$.

**Remark**. The solution by the presented formula (4.1) is calculated with the error $\varepsilon = 10^{-4}$.

| | t=0.25 | t=0.75 | t=1.25 | t=2,25 | t=T |
|---|---|---|---|---|---|
| x=-1 | 0 | 0 | 0 | 0 | 0 |
| x=-0.8 | -0.059 | -0.0123 | 0.0046 | -0.059 | -0.0891 |
| x=-0.4 | -0.1316 | -0.0102 | 0.0352 | -0.1316 | -0.2079 |
| x=0.2 | -0.1053 | 0.1156 | 0.2047 | -0.1053 | -0.2377 |
| x=0.6 | -0.0579 | 0.1096 | 0.1756 | -0.0579 | -0.1585 |

**Table 4.1.** The numerical solution at $T = 2.5$

| | t=0.25 | t=0.75 | t=1.25 | t=2.25 | t=T |
|---|---|---|---|---|---|
| x=-1 | 0 | 0 | 0 | 0 | 0 |
| x=-0.8 | -0.1946 | -0.0511 | 0.1694 | -0..1764 | -0..0891 |
| x=-0.4 | -53.5711 | -15.724 | 64.5786 | -43.6283 | -0..2079 |
| x=0.2 | 17375.4 | 5111.31 | -20982.9 | 14136.65 | -0..2377 |
| x=0.6 | 9134.79 | 2687.234 | -11031.3 | 7432.073 | -0..1585 |

**Table 4.2.** The numerical solution at $T = 2.4397728$

## 5. Conclusion

We tried to show that the problem under consideration belongs to the class of ill-posed problems of mathematical physics. Based on the idea of the theory of ill-posed problems, the initial problem is investigated for conditional correctness. Since the problem belongs to the class of weakly incorrect problems, pairs of spaces for which the problem becomes correct are obtained. The proved theorems provide an opportunity to construct an algorithm and to obtain the numerical solution on a computer.

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