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STRUCTURE OF MINIMUM-WEIGHT DIRECTED FORESTS: RELATED FORESTS AND CONVEXITY INEQUALITIES

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A toolkit has been developed that allows one to build directed forests from other directed forests. With its help, inequalities are proved, which connect the weights of minimal directed forests with different numbers of trees in them. A theorem on the minimum necessary changes that must be made in the minimal directed forest is also proved in order to obtain another minimal directed forest with the number of roots different by one. Bibliography: 10 *titles.*

1. Notation and definitions

For an (undirected) graph G, we denote the sets of its vertices and edges (unordered pairs of vertices) by $\mathcal{V}G$ and $\mathcal{E}G$, respectively. For a digraph G we denote the sets of its vertices and arcs (ordered pairs of vertices) by $\mathcal{V}G$ and $\mathcal{A}G$, respectively.

The *outdegree* $d_i^{\rm out}$ *(indegree* $d_i^{\rm in}$*)* of a vertex i is the number of arcs outgoing from i (entering i).

A graph H is a *subgraph* of a graph G (notation: $H \subseteq G$) if $\mathcal{V}H \subseteq \mathcal{V}G$ and $\mathcal{E}H \subseteq \mathcal{E}G$. H is a *spanning subgraph* of G if $\mathcal{V}H = \mathcal{V}G$. H is an *induced subgraph of* G *on the set* $\mathcal{U}$ (or, briefly, an *induced subgraph of* G) if $\mathcal{E}H$ consists of all edges of G joining vertices of $\mathcal{U} = \mathcal{V}H$. It is also called a *restriction* of G to $\mathcal{U}$ and is denoted by $G|_{\mathcal{U}} = H$. For digraphs, definitions of all types of subgraphs are similar (with arcs instead of edges).

A *route* of length k in a digraph is an alternating sequence of vertices v_i and arcs $a_i = (v_{i-1}, v_i)$: $v_0, a_1, v_1, \ldots, a_k, v_k$. A *cycle* is a route such that $v_0 = v_k$ and all other vertices are distinct and different from v_0. A *path* is a route all arcs of which are different. A *semiroute* is a sequence of vertices where, for all i, either (v_{i-1}, v_i) or (v_i, v_{i-1}) is an arc. A *semipath* is defined similarly.

A vertex j is *reachable* from a vertex i in the digraph G if there exists a path from i to j. Every vertex is reachable from itself.

A digraph is *weak (weakly connected)* if any two of its vertices are joined by a semipath. A maximal up to inclusion weak subgraph of G is its *connected component* (or simply *component*).

An undirected graph without cycles is called a *forest*, and its connected components are called *trees*.

In digraphs, two types of forests can be considered: *entering forests* and *outgoing forests*, which are results of reversing all arcs in entering forests. An *entering forest* is an acyclic digraph in which the outdegree of each vertex is either one or zero ($d_i^{\rm out} \in \{0,1\}$). In what follows, we use only entering forests, which we will call simply forests. Connected components of forests are *trees*. The only vertex of a tree with zero outdegree is its *root*. A tree with root i of a forest F will be denoted by T_i^F. The set of all roots of a forest F will be denoted by $\mathcal{K}_F$.

The *outgoing neighborhood* of a vertex i (notation: $\mathcal{N}_i^{\rm out}(F)$) is the set of all ends of arcs of F outgoing i. If it is clear what graph we consider, or it does not matter, we will write simply $\mathcal{N}_i^{\rm out}$. Similarly, the *entering neighborhood* of i (notation: $\mathcal{N}_i^{\rm in}$) is the set of all beginnings of arcs entering i.

We say that an arc *outgoes* $\mathcal{U}$ if it goes from a vertex of $\mathcal{U}$ to a vertex outside $\mathcal{U}$.

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For a subset $\mathcal{D}$ of the vertex set of a digraph G, its outgoing neighborhood $\mathcal{N}^{\text{out}}_{\mathcal{D}}(G)$ is the set of all ends of arcs of G outgoing $\mathcal{D}$. We will omit the reference to the graph G if it is clear or does not matter. Thus, $\mathcal{N}^{\text{out}}_{\mathcal{D}} = (\bigcup_{i\in\mathcal{D}} \mathcal{N}^{\text{out}}_i) \setminus \mathcal{D}$. Similarly, the *entering neighborhood* of a vertex set $\mathcal{D}$ is the set $\mathcal{N}^{\text{in}}_{\mathcal{D}} = (\bigcup_{i\in\mathcal{D}} \mathcal{N}^{\text{in}}_i) \setminus \mathcal{D}$ that consists of all beginnings of arcs entering $\mathcal{D}$.

We will use the following notation:

$i \overset{G}{\to} j$ means that the vertex j is reachable from i in the graph (digraph) G;

$i \overset{G}{\not\to} j$ means that the vertex j is not reachable from i in the graph (digraph) G;

$\mathcal{X} \overset{G}{\to} \mathcal{Y}$ means that the set $\mathcal{Y}$ is reachable from the set $\mathcal{X}$ in G, i. e., there exist vertices $i \in \mathcal{X}$ and $j \in \mathcal{Y}$ such that $i \overset{G}{\to} j$;

$\mathcal{X} \overset{G}{\not\to} \mathcal{Y}$ means that the set $\mathcal{Y}$ is not reachable from the set $\mathcal{X}$ in G, i. e., $i \overset{G}{\not\to} j$ for all $i \in \mathcal{X}$ and $j \in \mathcal{Y}$.

2. The operation of substituting outgoing arcs

Let F and G be two digraphs with the same vertex set $\mathcal{N}$, and $\mathcal{D} \subset \mathcal{N}$. Then $F^G_{\uparrow\mathcal{D}}$ is the graph obtained from F upon substituting all arcs outgoing the set $\mathcal{D}$ by arcs of the digraph G outgoing $\mathcal{D}$.

We are interested in the situation where both F and G are entering forests (in what follows we will call them simply forests) on the same vertex set and the digraph $F^G_{\uparrow\mathcal{D}}$ is also a forest. Note that if both F and G are forests, then, in $F^G_{\uparrow\mathcal{D}}$, the outgoing degree of a vertex can differ from its degree in F but is still equal to 1 or 0. Thus, if $F^G_{\uparrow\mathcal{D}}$ has no cycles, then it is a forest. Let us state a sufficient condition, which implies that the resulting graph is a forest.

Lemma 1. *Let F and G be forests on the same vertex set, and let $\mathcal{D}$ be its subset such that $\mathcal{N}^{\text{out}}_{\mathcal{D}}(G) \overset{F}{\not\to} \mathcal{N}^{\text{in}}_{\mathcal{D}}(F)$. Then $F^G_{\uparrow\mathcal{D}}$ is a forest.*

Proof. If a cycle appears after substituting arcs, then it must contain arcs outgoing $\mathcal{D}$ and arcs entering $\mathcal{D}$. Since $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F)$ is not reachable from $\mathcal{N}^{\text{out}}_{\mathcal{D}}(G)$ in F (see Fig. 1), this property will still hold after substituting arcs and no cycle can appear. □

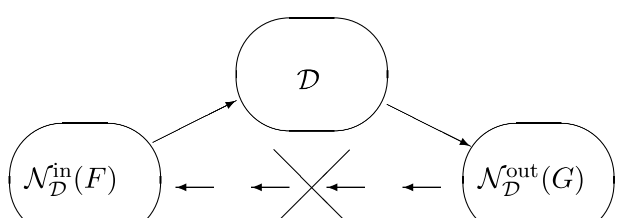

Fig. 1. No route leads from $\mathcal{N}^{\text{out}}_{\mathcal{D}}(G)$ to $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F)$ in F and, therefore, in $F^G_{\uparrow\mathcal{D}}$. Hence, no cycle can appear after substituting arcs.

Corollary 1. *Let F and G be forests on the same vertex set, and let $\mathcal{D}$ be its subset such that no arc enters $\mathcal{D}$ in F. Then $F^G_{\uparrow\mathcal{D}}$ is a forest.*

Proof. Since $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F)$ is empty, the condition of Lemma 1 holds. □

Corollary 2. *Let F and G be forests on the same vertex set, and let $\mathcal{D}$ be its subset such that no arc outgoes $\mathcal{D}$ in G. Then $F^G_{\uparrow\mathcal{D}}$ is a forest.*

Proof. Since $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G)$ is empty, the condition of Lemma 1 holds. □

Corollary 3. *Let F and G be forests on the same vertex set, and let T^F be a tree of F and $\mathcal{D} = \mathcal{V}T^F$. Then both $F^G_{\uparrow\mathcal{D}}$ and $G^F_{\uparrow\mathcal{D}}$ are forests.*

Proof. Since T^F is a connected component, $\mathcal{N}_{\mathcal{D}}^{\text{in}}(F) = \varnothing$ and, therefore, $F^G_{\uparrow\mathcal{D}}$ is a forest. Moreover, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(F) = \varnothing$ and, therefore, $G^F_{\uparrow\mathcal{D}}$ is also a forest. □

Corollary 4. *Let F and G be forests on the same vertex set, let T^F and T^G be trees of F and G, respectively, and $\mathcal{D} = \mathcal{V}T^F \cap \mathcal{V}T^G$. Then both $F^G_{\uparrow\mathcal{D}}$ and $G^F_{\uparrow\mathcal{D}}$ are forests.*

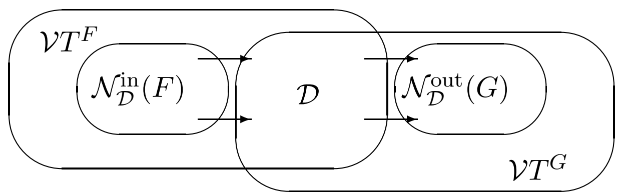

Fig. 2. $\mathcal{D} = \mathcal{V}T^F \cap \mathcal{V}T^G$.

Proof. By symmetry, it is enough to prove that $F^G_{\uparrow\mathcal{D}}$ is a forest. The set $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G)$ is a subset of $\mathcal{V}T^G \setminus \mathcal{V}T^F$ (see Fig. 2). Since T^F is a connected component of the forest F, there is no route in F from $\mathcal{V}T^G \setminus \mathcal{V}T^F$ to $\mathcal{V}T^F$ and, therefore, to $\mathcal{V}T^F \setminus \mathcal{V}T^G \supset \mathcal{N}_{\mathcal{D}}^{\text{in}}(F)$. Hence, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G) \stackrel{F}{\nrightarrow} \mathcal{N}_{\mathcal{D}}^{\text{in}}(F)$ and the condition of Lemma 1 holds. □

Corollary 5. *Let F and G be forests on the same vertex set $\mathcal{N}$, let T^F and T^G be trees of F and G, respectively, and $\mathcal{D} = \mathcal{V}T^F \setminus \mathcal{V}T^G$. Then both $F^G_{\uparrow\mathcal{D}}$ and $G^F_{\uparrow\mathcal{D}}$ are forests.*

Proof. First, let us prove that $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G)$ contains no vertex of the tree T^F. Indeed, the set $\mathcal{D} = \mathcal{V}T^F \setminus \mathcal{V}T^G$ contains no vertex of the tree T^G. Hence, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G)$ contains no vertex of T^G. In particular, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G) \cap (\mathcal{V}T^F \cap \mathcal{V}T^G) = \varnothing$. However, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G) \cap \mathcal{D} = \varnothing$ (this property holds for any graph by the definition). The vertex set of the tree T^F can be represented as $\mathcal{V}T^F = \mathcal{D} \cup \big(\mathcal{V}T^F \cap \mathcal{V}T^G\big)$. Thus, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G)$ intersects no part of this union and, therefore, contains no vertex of T^F (see Fig. 3(a)): $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G) \subset \mathcal{N} \setminus \mathcal{V}T^F$.

In the forest F, $\mathcal{V}T^F$ is not reachable from $\mathcal{N} \setminus \mathcal{V}T^F$ (since T^F is a connected component of F). The set $\mathcal{N}_{\mathcal{D}}^{\text{in}}(F)$ is a subset of the vertex set of T^F. Hence, no route goes from $\mathcal{N}_{\mathcal{D}}^{\text{out}}(G)$ to $\mathcal{N}_{\mathcal{D}}^{\text{in}}(F)$. Thus, $F^G_{\uparrow\mathcal{D}}$ is a forest.

Now consider $G^F_{\uparrow\mathcal{D}}$. Since $\mathcal{D} = \mathcal{V}T^F \setminus \mathcal{V}T^G$, the set $\mathcal{N}_{\mathcal{D}}^{\text{in}}(G)$ contains no vertex of T^G (see Fig. 3(b)). Further, if T^F contains an arc (i, j) such that $i \in \mathcal{D}$ and $j \notin \mathcal{D}$, then, since $\mathcal{V}T^F = \mathcal{D} \cup \big(\mathcal{V}T^F \cap \mathcal{V}T^G\big)$, we have $j \in \mathcal{V}T^F \cap \mathcal{V}T^G$. Thus, $\mathcal{N}_{\mathcal{D}}^{\text{out}}(F) \subseteq \mathcal{V}T^G$. In G, no arc outgoes $\mathcal{V}T^G$ (since it is a connected component of G). Hence, $\mathcal{N}_{\mathcal{D}}^{\text{in}}(G)$ is not reachable from $\mathcal{N}_{\mathcal{D}}^{\text{out}}(F)$ in G. □

Corollary 6. *Let F and G be forests on the same vertex set, and let T^F and T^G be trees of F and G, respectively. Assume that $\mathcal{D} \subset \mathcal{V}T^F \cap \mathcal{V}T^G$, $\mathcal{N}_{\mathcal{D}}^{\text{in}}(F) = \varnothing$ and $\mathcal{N}_{\mathcal{D}}^{\text{out}}(F) \subset \mathcal{V}T^F \setminus \mathcal{V}T^G$. Then both $F^G_{\uparrow\mathcal{D}}$ and $G^F_{\uparrow\mathcal{D}}$ are forests.*

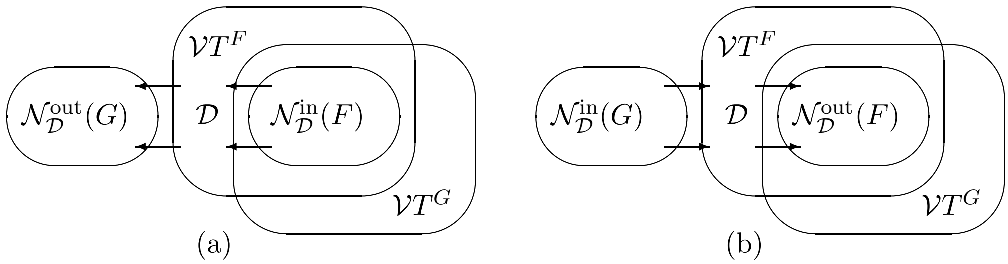

Fig. 3. $\mathcal{D} = \mathcal{V}T^F \setminus \mathcal{V}T^G$.

Proof. Since $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F) = \varnothing$, $F^G_{\uparrow\mathcal{D}}$ is a forest. By condition, $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F)$ contains no vertex of the tree T^G. At the same time, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(G)$ is contained in the vertex set of T^G (see Fig. 4). Hence, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(G)$ is not reachable from $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F)$ in G. Therefore, $G^F_{\uparrow\mathcal{D}}$ is a forest. □

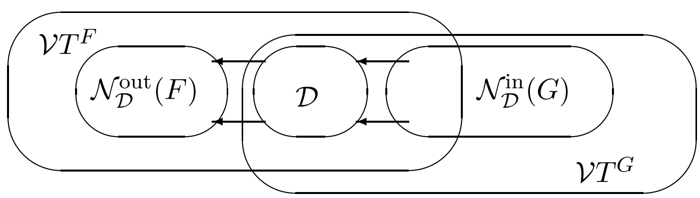

Fig. 4. $\mathcal{D} \subset \mathcal{V}T^F \cap \mathcal{V}T^G$, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F) = \varnothing$, and $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F) \subset \mathcal{V}T^F \setminus \mathcal{V}T^G$.

Corollary 7. *Let F and G be forests on the same vertex set, and let T^F and T^G be trees of F and G, respectively. Assume that $\mathcal{D} \subset \mathcal{V}T^F \setminus \mathcal{V}T^G$, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F) = \varnothing$, and $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F) \subset \mathcal{V}T^G$. Then both $F^G_{\uparrow\mathcal{D}}$ and $G^F_{\uparrow\mathcal{D}}$ are forests.*

Proof. Since $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F) = \varnothing$, $F^G_{\uparrow\mathcal{D}}$ is a forest. Further, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(G)$ cannot contain a vertex of T^G since $\mathcal{D} \subset \mathcal{V}T^F \setminus \mathcal{V}T^G$ (see Fig. 5). At the same time, by condition, $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F)$ is contained in the vertex set of T^G. Hence, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(G)$ is not reachable from $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F)$ in G. Therefore, $G^F_{\uparrow\mathcal{D}}$ is a forest. □

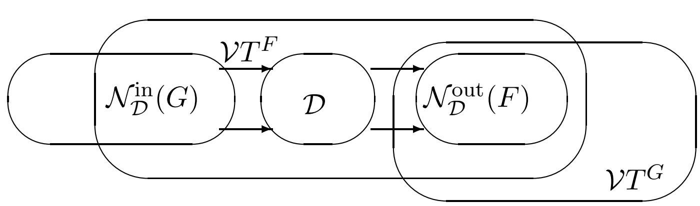

Fig. 5. $\mathcal{D} \subset \mathcal{V}T^F \setminus \mathcal{V}T^G$, $\mathcal{N}^{\text{in}}_{\mathcal{D}}(F) = \varnothing$, and $\mathcal{N}^{\text{out}}_{\mathcal{D}}(F) \subset \mathcal{V}T^G$.

3. Convexity inequalities

Let V be a weighted digraph with vertex set $\mathcal{N}$, $|\mathcal{N}| = N$. Let each arc (i,j) have a real weight v_{ij}. For each spanning subgraph G of V and $\mathcal{S} \subseteq \mathcal{N}$, set the notation

$$\Upsilon^G_{\mathcal{S}} = \sum_{\substack{i \in \mathcal{S} \\ (i,j) \in \mathcal{A}G}} v_{ij}\ , \quad \Upsilon^G = \Upsilon^G_{\mathcal{N}} = \sum_{(i,j) \in \mathcal{A}G} v_{ij}\ . \tag{1}$$

Note that $\Upsilon^G_{\mathcal{S}}$ is the sum of weights of all arcs with the beginning in $\mathcal{S}$, including those the ends of which do not belong to $\mathcal{S}$. If no arc of G outgoes $\mathcal{S}$, then $\Upsilon^G_{\mathcal{S}} = \Upsilon^{G|_{\mathcal{S}}}$.

We denote the set of all forests that consists of $k = 1, 2, \ldots, N$ trees by $\mathcal{F}^k$. The minimum of Υ^F over all forests $F \in \mathcal{F}^k$ will be denoted by ϕ^k:

$$\phi^k = \min_{F \in \mathcal{F}^k} \Upsilon^F\ .$$

If $\mathcal{F}^k = \varnothing$, then we set $\phi^k = \infty$. In particular, $\phi^0 = \infty$. Note that $\mathcal{F}^N$ consists only of the empty forest and, therefore, $\phi^N = 0$. Each forest of $\mathcal{F}^{N-1}$ contains exactly one arc and, therefore, $\phi^{N-1} = \min\limits_{(i,j) \in \mathcal{A}V} v_{ij}$.

Our tools allow us to obtain simple proofs of well-known convexity inequalities (2) on the values ϕ^k, which were proved by Ventsel [1]. Ventsel's proof was based on the analysis of the asymptotic spectrum of matrices with exponentially small coefficients (without any graph theory).

First, we will prove a simple lemma, which will be useful in what follows.

Lemma 2. *Let $F \in \mathcal{F}^m$, $H \in \mathcal{F}^k$, and let $\mathcal{D}$ be a subset of the vertex set such that the number of roots of F contained in $\mathcal{D}$ is greater by n than the number of roots of H contained in $\mathcal{D}$. Then the following statements hold.*

(a) If $F^H_{\uparrow\mathcal{D}}$ is a forest, then $F^H_{\uparrow\mathcal{D}} \in \mathcal{F}^{m-n}$.

(b) If $H^F_{\uparrow\mathcal{D}}$ is a forest, then $H^F_{\uparrow\mathcal{D}} \in \mathcal{F}^{k+n}$.

Proof. (a) By the choice of $\mathcal{D}$, the number of vertices with zero outdegree in $F^H_{\uparrow\mathcal{D}}$ is smaller by n than the number of vertices with zero outdegree in F. Since $F^H_{\uparrow\mathcal{D}}$ is a forest, its number of roots is smaller by n than the number of roots of the forest F. Therefore, $F^H_{\uparrow\mathcal{D}} \in \mathcal{F}^{m-n}$. The proof of item (b) is similar. □

Let $\widetilde{\mathcal{F}}^k$ be the subset of $\mathcal{F}^k$ consisting of all forests on which the minimum ϕ^k is attained: $F \in \widetilde{\mathcal{F}}^k \Leftrightarrow F \in \mathcal{F}^k$ and $\Upsilon^F = \phi^k$. We will call a forest of minimal weight *minimal.*

Theorem 1. *Let $\mathcal{F}^k \neq \varnothing$ for a certain $k \in \{1, 2, \ldots, N-1\}$. Then*

$$\phi^{k-1} - \phi^k \geq \phi^k - \phi^{k+1}. \tag{2}$$

Proof. Since $\mathcal{F}^k \neq \varnothing$, all sets $\mathcal{F}^l$, $l \in \{k, k+1, \ldots, N\}$, are also nonempty. In particular, there exists a forest $F \in \widetilde{\mathcal{F}}^{k+1}$. If $\mathcal{F}^{k-1} = \varnothing$, then, by definition, $\phi^{k-1} = \infty$ and (2) holds. Let $\mathcal{F}^{k-1}$ be nonempty. Then there exists a forest $H \in \widetilde{\mathcal{F}}^{k-1}$. Since the number of roots of F is greater by 2 than the number of roots of H, there exists a tree T^F of the forest F all vertices of which have nonzero outdegree in H. Let $P = F^H_{\uparrow\mathcal{D}}$ and $Q = H^F_{\uparrow\mathcal{D}}$, where $\mathcal{D} = \mathcal{V}T^F$. By Corollary 3, both P and Q are forests. Since $\mathcal{D}$ contains exactly one root of F and no roots of H, by Lemma 2 both forests P and Q belong to $\mathcal{F}^k$. Let $\Delta = \Upsilon^H_{\mathcal{D}} - \Upsilon^F_{\mathcal{D}}$. Then

$$\Upsilon^P = \Upsilon^F + \Delta = \phi^{k+1} + \Delta \geq \phi^k,$$
$$\Upsilon^Q = \Upsilon^H - \Delta = \phi^{k-1} - \Delta \geq \phi^k.$$

Hence,

$$\phi^{k-1} - \phi^k \geq \Delta \geq \phi^k - \phi^{k+1}. \qquad \square$$

The convexity inequalities (2) are important in the analysis of dynamic systems with small stochastic disturbance [2–4].

4. Relative forests of minimal weight

Let us study properties of minimal forests for different k.

Lemma 3. *Let $F \in \widetilde{\mathcal{F}}^{k+1}$, $H \in \widetilde{\mathcal{F}}^k$, and let a subset $\mathcal{D}$ of the vertex set be such that $F^H_{\uparrow\mathcal{D}} \in \mathcal{F}^k$ and $H^F_{\uparrow\mathcal{D}} \in \mathcal{F}^{k+1}$. Then $F^H_{\uparrow\mathcal{D}} \in \widetilde{\mathcal{F}}^k$ and $H^F_{\uparrow\mathcal{D}} \in \widetilde{\mathcal{F}}^{k+1}$.*

Proof. Let $H' = F^H_{\uparrow\mathcal{D}}$ and $F' = H^F_{\uparrow\mathcal{D}}$. Since $H' \in \mathcal{F}^k$, $\Upsilon^{H'} \geq \phi^k$. Similarly, $F' \in \mathcal{F}^{k+1}$ implies that $\Upsilon^{F'} \geq \phi^{k+1}$. Let $\Delta = \Upsilon^H_{\mathcal{D}} - \Upsilon^F_{\mathcal{D}}$. Then

$$\phi^k \leq \Upsilon^{H'} = \Upsilon^F + \Delta = \phi^{k+1} + \Delta,$$
$$\phi^{k+1} \leq \Upsilon^{F'} = \Upsilon^H - \Delta = \phi^k - \Delta.$$

Substituting Δ from the second inequality in the first one we obtain $\Upsilon^{H'} \leq \phi^k$ and, therefore, $\Upsilon^{H'} = \phi^k$. Hence, $H' \in \widetilde{\mathcal{F}}^k$. Similarly, $F' \in \widetilde{\mathcal{F}}^{k+1}$. $\square$

Remark 1. By simple calculus of roots, the number of roots of F in the set $\mathcal{D}$ from Lemma 3 must be greater by 1 than the number of roots of H in $\mathcal{D}$. In particular, if $\mathcal{D}$ contains one root of F, then all vertices of $\mathcal{D}$ have nonzero outdegrees in H.

There exists a "genetic" connection between minimal forests for different k. Let us start with a definition.

Definition 1. A forest $F \in \mathcal{F}^{k+1}$ with roots (up to numeration) $1, 2, \cdots, k+1$ is an *ancestor* of the forest $G \in \mathcal{F}^k$ with roots $1, 2, \cdots, k$ if $T^F_i = T^G_i$ for $i = 1, 2, \cdots, k-1$, $G|_{\mathcal{V}T^F_k} = T^F_k$, and $G|_{\mathcal{V}T^F_{k+1}}$ is a tree (see Fig. 6). In this case, we will also say that G is a *descendant* of F.

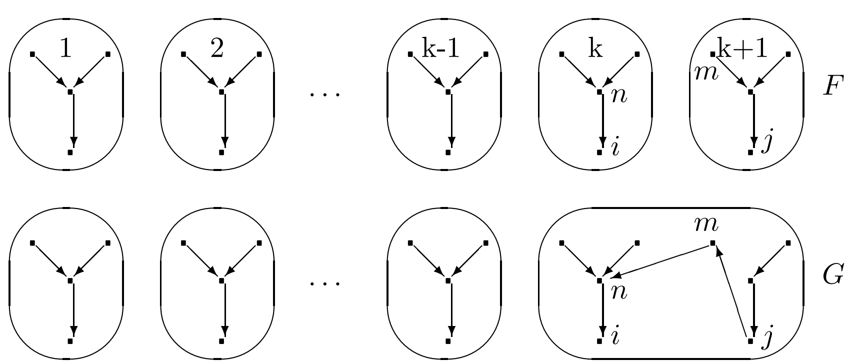

Fig. 6. G is a descendant of F, and F is an ancestor of G.

The following theorem describes minimal changes necessary to transform a forest of the set $\widetilde{\mathcal{F}}^{k+1}$ into a forest of the set $\widetilde{\mathcal{F}}^k$, and vice versa.

Theorem 2 (On relative forests). *Let $k \in \{1, 2, \cdots, N-1\}$ be such that $\mathcal{F}^k \neq \varnothing$. Then the following statements hold.*

(a) Each forest of $\widetilde{\mathcal{F}}^{k+1}$ has a descendant in $\widetilde{\mathcal{F}}^k$;

(b) Each forest of $\widetilde{\mathcal{F}}^k$ has an ancestor in $\widetilde{\mathcal{F}}^{k+1}$.

Proof. (a) Since $\mathcal{F}^k \neq \varnothing$, we also have $\mathcal{F}^{k+1} \neq \varnothing$. Let us prove that a forest $F \in \widetilde{\mathcal{F}}^{k+1}$ has a descendant in $\widetilde{\mathcal{F}}^k$. Consider an arbitrary forest $H \in \widetilde{\mathcal{F}}^k$. Since the number of roots of F is greater by 1 than the number of roots of H and the outdegree of each vertex does not exceed 1, there exists a root of F, say, j, such that the tree T_j^F with root j contains no root of H.

First, we will construct an auxiliary forest $H' \in \widetilde{\mathcal{F}}^k$, which will later help us to construct a descendant of F. Let $H' = F^H_{\uparrow\mathcal{D}}$, where $\mathcal{D} = \mathcal{V}T_j^F$. By Corollary 3, both H' and $H^F_{\uparrow\mathcal{D}}$ are forests. The set $\mathcal{D}$ contains no roots of H' and exactly one root of F, namely, j. Thus, by Lemma 2, $H' \in \mathcal{F}^k$ and $H^F_{\uparrow\mathcal{D}} \in \mathcal{F}^{k+1}$. By Lemma 3, $H' \in \widetilde{\mathcal{F}}^k$.

Note that all trees of the forest F except for T_j^F are subtrees of trees of H' with the same roots. The vertices of $\mathcal{V}T_j^F$ are divided between trees of H'. No arc of H' enters $\mathcal{V}T_j^F$ by construction. The induced subgraph $H'|_{\mathcal{V}T_j^F}$ is a forest. Let T be the tree of $H'|_{\mathcal{V}T_j^F}$ that contains j, and let m be the root of T. Denote by $T^{H'}$ the tree of H' such that T is its subtree. By construction, no arc of H' enters $\mathcal{D}' = \mathcal{V}T$. (We have constructed the forest H' just for this property. The forest H may not satisfy such a condition: see Example 1 and Fig. 11). Thus, $\mathcal{N}^{\text{in}}_{\mathcal{D}'}(H') = \varnothing$, $\mathcal{N}^{\text{out}}_{\mathcal{D}'}(H') \subset \mathcal{V}T^{H'} \setminus \mathcal{V}T^F$, and $\mathcal{D}' \subset \mathcal{V}T^{H'} \cap \mathcal{V}T_j^F$.

Let $G = F^{H'}_{\uparrow\mathcal{D}'}$. By Corollary 6, both graphs G and $H'^F_{\uparrow\mathcal{D}'}$ are trees. Since $\mathcal{D}'$ contains no root of H' and exactly one root of F (namely, j), by Lemma 2, we have $G \in \mathcal{F}^k$ and $H'^F_{\uparrow\mathcal{D}'} \in \mathcal{F}^{k+1}$. By Lemma 3, $G \in \widetilde{\mathcal{F}}^k$.

Let us prove that the forest G is a descendant of F. Indeed, in G, all vertices of $\mathcal{V}T_j^F$ have nonzero outdegrees. Since T_j^F is a tree of F, all arcs of the forest G outgoing $\mathcal{V}T_j^F \setminus \mathcal{D}'$ must enter vertices of $\mathcal{V}T_j^F$. By construction, $H'|_{\mathcal{D}'} = T$. Hence, exactly one arc of G outgoes $\mathcal{D}'$, and this arc outgoes the root m of T, let this arc be (m, n) (see Fig. 6). Therefore, exactly one arc of G outgoes $\mathcal{V}T_j^F$, and this arc is (m, n). The vertex n cannot belong to $\mathcal{D}'$ by the definition of an arc outgoing a set. Moreover, n cannot belong to $\mathcal{V}T_j^F \setminus \mathcal{D}'$ (otherwise, T is not a connected component of the induced subgraph $H'|_{\mathcal{V}T_j^F}$). Hence, $n \in \mathcal{N} \setminus \mathcal{V}T_j^F$. Thus, $G|_{\mathcal{V}T_j^F}$ is a tree with root m.

Let n be a vertex of a tree T_i^F. Then $G|_{\mathcal{V}T_i^F} = T_i^F$ and i is the root of a tree T_i^G of the forest G. The induced subgraph $G|_{\mathcal{V}T_j^F}$ is a tree with root m. All trees of the forests F and G with roots at $\mathcal{K}_F \setminus \{i, j\} = \mathcal{K}_G \setminus \{j\}$ coincide since arcs outgoing vertices of the set $\mathcal{N} \setminus \mathcal{D}'$ were not changed during the construction of the forest H' from the forest F and the only arc of H' outgoing the set $\mathcal{D}'$ enters T_i^F.

As a result, in the forest F, we have replaced exactly all arcs outgoing the set $\mathcal{D}' = \mathcal{V}T$ by arcs of the forest H. However, it was necessary to construct the intermediate forest H' in which no arc enters $\mathcal{D}'$ (see Example 1).

(b) As in the first part of the proof, let $F \in \widetilde{\mathcal{F}}^{k+1}$ and $H \in \widetilde{\mathcal{F}}^k$. Assume that a root j of the forest F is such that the tree T_j^F with the root j has no intersection with $\mathcal{K}_H$ – the set of roots of the forest H. Let j belong to the tree T_i^H (with the root i) of the forest H. We will construct the ancestor Q of H in several steps.

Let $F' = H^F_{\uparrow\mathcal{D}}$ where $\mathcal{D} = \mathcal{V}T_i^H \cap \mathcal{V}T_j^F$. By Corollary 4, both F' and $F^H_{\uparrow\mathcal{D}}$ are forests. All vertices of $\mathcal{D}$ have nonzero outdegrees in H and all vertices of $\mathcal{D}$ except for j have nonzero outdegrees in F. Hence, by Lemma 2, $F' \in \mathcal{F}^{k+1}$ and $F^H_{\uparrow\mathcal{D}} \in \mathcal{F}^k$. By Lemma 3, $F' \in \widetilde{\mathcal{F}}^{k+1}$. In the forest F', the vertex j remains a root. The tree $T_j^{F'}$ contains exactly those arcs of the forest F that form a tree with root j in the induced subgraph $F|_{\mathcal{D}}$. Also $T_j^{F'}$ can contain arcs of the forest H, these arcs must belong to the tree T_i^H (see Figs. 7 and 8). Hence, $\mathcal{V}T_j^{F'} \subset \mathcal{V}T_i^H$. The set $\mathcal{V}T_j^{F'} \setminus \mathcal{D}$ may be nonempty, because $T_j^{F'}$ may contain arcs of the tree T_i^H.

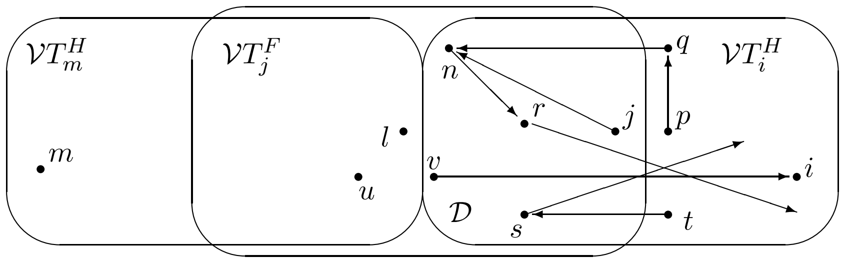

Fig. 7. $\mathcal{D} = \mathcal{V}T_j^F \cap \mathcal{V}T_i^H$. Arcs of the tree T_i^H outgoing the set $\mathcal{D}$ and entering vertices of $\mathcal{D}$ are shown. Arcs of other trees of H (in particular, arcs of the tree T_m^H shown in the figure) cannot enter $\mathcal{D}$.

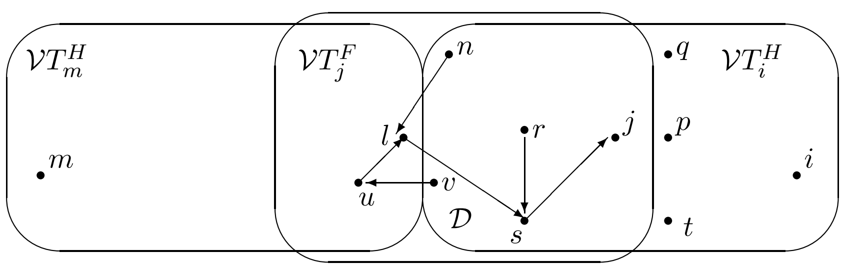

Fig. 8. $\mathcal{D} = \mathcal{V}T_j^F \cap \mathcal{V}T_i^H$. Only arcs of the tree T_j^F are shown. The vertices n and v do not belong to T_m^H, but they belong to the tree $T_m^{F'}$ of the forest $F' = H^F_{\uparrow\mathcal{D}}$. Thus, $T_m^{F'} \supseteq T_m^H$. At the same time, $\mathcal{V}T_j^{F'} \subset \mathcal{V}T_i^H$.

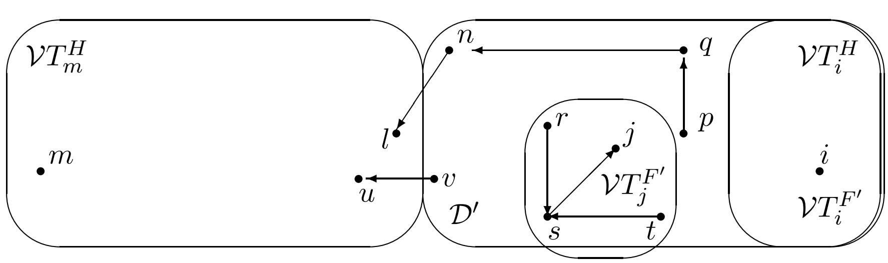

Fig. 9. $\mathcal{D}' = \mathcal{V}T_m^{F'} \setminus \mathcal{V}T_m^H = \{v, n, p, q\}$. The arcs (n, l) and (v, u) of the forest F' were arcs of F. Because of these arcs, the trees T_m^H and $T_m^{F'}$ do not coincide, we have only $T_m^{F'} \supseteq T_m^H$. We need to replace (n, l) by the arc H outgoing n in the forest H and the arc (v, u) by the arc (v, i) (see Fig. 7).

The set of roots of the forest F' is $\mathcal{K}_H \cup \{j\}$. However, in the forest F, arcs outgoing the set $\mathcal{V}T_i^H \cap \mathcal{V}T_j^F$ can exist. Thus, for $m \in \mathcal{K}_H \setminus \{i\}$, we have only $T_m^{F'} \supseteq T_m^H$, not necessary $T_m^H = T_m^{F'}$ (see Fig. 9). We have changed too many arcs of H during the construction of its ancestor, some of them are to be returned (see arcs (n,l) and (v,u) in Fig. 9: these arcs "extend" the tree T_m^H to the tree $T_m^{F'}$).

Let $m \in \mathcal{K}_H \setminus \{i\}$, and let $\mathcal{D}' = \mathcal{V}T_m^{F'} \setminus \mathcal{V}T_m^H \neq \varnothing$. We will start the procedure of moving surplus vertices of the tree $T_m^{F'}$ to trees with other roots.

Let $F'' = F'^{H}_{\uparrow\mathcal{D}'}$. By Corollary 5, F'' and $H^{F'}_{\uparrow\mathcal{D}'}$ are forests. The set $\mathcal{D}'$ contains neither root of F' nor root of H (see Fig. 9). By Lemma 2, $F'' \in \mathcal{F}^{k+1}$ and $H^{F'}_{\uparrow\mathcal{D}'} \in \mathcal{F}^k$. By Lemma 3, $F'' \in \widetilde{\mathcal{F}}^{k+1}$. Note that $\mathcal{V}T_m^{F'} = \mathcal{D}' \cup \mathcal{V}T_m^H$. By the construction of F'', arcs outgoing vertices of $\mathcal{D}'$ in F'' are the same as in H. By the construction of F', arcs outgoing vertices of $\mathcal{D}'$ in F' (and, therefore, in F'') are the same as in H. Thus, $T_m^{F''} = T_m^H$. Note that neither F'' nor H has arcs outgoing $\mathcal{V}T_m^{F''} = \mathcal{V}T_m^H$ or entering this set, and the induced subgraphs of F'' and H on this set coincide. Hence, any further replacement of arcs of F'' by arcs of H (for arbitrary vertex sets) cannot change the tree T_m^H.[1]

If there exists another vertex $l \in \mathcal{K}_H \setminus \{i,m\}$ such that $\mathcal{D}'' = \mathcal{V}T_l^{F''} \setminus \mathcal{V}T_l^H \neq \varnothing$, then we construct a forest $F''^{H}_{\uparrow\mathcal{D}''}$ and so on until we obtain a forest $F^* \in \widetilde{\mathcal{F}}^{k+1}$ such that $T_m^{F^*} = T_m^H$ for all $m \in \mathcal{K}_H \setminus \{i\}$. Now the union of vertex sets of the trees $T_i^{F^*}$ and $T_j^{F^*}$ is exactly the vertex set of T_i^H.

However, F^* still is not an ancestor of the forest H, since T_i^H can contain more than one arc outgoing the set $\mathcal{V}T_j^{F^*}$ (see Figs. 7 and 10). That is, $H|_{\mathcal{V}T_j^{F^*}}$ is not surely a tree, it may be a forest. This forest divides the vertex set $\mathcal{V}T_j^{F^*}$ into several trees, we choose among them the tree T that contains j. Let $\hat{\mathcal{D}} = \mathcal{V}T_j^{F^*} \setminus \mathcal{V}T$.

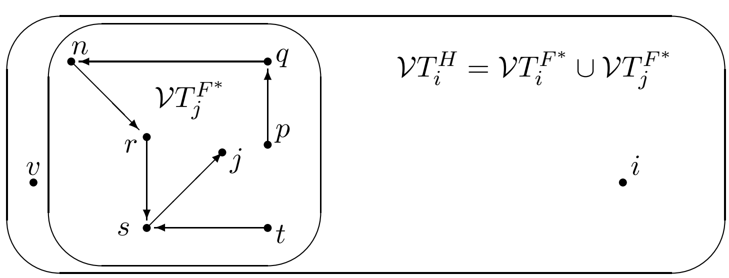

Fig. 10. Arcs of the tree $T_j^{F^*}$ are shown. Since H has two arcs outgoing the set $\mathcal{V}T_j^{F^*}$ (from the vertices r and s, see Fig. 7), the induced subgraph $H|_{\mathcal{V}T_j^{F^*}}$ is a forest with two connected components.

By the construction of the forest F^* (starting at the intermediate forest F', step by step, we have returned arcs of the forest H), all arcs of F^* outgoing vertices of $\mathcal{N} \setminus \mathcal{V}T_j^{F^*}$ (and also some arcs of the tree $T_j^{F^*}$) are the same as in H. Every arc we have returned belongs either to the tree $T_j^{F^*}$ or to the tree $T_i^{F^*}$ (which consists only of arcs of H). Hence, no arc of H enters $\mathcal{V}T_j^{F^*}$ (arcs of H entering $\mathcal{V}T_j^{F^*}$ have beginnings in $\mathcal{V}T_i^{F^*}$, but these arcs form the tree $T_i^{F^*}$). Therefore, no arc of H enters any connected component of the induced subgraph $H|_{\mathcal{V}T_j^{F^*}}$. Thus, $\mathcal{N}^{\text{in}}_{\hat{\mathcal{D}}}(H) = \varnothing$. In addition, $\mathcal{N}^{\text{out}}_{\hat{\mathcal{D}}}(H) \subset \mathcal{V}T_i^H \setminus \mathcal{V}T_j^{F^*}$ and $\hat{\mathcal{D}} \subset \mathcal{V}T_i^H \cap \mathcal{V}T_j^{F^*}$. Let

[1]If two forests have the same connected component, no operation of replacement of arcs of one of them by arcs of the other will change this component.

$Q = {F^*}^H_{\uparrow\hat{\mathcal{D}}}$. By Corollary 6, both Q and $H^{F^*}_{\uparrow\hat{\mathcal{D}}}$ are forests. Since $\hat{\mathcal{D}}$ contains neither roots of F^* nor roots of H, by Lemma 2, we have $Q \in \mathcal{F}^{k+1}$ and $H^{F^*}_{\uparrow\hat{\mathcal{D}}} \in \mathcal{F}^k$. By Lemma 3, $Q \in \widetilde{\mathcal{F}}^{k+1}$. Let us prove that Q is an ancestor of the forest H. Indeed, $\mathcal{V}T^Q_j = \mathcal{V}T$, $H|_{\mathcal{V}T^Q_j}$ is a tree. The vertex i is a root of the forest Q and $H|_{\mathcal{V}T^Q_i} = T^Q_i$. Finally, $T^H_l = T^Q_l$ for $l \in \mathcal{K}_H \setminus \{i\}$. $\square$

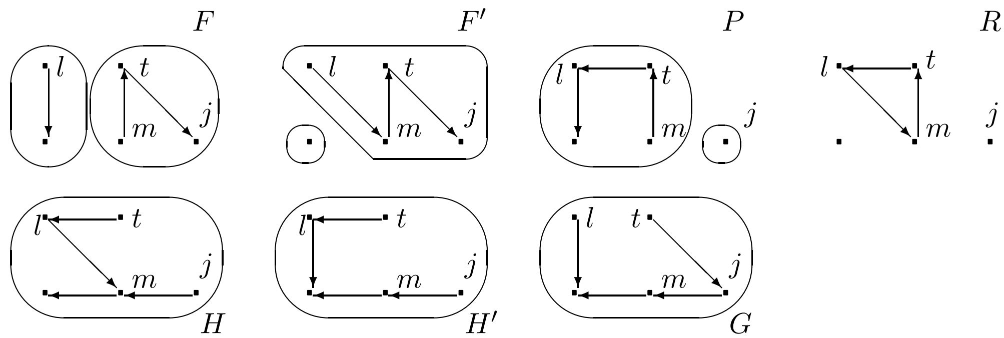

Fig. 11. The construction of a descendant G from the forest F. We use a forest H and the intermediate forest H'.

Example 1. The construction of a descendant G from the forest F. We use a forest H. Trees of forests F and H involved in the construction are shown in Fig. 11, at the left. In the second column of this figure, the forests $H' = F^H_{\uparrow\mathcal{D}}$ and $F' = H^F_{\uparrow\mathcal{D}}$ are shown, where $\mathcal{D} = \mathcal{V}T^F_j = \{j, t, m\}$. The induced subgraph $H'|_{\mathcal{D}}$ is a forest consisting of two trees: an empty tree with root t and the tree T with the only arc (j, m). Let $G = F^{H'}_{\uparrow\mathcal{D}'}$ and $P = {H'}^F_{\uparrow\mathcal{D}'}$, where $\mathcal{D}' = \{m, j\}$. The forest G is the desired descendant of F. In this example, the graph $F^H_{\uparrow\mathcal{D}'}$ coincides with the descendant G. However, the graph $R = H^F_{\uparrow\mathcal{D}'}$ contains a cycle. Therefore, we cannot apply Lemma 3 for the forests F, H and prove that G is a minimal forest. The cycle in R appears, since H has an arc entering $\mathcal{D}' = \{j, m\}$, namely, (l, m). In the forest H', no arc enters $\mathcal{D}'$. Thus, we need the intermediate forest H'.

Remark 2. In the proof of Theorem 2, we pick two arbitrary minimal forests $F \in \widetilde{\mathcal{F}}^{k+1}$ and $H \in \widetilde{\mathcal{F}}^k$ and, with the help of operations of substituting arcs, construct a minimal descendant of F and a minimal ancestor of H. Thus, two arbitrary forests $F \in \widetilde{\mathcal{F}}^{k+1}$ and $H \in \widetilde{\mathcal{F}}^k$ contain a lot of “genetic” information, since we can determine by them both a minimal ancestor of H and a minimal descendant of F.

5. Conclusions

Entering forests are important, since, in terms of them, we can express minors of Laplacian matrices (with zero sum in each row) [7–10], minors of arbitrary square matrices and their spectrum [5, 6]. This motivate us to choose entering forests (not outgoing ones).

More special properties of forests of minimal weight and their structure on some special subsets of the vertex set will be presented in the continuation of this paper.

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REFERENCES

1. A. D. Ventsel, "On the asymptotics of eigenvalues of matrices with elements of order $\exp\{-V_{ij}/2\varepsilon^2\}$," *Dokl. Akad. Nauk SSSR*, **202**, No. 2, 263–266 (1972).
2. A. D. Ventsel and M. I. Freidlin, *Fluctuations in Dynamic Systems Under Small Random Disturbance* [in Russian], Moscow (1979).
3. V. A. Buslov and K. A. Makarov, "Hierarchy of time scales in the case of weak diffusion," *Teor. Mat. Fiz.*, **76**, No. 2, 219–230 (1988).
4. V. A. Buslov and K. A. Makarov, "Lifetimes and lower eigenvalues of an operator of small diffusion," *Mat. Zam.*, **51**, No. 1, 20–31 (1992).
5. V. A. Buslov, "On characteristic polynomial and eigenvectors in terms of tree structure of a digraph," *Zap. Nauchn. Semin. POMI*, **450**, 14–36 (2016).
6. V. A. Buslov, "On the relationship between multiplicities of the matrix spectrum and signs of components of its eigenvector in a tree-like structure," *Zap. Nauchn. Semin. POMI*, **464**, 5–25 (2017).
7. V. A. Buslov, "On the characteristic polynomial coefficients of the Laplace matrix of a weighted digraph and the all minors theorem," *Zap. Nauchn. Semin. POMI*, **427**, 5–21 (2014).
8. S. Chaiken, "A combinatorial proof of the all minors matrix tree theorem," *SIAM J. Alg. Dis. Meth.*, **3**, No. 3, 319–329 (1982).
9. J. W. Moon, "Some determinant expansions and the matrix-tree theorem," *Discrete Mathematics*, **124**, 163–171 (1994).
10. P. Chebotarev and R. Agaev, "Forest matrices around the Laplacian matrix," *Lin. Algebra Appl.*, **356**, 253–274 (2002).