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GENERALIZED VERTEX OPERATORS OF HALL– LITTLEWOOD POLYNOMIALS AS TWISTS OF CHARGED FREE FERMIONS

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Using twists of fields of charged free fermions, we revise the generalized vertex operator presentation of Hall–Littlewood polynomials, propose a new version of the deformed boson-fermion correspondence, and construct new examples of tau-functions of the KP *hierarchy in the one-parameter deformation of the ring of symmetric functions* $\Lambda[t]$. *Bibliography*: 17 *titles.*

1 Introduction

The vertex operator realization of Hall–Littlewood polynomials through generalized fermions was first described in [1] followed by the construction of a deformed version of the boson-fermion correspondence in [2]. The ideas were further developed and applied in works of many authors [3]–[7].

The first goal of this paper is to show that the generalized fermions in [1] can be interpreted as a simple twist of charged free fermions that provide the vertex operator realization of Schur symmetric functions described in [8]–[10].

The deformed boson-fermion correspondence of [2] establishes a connection between the actions of generalized fermions and the twisted Heisenberg algebra. This correspondence found its applications in [6, 7]. Our second goal is to propose a version of the boson-fermion correspondence that relates generalized fermions with the action of the classical Heisenberg algebra. This construction is different from the deformed boson-fermion correspondence in [2] and has its own advantages: it does not require adjustments in the standard definition of the normal ordered product of fields; the action of the classical Heisenberg algebra and, as a consequence,

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of the Virasoro algebra are naturally present in the picture; the action of the twisted Heisenberg algebra appears as a renormalization of the action of the classical Heisenberg algebra; proofs of this version of the boson-fermion correspondence, the original deformed boson-fermion correspondence of [2], and its applications become simple implications of a twist of the classical boson-fermion correspondence [11].

The third result of this paper is the construction of tau-functions of the KP hierarchy in the deformed ring of symmetric functions $\Lambda[t]$. In [9], the symmetric functions S_λ are introduced as a basis dual to the basis of the classical Schur functions s_λ with respect to the natural scalar form in the deformed ring of symmetric functions $\Lambda[t]$. Extending the idea of twisting the fields of charged free fermions, we prove that S_λ are solutions of the bilinear KP identity and provide two different vertex operator realizations of this family of symmetric functions. Note that this is stronger than the result of [6] establishing that the element $\Lambda[t]$ of the form $\sum s_\lambda S_\lambda$ is a tau-function of the KP hierarchy.

The paper is organized as follows. In Sections 2 and 3, we review necessary facts about symmetric functions and the action of charged free fermions on the ring of symmetric functions. In Section 4, we connect generalized fermions with classical charged free fermions. In Section 5, we propose a version of the boson-fermion correspondence for generalized fermions and compare it with the construction of [2]. In Section 6, we prove that symmetric functions S_λ that form a basis for $\Lambda[t]$ orthogonal to the basis of Schur functions s_λ are tau-functions of the KP hierarchy. We provide two different versions of their vertex operator realization.

2 Preliminaries on Symmetric Functions

We review necessary facts about symmetric functions following [9, 12]. Let $\Lambda = \Lambda[x_1, x_2, \dots]$ be the ring of symmetric functions in variables $(x_1, x_2, \dots)$. *Schur symmetric functions* s_λ labeled by partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0)$ defined by

$$s_\lambda(x_1, x_2, \dots) = \frac{\det [x_i^{\lambda_j+n-j}]}{\det [x_i^{n-j}]}$$

form a linear basis of Λ. We define *complete symmetric functions* $h_r = s_{(r)}$ by

$$h_r(x_1, x_2 \dots) = \sum_{1\leqslant i_1 \leqslant \dots \leqslant i_r < \infty} x_{i_1} \dots x_{i_r},$$

elementary symmetric functions $e_r = s_{(1^r)}$ by

$$e_r(x_1, x_2 \dots) = \sum_{1< i_1 < \dots < i_r < \infty} x_{i_1} \dots x_{i_r},$$

and *power sums* p_k by

$$p_k(x_1, x_2, \dots) = \sum_i x_i^k.$$

It is convenient to set $h_{-k}(x_1, x_2 \dots) = e_{-k}(x_1, x_2 \dots) = p_{-k}(x_1, x_2 \dots) = 0$ for $k > 0$ and $h_0 = e_0 = p_0 = 1$. Each of these families generates the ring of symmetric functions Λ as a polynomial ring:

$$\Lambda = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[p_1, p_2, \dots]. \tag{2.1}$$

Based on the interpretation of a polynomial ring as the ring of symmetric functions, one defines the *boson Fock space* $\mathscr{B}$. Let $\mathscr{B} = \mathbb{C}[z, z^{-1}, p_1, p_2, \dots]$ be the graded space of polynomials $\mathscr{B} = \bigoplus_{m\in\mathbb{Z}} \mathscr{B}^{(m)}$, where $\mathscr{B}^{(m)} = z^m \cdot \mathbb{C}[p_1, p_2, \dots] = z^m\Lambda$. We write generating functions for complete, elementary symmetric functions and power sums:

$$H(u) = \sum_{k\geqslant 0} \frac{h_k}{u^k}, \quad E(u) = \sum_{k\geqslant 0} \frac{e_k}{u^k}, \quad P(u) = \sum_{k=1}^{\infty} p_k u^k. \tag{2.2}$$

Then

$$H(u) = \prod_{i\geqslant 1} \frac{1}{1 - x_i/u}, \quad E(u) = \prod_{i\geqslant 1}(1 + x_i/u), \quad H(u)E(-u) = 1 \tag{2.3}$$

and

$$H(u) = \exp\left(\sum_{n\geqslant 1} \frac{p_n}{n}\frac{1}{u^n}\right), \quad E(u) = \exp\left(-\sum_{n\geqslant 1} \frac{(-1)^n p_n}{n}\frac{1}{u^n}\right). \tag{2.4}$$

The *Heisenberg algebra* is a complex Lie algebra generated by elements $\{\alpha_m\}_{m\in\mathbb{Z}}$ and central element 1 with commutation relations

$$[\alpha_k, \alpha_n] = k\delta_{k,-n} \cdot 1. \tag{2.5}$$

Combining the generators into the formal distribution $\alpha(u) = \sum_k \alpha_k u^{-k-1}$, we can write this relation as

$$[\alpha(u), \alpha(v)] = \partial_v \delta(u, v), \tag{2.6}$$

where $\delta(u, v) = \sum_{k\in\mathbb{Z}} u^k v^{-k-1}$ is the formal delta-distribution. There is a natural action of the Heisenberg algebra on the graded components $\alpha_n\colon \mathscr{B}^{(m)} \to \mathscr{B}^{(m)}$ defined by the multiplication and differentiation operators:

$$\begin{aligned} &\alpha_{-n} = p_n/n, \quad n > 0,\\ &\alpha_n = \frac{\partial}{\partial p_n}, \quad n > 0,\\ &\alpha_0 = m. \end{aligned} \tag{2.7}$$

The ring of symmetric functions Λ possesses a natural scalar product, where the classical Schur functions s_λ constitute the orthonormal basis

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}. \tag{2.8}$$

Then for the operator of multiplication by a symmetric function f one can define the adjoint operator $f^\perp$ acting on the ring of symmetric functions by the standard rule

$$\langle f^\perp g, w\rangle = \langle g, fw\rangle, \quad g, f, w \in \Lambda.$$

We consider generating functions of the adjoint operators

$$E^\perp(u) = \sum_{k\geqslant 0} e_k^\perp u^k, \quad H^\perp(u) = \sum_{k\geqslant 0} h_k^\perp u^k.$$

One can prove that

$$E^{\perp}(u) = \exp\left(-\sum_{k\geqslant 1}(-1)^k \frac{\partial}{\partial p_k} u^k \right), \quad H^{\perp}(u) = \exp\left(\sum_{k\geqslant 1} \frac{\partial}{\partial p_k} u^k \right). \tag{2.9}$$

The following commutation relations serve as the foundation of most of calculations below.

Proposition 2.1 (cf. [9])**.**

$$\Big(1 - \frac{u}{v}\Big) E^{\perp}(u) E(v) = E(v) E^{\perp}(u),$$
$$\Big(1 - \frac{u}{v}\Big) H^{\perp}(u) H(v) = H(v) H^{\perp}(u),$$
$$H^{\perp}(u) E(v) = \Big(1 + \frac{u}{v}\Big) E(v) H^{\perp}(u),$$
$$E^{\perp}(u) H(v) = \Big(1 + \frac{u}{v}\Big) H(v) E^{\perp}(u).$$

Remark 2.1. Statements of Proposition 2.1 should be understood as equalities of series expansions in powers of $u^k v^{-m}$, $k, m \geqslant 0$.

3 Fermions and Schur Symmetric Functions

Following [11, 13, 14], we define the action of the algebra of charged free fermions on the boson Fock space.

Let $R(u) : \mathscr{B} \to \mathscr{B}$ act on elements $z^m f$, $f \in \Lambda$, by the rule (cf., for example, [14, 15])

$$R(u)(z^m f) = \Big(\frac{z}{u}\Big)^{m+1} f.$$

Then $R^{-1}(u)$ acts as $R^{-1}(u)(z^m f) = (z)^{m-1} u^m f$. One should think of $R^{\pm 1}(u)$ as operators that transport the action of other operators along the grading of the boson Fock space: $R^{\pm 1}(u) : \mathscr{B}^{(m)} \to \mathscr{B}^{(m\pm 1)}$. We set

$$\Phi^{+}(u) = u R(u) H(u) E^{\perp}(-u), \tag{3.1}$$

$$\Phi^{-}(u) = R^{-1}(u) E(-u) H^{\perp}(u). \tag{3.2}$$

Observe that

$$\Phi^{+}(u)|_{\mathscr{B}^{(m)}} = z u^{-m} H(u) E^{\perp}(-u),$$
$$\Phi^{-}(u)|_{\mathscr{B}^{(m)}} = z^{-1} u^{m} E(-u) H^{\perp}(u).$$

Proposition 3.1. *The quantum fields $\Phi^{\pm}(u)$ satisfy the relations of the algebra of charged free fermions*

$$\Phi^{\pm}(u)\Phi^{\pm}(v) + \Phi^{\pm}(v)\Phi^{\pm}(u) = 0, \tag{3.3}$$

$$\Phi^{+}(u)\Phi^{-}(v) + \Phi^{-}(v)\Phi^{+}(u) = \delta(u, v), \tag{3.4}$$

where $\delta(u, v) = \sum_{k\in\mathbb{Z}} \frac{u^k}{v^{k+1}}$ is the formal delta-distribution.

Proof. We use Proposition 2.1 to prove this classical result, thus illustrating the simplicity of this approach. Relations between other vertex operators further in this text are proved along the same line. Using the commutation relations of Proposition 2.1, for any $f \in \Lambda$ we have

$$\Phi^+(u)\Phi^+(v)(z^m f) = z^{m+2}(uv)^{-m-2}(v-u)H(u)H(v)E^\perp(-u)E^\perp(-v)(f),$$
$$\Phi^-(u)\Phi^-(v)(z^m f) = z^{m-2}(uv)^{m-2}(v-u)E(-u)E(-v)H^\perp(u)H^\perp(v)(f).$$

Changing the roles of u and v in these calculations, one gets (3.3). For (3.4) we observe that

$$\Big(1-\frac{u}{v}\Big)\Phi^+(u)\Phi^-(v)(z^m f) = z^m u^{-m} v^{m-1} H(u)E(-v)E^\perp(-u)H^\perp(v)(f),$$
$$\Big(1-\frac{v}{u}\Big)\Phi^-(v)\Phi^+(u)(z^m f) = z^m u^{-m-1} v^{m} H(u)E(-v)E^\perp(-u)H^\perp(v)(f).$$

By Remark 2.1,

$$\Big(1-\frac{u}{v}\Big)^{-1}\sum_{k\geqslant 0}\frac{u^k}{v^{k+1}}, \quad \Big(1-\frac{v}{u}\Big)^{-1} = \sum_{k\geqslant 0}\frac{v^k}{u^{k+1}}.$$

Then

$$\begin{aligned}
&(\Phi^+(u)\Phi^-(v) + \Phi^-(v)\Phi^+(u))(z^m f)\\
&= z^m \frac{v^m}{u^m}\Bigg(\sum_{k\geqslant 0}\frac{u^k}{v^{k+1}} + \sum_{k\geqslant 0}\frac{v^k}{u^{k+1}}\Bigg) H(u)E(-v)E^\perp(-u)H^\perp(v)(f)\\
&= z^m \frac{v^m}{u^m}\,\delta(u,v)H(u)E(-v)E^\perp(-u)H^\perp(v)(f) = \delta(u,v)\cdot z^m f.
\end{aligned}$$

We used (2.3) along with the property of the formal delta distribution $\delta(u,v)A(v) = \delta(u,v)A(u)$ for any formal distribution $A(u)$ (cf., for example, [14, 15]). □

Let $1 \in \mathscr{B}^{(0)}$ be the constant function.

Proposition 3.2. $\Phi^+(u_1)\dots\Phi^+(u_l)\,(1) = z^l u_1^{-l}\dots u_l^{-1} Q(u_1,\dots,u_l)$, *where*

$$Q(u_1,\dots,u_l) = \prod_{1\leqslant i<j\leqslant l}\Big(1-\frac{u_i}{u_j}\Big)\prod_{i=1}^{l} H(u_i).$$

Proof. Using Proposition 2.1 and taking into account that $E^\perp(-u)(1) = 1$, we write

$$\begin{aligned}
\Phi^+(u_1,)\dots\Phi^+(u_l)\,(1) &= z^l u_1^{-l}\dots u_l^{-1}\prod_{1\leqslant i<j\leqslant l}\Big(1-\frac{u_i}{u_j}\Big)\prod_{i=1}^{l} H(u_i)E^\perp(-u_i)(1)\\
&= z^l u_1^{-l}\dots u_l^{-1} Q(u_1,\dots,u_l).
\end{aligned}$$

The proposition is proved. □

It is known [9, 8] that $Q(u_1,\dots,u_l)$ is a generating function for Schur symmetric functions in the following sense. Consider the series expansion of the rational function

$$Q(u_1,\dots,u_l) = \sum_{(\alpha_1,\dots,\alpha_l)\in\mathbb{Z}^l} Q_\lambda\, u_1^{-\lambda_1}\dots u_l^{-\lambda_l}, \quad |u_1| < \dots < |u_l|.$$

Then for any partition $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_l \geqslant 0)$ the coefficient of $u_1^{-\lambda_1} \ldots u_l^{-\lambda_l}$ is exactly a Schur symmetric function: $Q_{(\lambda_1,\ldots,\lambda_l)} = s_\lambda$. Thus, Proposition 3.2 describes the vertex operator presentation of Schur symmetric functions (cf. [8]–[10]).

4 Generalized Fermions and Hall–Littlewood Symmetric Functions

Let λ be a partition of length at most n, and let t be a parameter. Hall–Littlewood polynomials are defined by

$$P_\lambda(x_1,\ldots,x_n;t) = \left(\prod_{i\geqslant 0}\prod_{j=1}^{m(i)} \frac{1-t}{1-t^j}\right) \sum_{\sigma\in S_n} \sigma\left(x_1^{\lambda_1}\ldots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\right),$$

where $m(i)$ is the number of parts of the partition λ that are equal to i and S_n is the symmetric group of n letters [9]. Labeled by partitions, the Hall–Littlewood polynomials form a linear basis of the deformed ring $\Lambda[t]$ of symmetric polynomials with coefficients in $\mathbb{C}[t]$.

In this section, we show that the vertex operators presentation of Hall–Littlewood polynomials P_λ is obtained from the vertex operators presentation of Schur symmetric functions s_λ by a simple twist of the fields of charged free fermions by multiplication by $E(-u/t)$ or $H(u/t)$. This approach significantly simplifies the technical proofs of [1, 2] and provides new insight into the original results of these papers.

Consider the deformed boson Fock space $\mathscr{B}(t) = \oplus \mathscr{B}^{(m)}[t]$, where $\mathscr{B}^{(m)}[t] = z^m\Lambda[t]$. We extend the action of the operators in Section 2 to $\mathscr{B}(t)$ by t-linearity. Define the quantum fields of operators acting on $\mathscr{B}(t)$ by

$$\Psi^+(u) = E(-u/t)\Phi^+(u) = uR(u)E(-u/t)H(u)E^\perp(-u), \tag{4.1}$$

$$\Psi^-(u) = H(u/t)\Phi^-(u) = R^{-1}(u)H(u/t)E(-u)H^\perp(u). \tag{4.2}$$

Proposition 4.1. *The quantum fields $\Psi^\pm(u)$ satisfy the relations of generalized fermions*

$$\left(1 - \frac{ut}{v}\right)\Psi^\pm(u)\Psi^\pm(v) + \left(1 - \frac{vt}{u}\right)\Psi^\pm(v)\Psi^\pm(u) = 0, \tag{4.3}$$

$$\left(1 - \frac{vt}{u}\right)\Psi^+(u)\Psi^-(v) + \left(1 - \frac{ut}{v}\right)\ \Psi^-(v)\Psi^+(u) = \delta(u,v)(1-t)^2. \tag{4.4}$$

The proof is based on the commutation relations of Proposition 2.1 and follows the same line as the proof of Proposition 3.1.

Proposition 4.1 immediately implies that the operators $\Psi^\pm(u)$ provide the vertex operators realization [1] of Hall–Littlewood polynomials. Let

$$F(u_1,\ldots,u_l;t) = \prod_{1\leqslant i<j\leqslant l} \frac{u_j - u_i}{u_j - u_i t} \prod_{i=1}^{l} H(u_i)E(-u_i/t),$$

where the expression $\prod\limits_{1\leqslant i<j\leqslant l} \frac{u_j-u_i}{u_j-u_it}$ is understood as the series expansion of this rational function in the region $|u_1| < \cdots < |u_l|$. Consider the expansion

$$F(u_1,\ldots,u_l;t) = \sum_{\lambda_1,\ldots\lambda_l\in\mathbb{Z}} F_\lambda\, u_1^{-\lambda_1}\ldots u_l^{-\lambda_l}, \quad |u_1| < \cdots < |u_l|.$$

As proved in [1] (cf. also [9]), for any partition $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_l \geqslant 0)$ the coefficient of $u_1^{-\lambda_1} \ldots u_l^{-\lambda_l}$ is exactly a Hall–Littlewood symmetric function: $F_\lambda = P_\lambda(x_1, x_2, \ldots; t)$.

Proposition 4.2 (the vertex operator presentation of Hall–Littlewood symmetric functions)**.**

$$\Psi^+(u_1) \ldots \Psi^+(u_l)\,(1) = z^l u_1^{-l} \ldots u_l^{-1} F(u_1, \ldots, u_l; t). \tag{4.5}$$

Proof. From the definitions (4.1), (4.2) and the commutation relations of Proposition 2.1 one immediately finds

$$\begin{aligned} &\Psi^+(u_1) \ldots \Psi^+(u_l)(1) \\ &= z^l \prod_{1 \leqslant i<j \leqslant l} \Big(1 - \frac{u_i}{u_j}\Big)\Big(1 - \frac{u_i t}{u_j}\Big)^{-1} \prod_{i=1}^{l} u_i^{i-l-1} H(u_i) E(-u_i/t) \prod_{i=1}^{l} E^\perp(-u_i)(1), \end{aligned}$$

which simplifies to $z^l u_1^{-l} \ldots u_l^{-1} F(u_1, \ldots, u_l; t)$ since $E^\perp(-u)(1) = 1$. □

Corollary 4.1. *The generating functions $F(u_1, \ldots, u_l; t)$ for Hall–Littlewood polynomials and $Q(u_1, \ldots, u_l)$ for Schur symmetric functions are related by*

$$F(u_1, \ldots, u_l; t) = \prod_{1 \leqslant i<j \leqslant l} \Big(1 - \frac{t u_i}{u_j}\Big)^{-1} \prod_{i=1}^{l} E(-u_i/t)\, Q(u_1, \ldots, u_l).$$

5 Boson–Fermion Correspondence for Hall–Litllewood Polynomials Revisited

The classical boson-fermion correspondence connects the action of charged free fermions and the action of the (classical) Heisenberg algebra with generators $\{\alpha_m\}_{m \in \mathbb{Z}}$, central element 1, and the relations (2.5):

(I) Heisenberg algebra → charged free fermions → Heisenberg algebra

In the same spirit, the deformed boson-fermion correspondence between the actions of generalized fermions and the twisted Heisenberg algebra was established in [2]:

(II) twisted Heisenberg algebra → generalized fermions → twisted Heisenberg algebra

The twisted Heisenberg algebra is an algebra with generators $\{\mathbf{h}_k\}$, central element c, and the relations

$$[\mathbf{h}_n, \mathbf{h}_m] = \frac{m \delta_{m,-n}}{1 - t^{|m|}} \cdot c. \tag{5.1}$$

This construction found its applications in [6, 7]. At the same time, it has certain disadvantageous deviations from the format of the classical boson-fermion correspondence. In particular, to obtain the bosonization [2] of generalized fermions, one has to change the standard definition of the normal ordered product of fields. Moreover, the natural presence of the action of the classical Heisenberg algebra and the Virasoro algebra is not reflected by this deformed version.

In this section, we propose another deformed construction of the boson-fermion correspondence, different from [2]. It establishes a connection between the actions of generalized fermions and the classical Heisenberg algebra:

(III) Heisenberg algebra → generalized fermions → Heisenberg algebra

Among the advantages of the boson-fermion correspondence (III) over (II) is that (a) the standard definition of the normal ordered product of fields is used in all definitions, (b) the action of the twisted Heisenberg algebra (5.1) is a certain renormalization of the action of the classical Heisenberg algebra, (c) the action of the Virasoro algebra remains naturally in the construction, (d) the proofs of statements of the correspondences (II) and (III) become simple implications of the results of the classical boson-fermion correspondence (I).

We recall the statement of the classical boson-fermion correspondence in the form convenient for our exposition. For a formal distribution $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ we define

$$a(z)_+ = \sum_{n\leqslant -1} a_n z^{-n-1}, \quad a(z)_- = \sum_{n\geqslant 0} a_n z^{-n-1}.$$

The normal ordered product of two formal distributions is a formal distribution defined by $:a(z)b(z): = a(z)_+ b(z) + b(z)a(z)_-$ (cf., for example, [15]).

Proposition 5.1 (the classical boson-fermion correspondence (I); [11, 13, 14])**.**

(a) *In the case of the action* (2.7) *of the Heisenberg algebra on the boson space* $\mathscr{B}$, *the fields*

$$\Phi^+(u) = uR(u)\exp\Big(\sum_{n\geqslant 1}\alpha_{-n}\frac{1}{u^n}\Big)\exp\Big(-\sum_{n\geqslant 1}\alpha_n u^n\Big), \tag{5.2}$$

$$\Phi^-(u) = R^{-1}(u)\exp\Big(-\sum_{n\geqslant 1}\alpha_{-n}\frac{1}{u^n}\Big)\exp\Big(\sum_{n\geqslant 1}\alpha_n u^n\Big) \tag{5.3}$$

satisfy the relations (3.3), (3.4) *and define the action of charged free fermions on* $\mathscr{B}$.

(b) *Let* $\Phi^\pm(u) = \sum_{k\in\mathbb{Z}} \Phi^\pm_{k+1/2} u^{\pm k}$ *satisfy the relations* (3.3) *and* (3.4), *and let*

$$\Phi^+(u)_+ = \sum_{k\geqslant 1}\Phi^+_{k+1/2}u^k, \quad \Phi^+(u)_- = \sum_{k\leqslant 0}\Phi^+_{k+1/2}u^k.$$

Then the coefficients of the formal distribution $\alpha(u) =: \Phi^+(u)\cdot\Phi^-(u) := \Phi^+(u)_+\Phi^-(u) - \Phi^-(u)\,\Phi^+(u)_-$ *satisfy the relations* (2.5) *of the Heisenberg algebra.*

(c) *Let* $\Phi^\pm(u) = \sum_{k\in\mathbb{Z}} \Phi^\pm_{k+1/2} u^{\pm k}$ *satisfy the relations* (3.3) *and* (3.4). *For any* $\beta\in\mathbb{C}$ *the coefficients of the formal distribution* $L^{(\beta)}(u) = \sum_{k\in\mathbb{Z}} L_k u^{-k-2}$ *defined by* $L^{(\beta)}(u) = \beta :\partial\Phi^+(u)\,\Phi^-(u): + (1-\beta) :\Phi^+(u)\partial\Phi^-(u):$ *satisfy the relations of the Virasoro algebra with central charge* $c_\beta = -12\beta^2 + 12\beta - 2$:

$$[L^{(\beta)}(u), L^{(\beta)}(v)] = \partial_v L(v)\,\delta(u,v) + 2L(v)\,\partial_v\delta(u,v) + \frac{c_\beta}{12}\,\partial_v^3\delta(u,v).$$

Proof. In the case of the action (2.7) of the Heisenberg algebra, we use (2.4) and (2.9) to compare (5.2) and (5.3) with the definitions (3.1) and (3.2) of $\Phi^\pm(u)$ to get part (a), while for the proof of (b) and (c) we refer to [15]. □

Proposition 5.2 (the revisited version of the boson-fermion correspondence (III)).

(a) *The action of the twisted Heisenberg algebra can be defined by renormalization of the action of the classical Heisenberg algebra*:

$$\mathbf{h}_{-n} = \alpha_{-n}, \quad n \geqslant 0,$$
$$\mathbf{h}_n = \frac{1}{1-t^n}\alpha_n \quad n > 0, \ c = 1. \tag{5.4}$$

(b) *Consider the action* (2.7) *of the Heisenberg algebra on the boson space* $\mathscr{B}$ *expanded to the deformed space* $\mathscr{B}(t)$ *by the rule* $\alpha_n(t^s f) = t^s\alpha_n(f)$. *Then the fields*

$$\Psi^+(u) = uR(u)\exp\left(\sum_{n\geqslant 1}(1-t^n)\alpha_{-n}\frac{1}{u^n}\right)\exp\left(-\sum_{n\geqslant 1}\alpha_n u^n\right), \tag{5.5}$$

$$\Psi^-(u) = R^{-1}(u)\exp\left(-\sum_{n\geqslant 1}(1-t^n)\alpha_{-n}\frac{1}{u^n}\right)\exp\left(\sum_{n\geqslant 1}\alpha_n u^n\right) \tag{5.6}$$

satisfy the relations (4.3), (4.4) *and define the action of generalized fermions on* $B(t)$.

(c) *The other way, consider the action of the generalized fermions* $\Psi^\pm(u)$ *on* $B(t)$ *defined by* (4.1), (4.2). *Then the coefficients of the normal ordered product* $\alpha(u) =: H(u/t)\Psi^+(u)\cdot E(-u/t)\Psi^-(u):$ *satisfy* (2.5) *and define the action of the classical Heisenberg algebra on* $\mathscr{B}(t)$.

(d) *For the action* (4.1), (4.2) *of the generalized fermions* $\Psi^\pm(u)$ *on* $\mathscr{B}(t)$ *the coefficients of the formal distribution* $L^{(\beta)}(u) = \sum_{k\in\mathbb{Z}} L_k u^{-k-2}$ *defined by the formula*

$$L^{(\beta)}(u) = \beta : H(u/t)(t^{-1}P(-u/t)\Psi^+(u) + \partial\Psi^+(u))\cdot E(-u/t)\Psi^-(u): \\ + (1-\beta) : H(u/t)\Psi^+(u)\cdot E(-u/t)(-t^{-1}P(-u/t)\Psi^-(u) + \partial\Psi^-(u)):$$

satisfy the relations of the Virasoro algebra with central charge $c_\beta = -12\beta^2 + 12\beta - 2$.

Proof. A direct verification of the commutation relations proves part (a). For the given action (2.7) of the Heisenberg algebra we use (2.4) and (2.9) to compare (5.5) and (5.6) with the definitions (4.1) and (4.2) of $\Psi^\pm(u)$ to obtain part (b). Note that, by (a) and (4.1), (4.2), the quantum fields

$$\Phi^+(u) = H(u/t)\Psi^+(u), \quad \Phi^-(u) = E(-u/t)\Phi^-(u) \tag{5.7}$$

satisfy the relations of charged free fermions. Then part (c) follows from Proposition 5.1 (b).

Part (d) follows by substituting (5.7) into the formula of $L^{(\beta)}(u)$ in Proposition 5.1 (c) and the properties (cf. [9]) $P(u) = \partial H(u)/H(u)$ and $P(-u) = \partial E(u)/E(u)$. □

6 Tau-Functions of the KP Hierarchy in $\Lambda[t]$

The ring $\Lambda[t]$ possesses the scalar product $\langle\cdot,\cdot\rangle_t$ which is a deformation of the scalar product (2.8) on Λ. Following [9, III.4], we define symmetric functions $S_\lambda = S_\lambda(x_1, x_2, \dots; t)$ as a basis dual to the classical Schur functions $s_\lambda = s_\lambda(x_1, x_2, \dots)$ with respect to the deformed scalar product $\langle S_\lambda, s_\mu\rangle_t = \delta_{\lambda,\mu}$.

Proposition 6.1. *Let*

$$S(u_1,\dots,u_l)=\prod_{i<j}\Big(1-\frac{u_i}{u_j}\Big)\prod_{i=1}^{l}H(u_i)E(-u_i/t).$$

Then for any partition λ the coefficient of $u_1^{-\lambda_1}\dots u_l^{-\lambda_l}$ is S_λ.

Proof. By [9, III.4, formula (4.3) and III.2, formula (2.10)], S_λ can be expressed through Hall–Littlewood polynomials $P_{(k)}$ by an analogue of the Jacobi–Trudi formula

$$S_\lambda=\det\,[(1-t)P_{(\lambda_i-i+j)}]. \tag{6.1}$$

Here, $P_{(k)}=P_{(k)}(x_1,x_2,\dots;t)$ is the coefficient of the expansion

$$S(u;t)=H(u)E(-u/t)=(1-t)\sum_{k=0}^{\infty}P_{(k)}\frac{1}{u^k}.$$

Then

$$\begin{aligned}S(u_1,\dots,u_l)&=\prod_{i<j}\Big(1-\frac{u_i}{u_j}\Big)\prod_{i=1}^{l}S(u_i;t)=\det\,[u_i^{-i+j}]\prod_{i}S(u_i;t)=\det\,[u_i^{-i+j}S(u_i;t)]\\&=(1-t)^l\sum_{\sigma\in S_l}\operatorname{sgn}(\sigma)\sum_{a_1\dots a_l}P_{(a_1)}u_1^{-a_1-1+\sigma(1)}\dots P_{(a_l)}u_l^{-a_l-l+\sigma(l)}\\&=(1-t)^l\sum_{\lambda_1\dots\lambda_l}\sum_{\sigma\in S_l}\operatorname{sgn}(\sigma)P_{(\lambda_1-1+\sigma(1))}\dots P_{(\lambda_l-l+\sigma(l))}u_1^{-\lambda_1}\dots u_l^{-\lambda_l}\\&=\sum_{\lambda_1\dots\lambda_l}\det\,[(1-t)P_{(\lambda_i-i+j)}]u_1^{-\lambda_1}\dots u_l^{-\lambda_l}.\end{aligned}$$

The proposition is proved. □

Corollary 6.1. *The generating functions $S(u_1,\dots,u_l)$ for symmetric functions S_λ and $Q(u_1,\dots,u_l)$ for Schur symmetric functions s_λ are related by the formula*

$$S(u_1,\dots,u_l)=E(-u_l/t)\dots E(-u_1/t)Q(u_1,\dots u_l).$$

The vertex operator presentation of the generating function $S(u_1,\dots,u_l)$ can be written as

$$z^l u_1^{-l}\dots u_l^{-1}S(u_1,\dots,u_l)=E(-u_l/t)\dots E(-u_1/t)\Phi^+(u_1)\dots\Phi^+(u_l)\,(1).$$

The relation of the symmetric functions S_λ to tau-functions of the KP hierarchy is discussed in [6]. The determinant type property (6.1) of symmetric functions S_λ is interpreted as the Plücker coordinates type property. This observation allows the authors of [6] to conclude that the expression $\sum_\lambda s_\lambda S_\lambda$ is an example of tau-functions of the KP hierarchy in $\Lambda[t]$.

Here, we show that the symmetric functions S_λ themselves are tau-functions of the KP hierarchy. This result is not present in [6] and does not follow from the above-mentioned statement. Moreover, we provide an explicit formula for the charged free fermions action that realizes the KP hierarchy for these tau-functions and conclude with one more vertex operator presentation of the generating function of S_λ.

It is well known that the bilinear form of the KP hierarchy is the equation $\Omega(\tau\otimes\tau)=0$, where $\tau\in\mathscr{B}^{(0)}=\Lambda=\mathbb{C}[p_1,p_2,\dots]$ and

$$\Omega=\operatorname{Res}_{u=0}\Big(\frac{1}{u}\Phi^+(u)\otimes\Phi^-(u)\Big). \tag{6.2}$$

It is known that Schur symmetric functions $s_\lambda \in \mathscr{B}^{(0)}$ are solutions of the KP hierarchy [11, 16, 17]. We formally define

$$\Phi_t^+(u) = uR(u)H(u)E(-u/t)\prod_{i=0}^{\infty} E^{\perp}(-u/t^i), \tag{6.3}$$

$$\Phi_t^-(u) = R(u)^{-1}E(-u)H(u/t)\prod_{i=0}^{\infty} H^{\perp}(u/t^i) \tag{6.4}$$

and consider

$$\Omega_t = \mathrm{Res}_{u=0}\Big(\frac{1}{u}\Phi_t^+(u)\otimes\Phi_t^-(u)\Big), \tag{6.5}$$

$$\Omega_t(\tau\otimes\tau) = 0, \quad \tau\in\mathscr{B}^{(0)}(t). \tag{6.6}$$

We summarize the statements in the following assertion.

Proposition 6.2. (a) *Let $\Phi_t^{\pm}(u)$ be defined by* (6.3), (6.4) *with the expansion to the region $|t|<1$. The operators $\Phi_t^{\pm}(u)$ satisfy exactly the same relations as the classical charged free fermions $\Phi^{\pm}(u)$ in Proposition* 3.1. *Thus, the operators $\Phi_t^{\pm}(u)$ provide the action of charged free fermions on the deformed space $\mathscr{B}(t)$. Consequently, Equation* (6.6) *is the bilinear identity of the* KP *hierarchy on functions $\tau\in\mathscr{B}(t)$.*

(b) *The symmetric functions S_λ are solutions of the bilinear identity* (6.6)*. Consequently, the symmetric functions S_λ are tau-functions of the* KP *hierarchy.*

(c) *The generating function $S(u_1,\dots,u_l)$ of symmetric functions S_λ has the second vertex operator presentation $\Phi_t^+(u_1)\dots\Phi_t^+(u_l)\,(1) = z^l u_1^{-l}\dots u_l^{-1}S(u_1,\dots,u_l)$.*

Proof. The Schur symmetric functions s_λ are expressed in terms of complete symmetric functions by the Jacobi–Trudi formula $s_\lambda = \det[h_{\lambda_i-i+j}]$, whereas the complete symmetric functions h_k in the determinant can be expressed as polynomial functions of power sums $h_k = h_k(p_1,p_2,\dots)$ through the relation

$$\sum_{k=0}^{\infty}\frac{h_k}{u^k} = H(u) = \exp\Bigg(\sum_{n\geqslant 1}\frac{p_n}{n}\frac{1}{u^n}\Bigg).$$

At the same time, from (6.1) the dual symmetric functions S_λ are given by $S_\lambda = \det[q_{\lambda_i-i+j}]$, where the symmetric functions $q_k = (1-t)P_{(k)}$ are expressed as functions of power sums $q_k = q_k(p_1,p_2,\dots)$ through the relation

$$\sum_{k=0}^{\infty}\frac{q_k}{u_k} = H(u)E(-u/t) = \exp\Bigg(\sum_{n\geqslant 1}(1-t^n)\frac{p_n}{n}\frac{1}{u^n}\Bigg).$$

Thus, S_λ as a function of power sums $(p_1,p_2,\dots)$ can be obtained from s_λ by the substitution of variables $p_n\to(1-t^n)p_n$.

Recall that the Schur symmetric functions s_λ are solutions of the bilinear KP identity (6.2) and $\Phi^{\pm}(u)$ in (6.2) expressed in terms of the operators p_i and $\partial/\partial p_i$ have the form (5.2), (5.3). Hence S_λ satisfies the bilinear identity $\Omega_t(S_\lambda\otimes S_\lambda)=0$, where

$$\Omega_t = \mathrm{Res}_{u=0}\Big(\frac{1}{u}\Phi_t^+(u)\otimes\Phi_t^-(u)\Big), \tag{6.7}$$

and $\Phi_t^{\pm}(u)$ is obtained from (5.2), (5.3) by the same substitution $p_n \to (1-t^n)p_n$:

$$\Phi_t^{+}(u) = uR(u)\exp\left(\sum_{n\geqslant 1}\frac{(1-t^n)p_n}{n}\frac{1}{u^n}\right)\exp\left(\sum_{n\geqslant 1}\frac{1}{(1-t^n)}\frac{\partial}{\partial p_n}u^n\right),$$

$$\Phi_t^{-}(u) = R(u)^{-1}\exp\left(-\sum_{n\geqslant 1}\frac{(1-t^n)p_n}{n}\frac{1}{u^n}\right)\exp\left(\sum_{n\geqslant 1}\frac{1}{(1-t^n)}\frac{\partial}{\partial p_n}u^n\right).$$

Using the geometric series expansion

$$\frac{1}{1-t^n} = \sum_i (t^i)^n, \quad |t| < 1,$$

we can write the second exponential factor as

$$\exp\left(\mp\sum_{n\geqslant 1}\frac{1}{(1-t^n)}\frac{\partial}{\partial p_n}u^n\right) = \prod_{i=0}^{\infty}\exp\left(\mp\sum_{n\geqslant 1}\frac{\partial}{\partial p_n}(t^i u)^n\right) = \prod_{i=0}^{\infty}E^{\perp}(-u/t^i).$$

Note that

$$\exp\left(\sum_{n\geqslant 1}\frac{(1-t^n)p_n}{n}\frac{1}{u^n}\right) = H(u)E(-u/t),$$

which gives (6.3) for $\Phi_t^{+}(u)$ and, similarly, (6.4) for $\Phi_t^{-}(u)$.

Applying the same substitution $p_n \to (1-t^n)p_n$ to the result of Proposition 3.2, we get (c). Proposition 6.2 is proved. □

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