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ON THE CHROMATIC NUMBERS CORRESPONDING TO EXPONENTIALLY RAMSEY SETS

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In this paper, nontrivial upper bounds on the chromatic numbers of the spaces $\mathbb{R}_p^n = (\mathbb{R}^n, l_p)$ with forbidden monochromatic sets are proved. In the case of a forbidden rectangular parallelepiped or a regular simplex, explicit exponential lower bounds on the chromatic numbers are obtained. Exact values of the chromatic numbers of the spaces $\mathbb{R}_p^n$ with a forbidden regular simplex in the case $p = \infty$ are found. Bibliography: 39 titles.

1. Introduction

In 1950, Nelson formulated his famous question (see [1]): "What is the minimal number $\chi\left(\mathbb{R}^2\right)$ of colors enough to color all points of a plane such that no two points at distance 1 have the same color?" Unfortunately, this question remains open. It is known that

$$5 \le \chi\left(\mathbb{R}^2\right) \le 7$$

(see [1] for the proof of the upper bound; the lower bound has been proved in a recent preprint [2]).

There are different ways to generalize the classical Nelson's problem. For example, instead of the plane $\mathbb{R}^2$, one can color the Euclidean space $\mathbb{R}^n$ or, more general, the metric space $\mathbb{R}_p^n$, for $p \in [1; \infty]$, which is defined as $\mathbb{R}^n$ with the following metric l_p:

$$\mathbf{x} = (x_1, \ \dots \ , x_n), \ \mathbf{y} = (y_1, \ \dots \ , y_n) \ \Rightarrow \ l_p(\mathbf{x}, \mathbf{y}) = \sqrt[p]{|x_1 - y_1|^p + \dots + |x_n - y_n|^p}.$$

The chromatic numbers $\chi\left(\mathbb{R}_p^n\right)$ were studied in several papers, and a lot of facts about them are known. In particular, for $\chi\left(\mathbb{R}_p^n\right)$ where n is large, the following is known:

Theorem 1. *The following four statements are valid:*
(1) *for $p \in [1, \infty]$ and $n \to \infty$, one has*

$$\left(\frac{1+\sqrt{2}}{2} + o(1)\right)^n = (1.207... + o(1))^n \le \chi\left(\mathbb{R}_p^n\right) \le (4 + o(1))^n,$$

(2) *for $p = 1$ and $n \to \infty$, one has*

$$\left(\frac{1+\sqrt{3}}{2} + o(1)\right)^n = (1.366... + o(1))^n \le \chi\left(\mathbb{R}_1^n\right) \le (4 + o(1))^n,$$

(3) *for $p = 2$ and $n \to \infty$, one has*

$$(1.239... + o(1))^n \le \chi\left(\mathbb{R}_2^n\right) \le (3 + o(1))^n,$$

(4) *For $p = \infty$ and arbitrary n, one has*

$$\chi\left(\mathbb{R}_\infty^n\right) = 2^n.$$

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For proofs of upper and lower bounds in item 1, see [3] and [4], respectively. The proof of item 2 can be found in [5] and [4]. Item 3 is proved in [6] and [7]. The equality of item 4 is classical, and its proof can be found in [8].

Euclidean Ramsey Theory studies other generalizations of Nelson's question, see [9–11] and book [12]. Let the parameters p and n be as defined above, and $\mathcal{Y} = (Y, d_Y)$ be a metric space. A subset $X \subset \mathbb{R}^n$ is called a *copy* of the space $\mathcal{Y}$ if there exists a bijection $f : Y \to X$ such that for any $y_1,\, y_2 \in Y$ one has $d_Y(y_1, y_2) = l_p(f(y_1), f(y_2))$.

The *chromatic number* $\chi\left(\mathbb{R}^n_p; \mathcal{Y}\right)$ *of the space* $\mathbb{R}^n_p$ *with a forbidden space* $\mathcal{Y}$ is the minimal number of colors enough to color $\mathbb{R}^n_p$ such that no copy $X \subset \mathbb{R}^n$ of the space $\mathcal{Y}$ is colored with only one color.

Clearly, the initial Nelson's question of finding $\chi\left(\mathbb{R}^2\right)$ is a special case of finding $\chi\left(\mathbb{R}^n_p; \mathcal{Y}\right)$. Indeed, one can set $n = p = 2$ and choose a two-point space as $\mathcal{Y}$. Since the exact value of $\chi\left(\mathbb{R}^2\right)$ has not been found yet, we cannot hope to obtain the exact value of $\chi\left(\mathbb{R}^n_p; \mathcal{Y}\right)$ in the general case. However, we will try to solve simpler questions.

For example, we can try to ask a natural question, which is similar to the result of Theorem 1: does $\chi\left(\mathbb{R}^n_p; \mathcal{Y}\right)$ tend to infinity (exponentially) as n tends to infinity? If this property holds, the metric space $\mathcal{Y}$ is called an *l_p-(exponentially) Ramsey space.*

No criterion that allows one to describe all Ramsey and exponentially Ramsey sets is known. However, there are two popular conjectures on this topic. The first conjecture claims that each Ramsey set is exponentially Ramsey. The second conjecture claims that for $p = 2$ (i. e., for Euclidean metric) a set Y is Ramsey if and only if it is finite and "spherical" (i. e., lies on a certain multidimensional sphere). Now it is known that these two conditions are necessary, but their sufficiency is not proved, see [12].

Only a few types of sets are known to be Ramsey or exponentially Ramsey. We will list them after giving some definitions.

An *l_p-Cartesian product* $\times_p$ of metric spaces $\mathcal{X} = (X, d_X)$ and $\mathcal{Y} = (Y, d_Y)$ is a metric space $\mathcal{X} \times_p \mathcal{Y} = (X \times Y, d)$, where the metric d is defined as follows:

$$\forall\, x_1, x_2 \in X \quad \forall\, y_1, y_2 \in Y \quad d((x_1, y_1), (x_2, y_2)) = \sqrt[p]{d_X^p(x_1, x_2) + d_Y^p(y_1, y_2)}.$$

A *k-dimensional l_p-rectangular parallelepiped* $\mathcal{I}_p^k = \mathcal{I}_p^k\,(a_1,\ \ldots\ , a_k)$ is a metric space, which is an l_p-Cartesian product of k " intervals" (2-point metric spaces) such that for all i, the length of the ith interval is equal to a_i.

A *k-dimensional simplex* is any metric space that consists of $k+1$ points. If, in addition, all distances between two of these points are equal, then we call such a simplex $\mathcal{S}_k$ *regular.* Note that this definition is natural in l_2-metric, but there is no obstacle to define a simplex in this way for $p \neq 2$.

Now we are ready to list all known exponentially Ramsey sets.

It was proved in [13] that for all $p \in [1; \infty]$, the set of vertices of any l_p-rectangular parallelepiped is l_p-exponentially Ramsey. Also it was proved there that for $p = 2$, each nondegenerate simplex is exponentially Ramsey. Clearly, each subset of an exponentially Ramsey set is also exponentially Ramsey: if $\mathcal{Y}_1 \subset \mathcal{Y}_2$ then $\chi\left(\mathbb{R}^n_p; \mathcal{Y}_1\right) \geq \chi\left(\mathbb{R}^n_p; \mathcal{Y}_2\right)$. These are all known examples of exponentially Ramsey sets.

A weaker Ramsey property is proved for some other sets. In [14] and [15], it was proved that the set of vertices of an arbitrary regular polytope is l_2-Ramsey. In [16], it was proved that an arbitrary trapezoid is l_2-Ramsey.

However, despite the fact that some sets have been known to be exponentially Ramsey for a long time, only a few papers studied exponentially increasing lower bounds similar to bounds from Theorem 1 for these exponentially Ramsey sets.

In [17] and [18], the case $\mathcal{Y} = \mathcal{S}_k$ was studied and the following theorem was proved.

Theorem 2. *Given $p \in [1, \infty]$, $k \geq 1$, for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}_p^n; \mathcal{S}_k\right) \geq \left(1 + \frac{1}{2^{2^{k+4}}} + o(1)\right)^n.$$

In [19], a new method allowed one to obtain several bounds and, in particular, the following Theorem.

Theorem 3. *Let $p \in [1, \infty]$, and let $\mathcal{Y} = \mathcal{S}_k$ or $\mathcal{Y} = \mathcal{I}_p^k$. Then there is a function $\varepsilon(k)$ that approaches 0 as $k \to \infty$ such that the following holds. Given $k \geq 1$, for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}_p^n; \mathcal{Y}\right) \geq \left(1 + \left(\frac{1}{2 + \varepsilon(k)}\right)^k + o(1)\right)^n.$$

It is clear that for large k, Theorem 3 gives a more powerful bound than Theorem 2. However, since there is no explicit form of $\varepsilon(k)$, we do not know whether it is true for small k or not. In Sec. 2, we state Theorem 6 and its advantage over Theorem 2 becomes clear for all k.

An interesting result is proved in [19] (compare with item 4 of Theorem 1):

Theorem 4. *Given $k \geq 1$, one has $\chi\left(\mathbb{R}_\infty^n; \mathcal{S}_k\right) = (2 + o(1))^n$ as $n \to \infty$.*

In this paper, we find the *exact value* of $\chi\left(\mathbb{R}_\infty^n; \mathcal{S}_k\right)$ for every n and k via Theorem 7 (see Sec. 2).

This is what is known about lower exponentially increasing bounds on $\chi\left(\mathbb{R}_p^n; \mathcal{Y}\right)$. Now let us discuss upper bounds, some of them can be found in [20] and [21]. In [21], the following theorem was proved.

Theorem 5. *Let $\mathcal{Y} = (Y, d_Y)$ be a finite metric space. Define $l(\mathcal{Y})$ as a minimal positive l such that any two elements $u, v \in Y$ can be joined by a path $\{u = y_0, y_1, \ \ldots \ , y_m = v\}$, where $y_i \in Y$ and*

$$\max_{0 \leq i \leq m-1} d_Y(y_i, y_{i+1}) \leq l.$$

Define $R(l_2; \mathcal{Y})$ as the minimal radius of an l_2-ball that contains a copy of $\mathcal{Y}$. Then, for $n \to \infty$, one has

$$\chi\left(\mathbb{R}_2^n; \mathcal{Y}\right) \leq \left(1 + \frac{l(\mathcal{Y})}{R(l_2; \mathcal{Y})} + o(1)\right)^n.$$

This theorem allows one to obtain explicit nontrivial upper bounds for any fixed space $\mathcal{Y}$, since in each special case one can easily calculate the values $l(\mathcal{Y})$ and $R(l_2; \mathcal{Y})$ (see, for example, several corollaries to this theorem in [21]). However, this theorem holds only for l_2-metric.

In the present paper, we apply the method from [21] to a more general case. The bound obtained in Theorem 8 (see Sec. 2) is valid for all l_p-metrics.

Note that some other similar problems of combinatorial geometry and Euclidean Ramsey theory were considered in papers [22–33], surveys [34–37], and in book [12].

2. Main results

As was mentioned in Sec. 1, this paper contains three main results. First of them is the following Theorem 6, which is a detailed version of Theorem 3.

Theorem 6. *For all $p \in [1, \infty]$, $k \geq 2$, and for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}_p^n; \mathcal{I}_p^k\right) \geq \left(1 + \frac{1}{2^{k+1} \cdot k^2} + o(1)\right)^n.$$

Since a regular k-dimensional simplex $\mathcal{S}_k$ is a subset of a $(k+1)$-dimensional cube $\mathcal{I}_p^{k+1}$ (a $k+1$ $\times_p$-power of an interval), Theorem 6 implies the following corollary.

Corollary 1. *For all $p \in [1, \infty]$, $k \geq 1$, and for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}_p^n; \mathcal{S}_k\right) \geq \left(1 + \frac{1}{2^{k+2} \cdot (k+1)^2} + o(1)\right)^n.$$

It is clear that the bound from Corollary 1 is more powerful than the bound from Theorem 2 for all k.

The second main result of our paper is the following Theorem 7.

Theorem 7. *Given positive integers k and n, one has*

$$\chi\left(\mathbb{R}_\infty^n; \mathcal{S}_k\right) = \left\lceil \frac{2^n}{k} \right\rceil.$$

In the special case $k = 1$, the result of Theorem 7 coincides with the previously known result (item 4 of Theorem 1).

To state the third main result of our paper, we need several new definitions concerned with the theory of dense packings.

Let K be a bounded convex body in $\mathbb{R}^n$, and let Ω be a lattice in $\mathbb{R}^n$ such that

$$\mathcal{K} = K + \Omega = \bigcup_{\omega \in \Omega} (K + \omega)$$

is a packing, i. e., the union is disjoint. Let $\delta(\mathcal{K}) = \frac{\text{vol } K}{\det \Omega}$ be the density of our packing and

$$\delta(K) = \sup_{\Omega} \delta(\mathcal{K}).$$

Let $r > 0$ and $B_p^n = B_p^n(r)$ be an n-dimensional l_p-ball of radius r. Let $c(p)$ be the least c for which the following inequality holds:

$$\delta(B_p^n) \geq (2^c + o(1))^{-n},$$

or, in other words,

$$c(p) = \limsup_{n \to \infty} \frac{\log_2\left(\delta^{-1}(B_p^n)\right)}{n}.$$

Now we are ready to state the third main result of our paper.

Theorem 8. *Let $p \in [1, \infty]$, and let $\mathcal{Y} = (Y, d_Y)$ be a finite metric space. The values $l(\mathcal{Y})$ and $R(l_p; \mathcal{Y})$ are defined as in Theorem 5. Then, for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}_p^n; \mathcal{Y}\right) \leq \left(2^{c(p)} \cdot \left(1 + \frac{l(\mathcal{Y})}{2 \cdot R(l_p; \mathcal{Y})}\right) + o(1)\right)^n.$$

Unfortunately, it is difficult to apply this theorem even in special cases. The exact value of $c(p)$ is known only for $p = \infty$: $c(\infty) = 0$ (see [38]). However, we need not the exact value of $c(p)$ to apply Theorem 8, it is enough to have an upper bound on it. Such upper bounds can be found in [38]. In particular, it is known that $c(p) \leq 1$ for all p.

One more problem with applying Theorem 8 in special cases is the fact that for $p \neq 2$ it is not clear how to calculate $R(l_p; \mathcal{Y})$ (or, even, how to give a lower bound on it). Of course, we can use the obvious inequality $2 \cdot R(l_p; \mathcal{Y}) \geq \operatorname{diam} \mathcal{Y}$, but this bound is often far from the optimal one.

Applying what was written above, we immediately obtain two corollaries to Theorem 8.

Corollary 2. *Let $p < \infty$ and $\mathcal{Y} = (Y, d_Y)$ be a finite metric space. Then for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}^n_p; \mathcal{Y}\right) \le \left(2 + \frac{l(\mathcal{Y})}{R(l_p; \mathcal{Y})} + o(1)\right)^n \le \left(2 + \frac{2 \cdot l(\mathcal{Y})}{\operatorname{diam} \mathcal{Y}} + o(1)\right)^n.$$

Corollary 3. *Let $p = \infty$ and $\mathcal{Y} = (Y, d_Y)$ be a finite metric space. Then for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}^n_\infty; \mathcal{Y}\right) \le \left(1 + \frac{l(\mathcal{Y})}{2 \cdot R(l_\infty; \mathcal{Y})} + o(1)\right)^n \le \left(1 + \frac{l(\mathcal{Y})}{\operatorname{diam} \mathcal{Y}} + o(1)\right)^n.$$

Remark 1. Note that, in "the most interesting case" $p = 2$, no better bound than $c(2) \le 1$ is known. Moreover, the conjecture that $c(2) = 1$ is popular. Thus, probably, for $p = 2$, the bound from Corollary 2 is the best possible that can be obtained by our method. However, this bound is much weaker than the previously known bound from Theorem 5. In the proof of Theorem 5, more powerful technique is used. The proof is based on the fact that each Voronoj decomposition consists of convex bodies. Unfortunately, this fact is not true for l_p-metrics in the general case. To obtain a bound valid for all p, we apply a less powerful but more universal method. Thus, in the case $p = 2$, our bound is weaker than the previously known one.

In Theorem 6, we have written an explicit lower bound on the chromatic number in the case where $\mathcal{Y} = \mathcal{I}^k_p(a_1, \ \dots \ , a_k)$. To illustrate this case, let us also write an upper bound, which trivially follows from Corollary 2.

Corollary 4. *Let $p < \infty$ and $\mathcal{Y} = \mathcal{I}^k_p = \mathcal{I}^k_p(a_1, \ \dots \ , a_k)$, where the a_i are sorted in ascending order. Then, for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}^n_p; \mathcal{I}^k_p\right) \le \left(2 + \frac{a_k}{\sqrt[p]{a_1^p + \ \cdots \ + a_k^p}} + o(1)\right)^n.$$

In particular, if all the a_i are equal, i. e., $\mathcal{I}^k_p$ is a k-dimensional cube, then

$$\chi\left(\mathbb{R}^n_p; \mathcal{I}^k_p\right) \le \left(2 + \frac{1}{\sqrt[p]{k}} + o(1)\right)^n.$$

The remaining part of our paper is organized as follows. In Sec. 3, we prove Theorem 6. In Sec. 4, we prove Theorem 7. Finally, in Sec. 5, we prove Theorem 8.

3. The proof of Theorem 6

3.1. Statement of an auxiliary Theorem 9 and derivation of Theorem 6 from Theorem 9. The proof of Theorem 6 is close to the proof of Theorem 3 from [19], but now we perform the main calculations more carefully. In fact, this leads to a more powerful statement than Theorem 6.

Theorem 9. *For all $p \in [1, \infty]$, $k \ge 3$, and for $n \to \infty$, one has*

$$\chi\left(\mathbb{R}^n_p; \mathcal{I}^k_p\right) \ge \left(e^{\frac{1}{2^{k+1} \cdot k \cdot (k-1)}} + o(1)\right)^n.$$

For all $k \ge 3$, Theorem 6 follows from Theorem 9, since

$$e^{\frac{1}{2^{k+1} \cdot k \cdot (k-1)}} > 1 + \frac{1}{2^{k+1} \cdot k \cdot (k-1)} > 1 + \frac{1}{2^{k+1} \cdot k^2}.$$

The only remaining case of Theorem 6 is the case $k = 2$. Here we need another result from [19]:

$$\chi\left(\mathbb{R}^n_p; \mathcal{I}^2_p\right) \ge (1.0428 \cdots + o(1))^n.$$

Clearly, this bound is even better than is required. Therefore, it remains only to prove Theorem 9.

3.2. The proof of Theorem 9. Let us give a proof by induction on k. The base of induction — the case $k = 3$ — follows from the bound proved in [19]:

$$\chi\left(\mathbb{R}_p^n; \mathcal{I}_p^3\right) \geq (1.0126\cdots + o(1))^n$$

and the trivial fact that

$$\ln 1.0126 > \frac{1}{2^{3+1} \cdot 3 \cdot 2}.$$

To prove the inductive step, we apply the following theorem proved in [19].

Theorem 10. *Let $\mathcal{A} = (A, d_A)$ be a finite metric l_p-exponentially Ramsey space for a certain $p \geq 1$. Assume that*

$$\chi\left(\mathbb{R}_p^n; \mathcal{A}\right) \geq (\chi_{\mathcal{A}} + o(1))^n$$

as $n \to \infty$. Let $\mathcal{I}$ be a two-point metric space. Then $\mathcal{A} \times_p \mathcal{I}$ is also an l_p-exponentially Ramsey space and

$$\chi\left(\mathbb{R}_p^n; \mathcal{A} \times_p \mathcal{I}\right) \geq \left(\chi_{\mathcal{A}}^{\delta} + o(1)\right)^n$$

as $n \to \infty$, where

$$c_a^b = \frac{a^a}{b^b \cdot (a-b)^{(a-b)}}, \quad \chi(x,y) = \frac{c_1^{\min(x, 2x-2y)}}{c_1^{x-y}}, \quad \delta = \frac{\ln \chi(x,y)}{\ln \chi_{\mathcal{A}} + \ln \chi(x,y) + \ln c_x^y + \ln c_{1-x}^{x-y}},$$

and the values of the auxiliary parameters $0 < y < x \leq \frac{1}{2}$ can be chosen arbitrary.

Remark 2. In [19], this theorem was stated under a stronger condition than the l_p-exponentially Ramsey property on $\mathcal{A}$. Informally speaking, it was required in [19] that the fact that $\mathcal{A}$ is l_p-exponentially Ramsey could be proved by showing a sequence of hypergraphs such that its number of vertices grew "rather slow". However, by looking over the proof of that theorem, one can see that a "rather slow" increasing of the number of vertices is not necessary, and it is sufficient to have any sequence of hypergraphs showing that $\mathcal{A}$ is l_p-exponentially Ramsey. One can also note that Erdős–de Breun theorem (which holds not only for graphs, but also for hypergraphs, see [34]) implies that such a sequence of hypergraphs exists for every l_p-exponentially Ramsey set. Hence, the theorem proved in [19] is equivalent to our Theorem 10.

Theorem 10 is rather complicated and not explicit, because the optimal values of the auxiliary parameters x and y still have to be calculated. That is why we use a simplified version of this theorem.

Corollary 5. *Let $p \geq 1$ and let $\mathcal{A} = (A, d_A)$ be a finite metric l_p-exponentially Ramsey space. Assume that*

$$\chi\left(\mathbb{R}_p^n; \mathcal{A}\right) \geq (e^r + o(1))^n$$

as $n \to \infty$, where $0 \leq r \leq \frac{1}{2}$. Let $\mathcal{I}$ be a two-point metric space. Then $\mathcal{A} \times_p \mathcal{I}$ is also an l_p-exponentially Ramsey space and

$$\chi\left(\mathbb{R}_p^n; \mathcal{A} \times_p \mathcal{I}\right) \geq \left(e^{r \cdot s(r)} + o(1)\right)^n$$

as $n \to \infty$, where

$$s(r) = \frac{1}{2} - \frac{\ln 2}{\ln\left(\frac{1}{r}\right)} - \frac{\ln 2 \cdot \ln\left(\ln\left(\frac{1}{r}\right)\right)}{\ln^2\left(\frac{1}{r}\right)}.$$

Proof. To obtain this result, we apply Theorem 10 to our situation with

$$x = \frac{r \ln\left(\frac{1}{r}\right)}{2\ln 2}, \quad y = \frac{r \ln\left(\frac{1}{r}\right)}{4\ln 2}.$$

After that it is sufficient to verify that the number δ from Theorem 10 is greater than the required quantity $s(r)$. □

Now to prove the inductive step, we apply Corollary 5 to

$$\mathcal{A} = \mathcal{I}_p^k \quad \text{and} \quad r = \frac{1}{2^{k+1} \cdot k \cdot (k-1)}.$$

It is easy to check that $s(r) > \frac{1}{2} - \frac{1}{k+1}$. Therefore,

$$r \cdot s(r) > \frac{1}{2^{k+1} \cdot k \cdot (k-1)} \cdot \left(\frac{1}{2} - \frac{1}{k+1}\right) = \frac{1}{2^{k+2} \cdot (k+1) \cdot k}.$$

This completes the inductive step. Thus, Theorem 9 holds by induction.

4. Proof of Theorem 7

Since the space $\mathbb{R}^n_\infty$ is homothetic to itself, we may assume without loss of generality that the length of each edge of $\mathcal{S}_k$ is equal to 1. Let $m = 2^n$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be all points of $\mathbb{R}^n$ such that each of its coordinates is equal to either 0 or 1.

Now we are ready to give a proof, which we divided into two simple lemmas.

Lemma 1. *Given positive integers k and n, one has*

$$\chi\left(\mathbb{R}^n_\infty; \mathcal{S}_k\right) \geq \left\lceil \frac{2^n}{k} \right\rceil.$$

Proof. Given positive integers k and n, let us assume the contrary. Then, there is a "proper" coloring of $\mathbb{R}^n_\infty$ with $s < \left\lceil \frac{2^n}{k} \right\rceil$ colors. In particular, the points $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are colored with s colors. By the pigeonhole principle, there are $k+1$ points of the same color among them. It is easy to see that this set of $k+1$ points is a copy of $\mathcal{S}_k$. But this situation is forbidden for proper colorings. Hence, it contradicts the assumption that there is a proper coloring. □

Lemma 2. *Given positive integers k and n, one has*

$$\chi\left(\mathbb{R}^n_\infty; \mathcal{S}_k\right) \leq \left\lceil \frac{2^n}{k} \right\rceil.$$

Proof. We prove the desired inequality by constructing an explicit coloring. Let

$$V_i = \bigcup_{\mathbf{z} \in \mathbb{Z}^n} \left\{\mathbf{v}_i + 2\mathbf{z} + [0;1)^n\right\} \subset \mathbb{R}^n, \quad 1 \leq i \leq m.$$

Clearly,

$$\bigsqcup_{i=1}^{m} V_i = \mathbb{R}^n. \tag{1}$$

It is easy to see that

$$\forall\, i\ \forall\, x, y \in V_i \quad l_\infty(x, y) \neq 1. \tag{2}$$

Now we color $\mathbb{R}^n$ with $\left\lceil \frac{2^n}{k} \right\rceil$ colors as follows: we join sets V_i into $\left\lceil \frac{2^n}{k} \right\rceil$ classes such that each of these classes contains exactly k sets V_i (maybe, except for the last class that contains remaining $2^n - k \cdot \left\lfloor \frac{2^n}{k} \right\rfloor$ sets in the case where 2^n is not divisible by k). We color each class with its own color and obtain the desired coloring.

Indeed, (1) implies that each point of $\mathbb{R}^n$ is colored. It follows from (2) that for each $(k+1)$-element set X which is a copy of $\mathcal{S}_k$ in $\mathbb{R}^n_\infty$, and for each V_i, we have $|X \cap V_i| \le 1$ and, therefore, X cannot be colored with one color.

Thus, we have constructed a "proper" coloring of $\mathbb{R}^n_\infty$ with $\lceil \frac{2^n}{k} \rceil$ colors and Lemma 2 is proved. □

5. Proof of Theorem 8

Let a number p and a space $\mathcal{Y}$ be fixed. Let $\epsilon(n)$ be a positive function that approaches 0 as $n \to \infty$. Given $n \in \mathbb{N}$, set

$$\mu = 1 + \frac{l(\mathcal{Y})}{2 \cdot R(l_p; \mathcal{Y})} + \epsilon(n) \quad \text{and} \quad r = R(l_p; \mathcal{Y}) + \frac{l(\mathcal{Y})}{2}.$$

The reason for such a choice of parameters will become clear later.

By the definition of $c(p)$, there is a function $\varepsilon(p, n)$ that approaches 0 as $n \to \infty$ and such that the following holds. Given n, there is a lattice Ω such that

$$\delta(B^n_p(r) + \Omega) \ge \left(2^{c(p)} + \varepsilon(p,n)\right)^{-n}.$$

Let $\mathcal{K}(\mu) = \frac{1}{\mu} B^n_p(r) + \Omega = B^n_p(\frac{r}{\mu}) + \Omega$. We are going to prove that $\mathcal{K}(\mu)$ does not contain a copy of $\mathcal{Y}$. Let us consider two cases.

Assume that a copy of $\mathcal{Y}$ is a subset of one l_p-ball of radius $\frac{r}{\mu}$. However, this contradicts the definition of $R(l_p; \mathcal{Y})$, since $\frac{r}{\mu} < R(l_p; \mathcal{Y})$.

Now assume that a copy of $\mathcal{Y}$ is a subset of $\mathcal{K}(\mu)$ and intersects several l_p-balls. By the definition of $l(\mathcal{Y})$, there are its two points in different balls at distance at most $l(\mathcal{Y})$. However, this is impossible as well, since the distance between any two balls of $\mathcal{K}(\mu)$ is greater than $l(\mathcal{Y})$. Indeed, let the points x_1 and x_2 belong to balls with centers O_1 and O_2, respectively. Then, by the triangle inequality,

$$l_p(x_1, x_2) \ge l_p(O_1, O_2) - l_p(O_1, x_2) - l_p(x_2, O_2) \ge 2r - \frac{r}{\mu} - \frac{r}{\mu} > l(\mathcal{Y}).$$

Thus, $\mathcal{K}(\mu)$ contains no copy of the space $\mathcal{Y}$ and, therefore, we can color $\mathcal{K}(\mu)$ with one color in the desired coloring. To complete the proof we need a statement, which is a simplified version of the main result in [39].

Theorem 11. *There is a subexponential function $\omega(n) = (1 + o(1))^n$ such that for any convex body $K \subset \mathbb{R}^n$ and for any lattice such that $\mathcal{K} = K + \Omega$ is a packing, the space $\mathbb{R}^n$ can be covered with at most $\delta^{-1}(\mathcal{K}) \cdot \omega(n)$ copies of the set $\mathcal{K}$.*

Now we apply Theorem 11 to $\mathcal{K}(\mu)$. It gives us that there is a proper coloring with at most

$$\left(2^{c(p)} + \varepsilon(p,n)\right)^n \cdot \mu^n \cdot \omega(n) = \left(2^{c(p)} \cdot \left(1 + \frac{l(\mathcal{Y})}{2 \cdot R(l_p; \mathcal{Y})}\right) + o(1)\right)^n$$

colors. The proof of Theorem 8 is complete.

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