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LINEAR BOUNDARY-VALUE PROBLEMS FOR WEAKLY SINGULAR INTEGRAL EQUATIONS

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We establish necessary and sufficient conditions for the solvability of the linear boundary-value problem for a weakly singular integral equation and find the general form of the solution to this problem.

The extensive application of integral equations and boundary-value problems for these equations promotes the development of the theory and the appearance of numerous publications in this field. In particular, there are many works devoted to the investigation of weakly singular integral equations called equations with weak singularity or equations with polar kernels and encountered in various fields of natural sciences [1, 2]. The foundations of the theory of equations of this kind were presented in the monographs by Goursat [3], Mikhlin [4], Tricomi [5], and Smirnov [6]. The works by Vainikko [7], Graham [8], Tang [9], and other researchers [10–14] were devoted to the study of differential properties of the solutions of weakly singular integral equations and the development of approximate methods for their solution. The problem of solvability of equations of this kind with unbounded kernels and singular equations with Cauchy-type kernels is investigated by using the procedure of regularization of these equations, i.e., by reduction to the Fredholm equation [15]. However, the problem of finding the solvability conditions and construction of the solutions of specific equations with the help of the Fredholm alternative encounters significant technical difficulties. In the present paper, by using the methods of the theory of pseudoinverse matrices, we consider an alternative approach and establish, in the constructive way, the conditions for the existence of solutions of weakly singular integral equations and Noetherian boundary-value problems for these equations and determine the structure of their solutions.

1. Boundary-Value Problem for an Integral Equation with Weakly Singular Kernel

Consider a linear inhomogeneous boundary-value problem for the integral equation with weakly singular kernel

$$x(t) = f(t) + \int_a^b K(t,s)x(s)ds, \tag{1}$$

$$lx(\cdot) = \alpha. \tag{2}$$

Here,

$$K(t,s) = \frac{H(t,s)}{|t-s|^\gamma},$$

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where $H(t,s)$ is a function bounded in the domain $[a,b]\times[a,b]$, $0<\gamma<1$, $f\in L_2[a,b]$, $x\in L_2[a,b]$, l is a bounded linear functional defined in $L_2[a,b]$, $l = \operatorname{col}\left(l_1, l_2, \ldots, l_p\right): L_2[a,b]\to R^p$, $l_\nu: L_2[a,b]\to R$, and $\alpha = \operatorname{col}\left(\alpha_1, \alpha_2, \ldots, \alpha_p\right)\in R^p$.

2. Reduction of the Integral Equation (1) to an Equation with Iterated Kernel

The kernel $K(t,s)$ of the integral operator

$$(Kw)(t) = \int_a^b K(t,s)w(s)ds$$

in Eq. (1) is unbounded. However, Eq. (1) can be reduced to a certain equivalent equation with integral operator whose kernel is square summable. This enables us to pass from the investigation of the boundary-value problem for the integral equation with unbounded kernel (1), (2) to the investigation of the boundary-value problem for the Fredholm integral equation.

To substantiate the above-mentioned transition, we present the required information from the theory of weakly singular integral operators. It is known [3–6] that, for two given integral operators H_1 and H_2 with weakly singular kernels

$$\frac{H_1(t,s)}{|t-s|^{\gamma_1}} \quad \text{and} \quad \frac{H_2(t,s)}{|t-s|^{\gamma_2}}$$

with exponents γ_1 and γ_2, respectively, their product H_1H_2 has the following kernel:

$$F(t,s) = \int_a^b \frac{H_1(t,\xi)H_2(\xi,s)}{|t-\xi|^{\gamma_1}|\xi-s|^{\gamma_2}}\, d\xi$$

of the same structure and the exponent that does not exceed $\gamma_1+\gamma_2-1$. Under the condition

$$\gamma_1+\gamma_2-1<\frac{1}{2} \tag{3}$$

the kernel $F(t,s)$ is square summable and, for

$$\gamma_1+\gamma_2-1<0, \tag{4}$$

the kernel $F(t,s)$ is bounded.

We now consider the iterated kernels $K_n(t,s)$, $n\in\mathbb{N}$, given by the recurrence relation

$$K_{n+1}(t,s) = \int_a^b K(t,\xi)K_n(\xi,s)d\xi, \quad K_1(t,s) = K(t,s). \tag{5}$$

According to the results presented above, the iterated kernels $K_n(t,s)$ have the same structure as the weakly singular kernel

$$K(t,s) = \frac{H(t,s)}{|t-s|^{\gamma}}$$

but the number γ is replaced with $1 - n(1-\gamma)$, which becomes negative for sufficiently large n. Thus, according to (3) and (4), for all n such that the condition

$$n > \frac{1}{2(1-\gamma)} \tag{6}$$

is satisfied, the kernels $K_n(t,s)$ are square summable and, for

$$n > \frac{1}{1-\gamma}, \tag{7}$$

the kernels $K_n(t,s)$ are bounded.

We now show that Eq. (1) can be reduced to the Fredholm equation with the kernel $K_n(t,s)$. Indeed, multiplying both sides of Eq. (1) from the left by $K(t,s)$ and integrating the left- and right-hand sides of the obtained equality over the segment $[a,b]$, we get

$$\int_a^b K(t,s)x(s)ds = \int_a^b K(t,s)f(s)ds + \int_a^b K_2(t,s)x(s)ds.$$

Continuing this process, we obtain

$$\int_a^b K_2(t,s)x(s)\,ds = \int_a^b K_2(t,s)f(s)\,ds + \int_a^b K_3(t,s)x(s)\,ds,$$

. . . .

$$\int_a^b K_{n-1}(t,s)x(s)\,ds = \int_a^b K_{n-1}(t,s)f(s)\,ds + \int_a^b K_n(t,s)x(s)\,ds.$$

Further, we add the obtained equations to Eq. (1) (term by term), this enables us to conclude that the function $x(t)$ is a solution of the equation

$$x(t) = f_n(t) + \int_a^b K_n(t,s)x(s)ds, \tag{8}$$

$$f_n(t) = f(t) + \sum_{k=1}^{n-1} \int_a^b K_k(t,s)f(s)ds.$$

Thus [3–6], according to condition (6), after finitely many steps, we arrive at Eq. (8) with square summable kernel. It is clear that any solution of Eq. (1) is a solution of Eq. (8). Generally speaking, the converse statement is not true. However, it is possible to choose a number n such that condition (7) and, hence, condition (6) are satisfied and an arbitrary solution of Eq. (8) is a solution of Eq. (1), i.e., Eqs. (1) and (8) are equivalent [4, p. 98]. In what follows, we assume that the number n is chosen in this way. Thus, for fixed n, we pass from the investigation of the boundary-value problem for the integral equation with unbounded kernel (1), (2) to the investigation of the boundary-value problem for the Fredholm integral equation (8), (2).

Remark. If, for some fixed n for which condition is satisfied, Eq. (8) possesses a single solution, then Eqs. (1) and (8) are equivalent. In particular, Eqs. (1) and (8) are equivalent if the Fredholm operator in Eq. (1) is replaced with the Volterra operator [3, p. 36].

3. Criterion of Solvability of the Boundary-Value Problem (1), (2)

To study problem (8), (2), we use the approach described in [16, 17]. Problem (8), (2) can be reduced to a countable system of linear algebraic equations. Let $\{\varphi_i(t)\}_{i=1}^{\infty}$ be a complete orthonormal system of functions in $L_2[a,b]$. We introduce the quantities

$$x_i = \int_a^b x(t)\varphi_i(t)\,dt, \qquad a_{ij} = \int_a^b\int_a^b K_n(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds, \tag{9}$$

$$f_i = \int_a^b f_n(t)\varphi_i(t)dt = \int_a^b f(t)\varphi_i(t)\,dt + \sum_{k=1}^{n-1}\int_a^b\int_a^b K_k(t,s)f(s)\varphi_i(t)\,dt\,ds. \tag{10}$$

Applying expressions (9) and (10) to problem (8), (2), we arrive at the following countable system of algebraic equations:

$$x_i - \sum_{j=1}^{\infty} a_{ij}x_j = f_i, \quad i = \overline{1,\infty}, \tag{11}$$

$$\sum_{j=1}^{\infty} l_\nu\varphi_j(\cdot)x_j = \alpha_\nu, \quad \nu = \overline{1,p}, \tag{12}$$

$$\sum_{i=1}^{\infty}|x_i|^2 < +\infty.$$

We rewrite system (11), (12) in the form of an operator equation in the space ℓ_2:

$$Uz = \begin{bmatrix}\Lambda\\ W\end{bmatrix} z = \begin{bmatrix} g\\ \alpha\end{bmatrix} = q, \tag{13}$$

where

$$z = \operatorname{col}\left(x_1,\ x_2,\ \ldots,\ x_i,\ \ldots\right), \qquad g = \operatorname{col}\left(f_1,\ f_2,\ \ldots,\ f_i,\ \ldots\right), \tag{14}$$

$$\Lambda = \begin{pmatrix} 1-a_{11} & -a_{12} & \dots & -a_{1i} & \dots \\ -a_{21} & 1-a_{22} & \dots & -a_{2i} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -a_{i1} & -a_{i2} & \dots & 1-a_{ii} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad W = l\Phi(\cdot), \tag{15}$$

$$\Phi(t) = \big(\varphi_1(t),\ \varphi_2(t),\ \dots,\ \varphi_i(t),\ \dots\big).$$

The operator $\Lambda\colon \ell_2 \to \ell_2$ on the left-hand side of the operator equation (13) has the form $\Lambda = I - A$, where $I\colon \ell_2 \to \ell_2$ is the identity operator and $A\colon \ell_2 \to \ell_2$ is a completely continuous operator. According to S. Krein's classification, $\Lambda\colon \ell_2 \to \ell_2$ is a Fredholm operator

$$(\dim\ker\Lambda = \dim\ker\Lambda^* < \infty)$$

and $U\colon \ell_2 \to \ell_2 \times \mathbb{R}^p$ is a Fredholm operator ($\dim\ker U < \infty$ and $\dim\ker U^* < \infty$).

Thus, the following theorem is true for Eq. (13) [18]:

Theorem 1. *The homogeneous equation (13)* ($q = 0$) *possesses a* d_2*-parameter family of solutions* $z \in \ell_2$ *of the form*

$$z = P_{\Lambda_r} P_{Q_{d_2}} c_{d_2} \quad \forall c_{d_2} \in \mathbb{R}^{d_2}.$$

The inhomogeneous equation (13) is solvable if and only if $r + d_1$ *linearly independent conditions*

$$P_{\Lambda_r^*} g = 0,$$

$$P_{Q_{d_1}^*}\big(\alpha - W\Lambda^+ g\big) = 0$$

are satisfied and possesses a d_2*-parameter family of solutions* $z \in \ell_2$ *of the form*

$$z = P_{\Lambda_r} P_{Q_{d_2}} c_{d_2} + P_{\Lambda_r} Q^+(\alpha - W\Lambda^+ g) + \Lambda^+ g \quad \forall c_{d_2} \in \mathbb{R}^{d_2}.$$

Here, $Q = WP_{\Lambda_r}$, P_{Λ_r} $\big(P_{\Lambda_r^*}\big)$ is a matrix formed by a complete system of r linearly independent columns (rows) of the matrix projector P_Λ (P_{Λ^*}), where P_Λ (P_{Λ^*}) is the projector onto the kernel (cokernel) of the matrix Λ and $P_{Q_{d_2}}$ $\Big(P_{Q_{d_1}^*}\Big)$ is a matrix formed by the complete system of d_2 (d_1) linearly independent columns (rows) of the matrix projector P_Q $\big(P_{Q^*}\big)$; here, P_Q $\big(P_{Q^*}\big)$ is the projector onto the kernel (cokernel) of the matrix Q, and Λ^+ $\big(Q^+\big)$ is the pseudoinverse (in the Moore–Penrose sense) matrix for the matrix Λ (Q).

If Eq. (13) has a solution, then, by the Riesz–Fischer theorem, there exists an element $x \in L_2[a,b]$ such that x_i, $i = \overline{1,\infty}$, determined from system (11), (12), are its Fourier coefficients, i.e., the representation

$$x(t) = \sum_{i=1}^{\infty} x_i \varphi_i(t) = \Phi(t) z \tag{16}$$

is true. The element $x(t)$ determined by relation (16) is the required solution of the boundary-value problem (8), (2) and, hence, of the original problem(1), (2).

According to [16, 18], the following assertion is true:

Theorem 2. *The homogeneous boundary-value problem (1), (2)* $(f(t) = 0, \alpha = 0)$ *possesses the solution* $x \in L_2[a, b]$

$$x(t) = \Phi(t) P_{\Lambda_r} P_{Q_{d_2}} c_{d_2} \quad \forall c_{d_2} \in \mathbb{R}^{d_2}.$$

The inhomogeneous boundary-value problem (1), (2) is solvable if and only if r *linearly independent conditions*

$$P_{\Lambda_r^*} g = 0 \tag{17}$$

and d_1 *linearly independent conditions*

$$P_{Q_{d_1}^*}\big(\alpha - W\Lambda^+ g\big) = 0 \tag{18}$$

are satisfied and has a d_2*-parameter family of solutions* $x \in L_2[a, b]$ *of the form*

$$x(t) = \Phi(t)\Big(P_{\Lambda_r} P_{Q_{d_2}} c_{d_2} + P_{\Lambda_r} Q^+ \left(\alpha - W\Lambda^+ g\right) + \Lambda^+ g\Big) \quad \forall c_{d_2} \in \mathbb{R}^{d_2}. \tag{19}$$

4. Example

We now illustrate the theoretical results obtained above by analyzing a specific example. Consider a boundary-value problem for the Volterra integral equation with weakly singular kernel

$$x(t) = (t + 1)(3 - 4\sqrt{t + 1}) + \int_{-1}^{t} \frac{x(s)ds}{\sqrt{t - s}}, \quad t \in [-1, 1], \tag{20}$$

$$\int_{-1}^{1} t x(t)\, dt = 2. \tag{21}$$

In this case,

$$K(t, s) = \frac{1}{\sqrt{t - s}},$$

i.e.,

$$\gamma = \frac{1}{2}$$

and, according to condition (6), all iterated kernels starting from $K_2(t, s)$ are square summable. Thus, in view of the remark made in Sec. 2, we can pass from the boundary-value problem for the integral equation with unbounded kernel (20), (21) to the equivalent boundary-value problem for the integral equation with square summable kernel $K_2(t,s)$ of the form [5, p. 58]

$$K_2(t,s) = \int_s^t \frac{d\xi}{\sqrt{(t-\xi)(\xi-s)}} = \pi$$

[in view of relation (5)]. Indeed, multiplying both sides of Eq. (20) from the left by $K(t,s)$, integrating the left- and right-hand sides of the obtained equality over the segment $[-1,1]$, and adding the obtained equation to Eq. (20) (term by term), we get

$$x(t) = \frac{3}{2}(t+1)(2-\pi-\pi t) + \pi\int_{-1}^{t} x(s)\,ds, \quad t\in[-1,1], \tag{22}$$

$$\int_{-1}^{1} t x(t)\,dt = 2. \tag{23}$$

Problem (22), (23) can be reduced to a countable system of linear algebraic equations. We introduce a function

$$\varphi_i(t) = \sqrt{\frac{2i-1}{2}}P_{i-1}(t),$$

where $P_i(t)$ are the Legendre polynomials. The system $\{\varphi_i(t)\}_{i=1}^{\infty}$ is a complete orthonormal system of functions in $L_2[-1,1]$. We compute the quantities a_{ij} and f_i. According to (9) and (10), by using the properties of the Legendre polynomials [19, p. 142], we obtain

$$\begin{aligned} a_{ij} &= \pi\int_{-1}^{1}\varphi_i(t)\int_{-1}^{t}\varphi_j(s)\,ds\,dt \\ &= \frac{\pi}{2}\sqrt{(2i-1)(2j-1)}\int_{-1}^{1}P_{i-1}(t)\int_{-1}^{t}P_{j-1}(s)\,ds\,dt \\ &= \frac{\pi}{2}\sqrt{\frac{2i-1}{2j-1}}\int_{-1}^{1}P_{i-1}(t)\big(P_j(t)-P_{j-2}(t)\big)\,dt \\ &= \begin{cases} \pi, & i=j=1, \\ \dfrac{\pi}{\sqrt{(2i+1)(2i-1)}}, & j=i-1, \\ -\dfrac{\pi}{\sqrt{(2i+1)(2i-1)}}, & j=i+1, \\ 0, & i=j>1,\ j=i\pm k,\ k>1, \end{cases} \end{aligned}$$

$$f_i = \frac{3}{2}\int_{-1}^{1}(t+1)(2-\pi-\pi t)\varphi_i(t)\,dt$$

$$= \frac{3}{2}\sqrt{\frac{2i-1}{2}}\int_{-1}^{1}(t+1)(2-\pi-\pi t)P_{i-1}(t)dt$$

$$= \begin{cases} \sqrt{2}(3-2\pi), & i=1, \\ \sqrt{6}(1-\pi), & i=2, \\ -\dfrac{\sqrt{10}}{5}\pi, & i=3, \\ 0, & i>3. \end{cases}$$

In view of notation (14), (15), the operator equation (13) takes the form

$$Uz = \begin{bmatrix} \Lambda \\ W \end{bmatrix} z = \begin{bmatrix} g \\ \alpha \end{bmatrix} = q,$$

where

$$z = \operatorname{col}\left(x_1, \quad x_2, \quad \ldots, \quad x_i, \quad \ldots\right), \qquad x_i = \sqrt{\frac{2i-1}{2}}\int_{-1}^{1} x(t)P_{i-1}(t)dt,$$

$$\Lambda = \begin{pmatrix} 1-\pi & \dfrac{\pi}{\sqrt{3}} & 0 & 0 & 0 & \ldots \\ -\dfrac{\pi}{\sqrt{3}} & 1 & \dfrac{\pi}{\sqrt{15}} & 0 & 0 & \ldots \\ 0 & -\dfrac{\pi}{\sqrt{15}} & 1 & \dfrac{\pi}{\sqrt{35}} & 0 & \ldots \\ 0 & 0 & -\dfrac{\pi}{\sqrt{35}} & 1 & \dfrac{\pi}{\sqrt{63}} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \qquad g = -\frac{\sqrt{10}}{5}\begin{pmatrix} \sqrt{5}(2\pi-3) \\ \sqrt{15}(\pi-1) \\ \pi \\ 0 \\ \ldots \end{pmatrix},$$

$$W = \left(0, \quad \frac{\sqrt{6}}{3}, \quad 0, \quad \ldots, \quad 0, \quad \ldots\right), \quad \alpha = 2.$$

Equation (22) is an equation with degenerate kernel and a polynomial right-hand side. Hence [3–6], its solution is a polynomial of degree at most 2. This means that, in the construction of the operator equation (13), we can restrict ourselves to the functions $\{\varphi_i(t)\}_{i=1}^{3}$. In this case,

$$\det\Lambda = \frac{1}{15}\left(15 - 15\pi + 6\pi^2 - \pi^3\right) = \beta \approx -0.26$$

and the tridiagonal matrix Λ is nonsingular. Therefore,

$$\Lambda^{+} = \Lambda^{-1} = \frac{1}{15\beta}\begin{pmatrix} 15+\pi^2 & -5\sqrt{3}\pi & \sqrt{5}\pi^2 \\ 5\sqrt{3}\pi & 15(1-\pi) & \sqrt{15}\left(\pi^2-\pi\right) \\ \sqrt{5}\pi^2 & -\sqrt{15}\left(\pi^2-\pi\right) & 5\left(3-3\pi+\pi^2\right) \end{pmatrix}$$

and

$$P_{\Lambda} = P_{\Lambda^*} = O_{3\times 3}, \quad Q = O_{1\times 3}, \quad Q^{+} = O_{3\times 1}, \quad P_{Q_{d_2}} = I_3, \quad P_{Q^*_{d_1}} = 1, \quad d_1 = 1, \quad d_2 = 3,$$

where $O_{3\times 3}$, $O_{1\times 3}$, and $O_{3\times 1}$ are 3×3, 1×3, and 3×1 zero matrices, respectively, and I_3 is the identity matrix of order 3.

We now check the conditions of solvability formulated in Theorem 2 for the boundary-value problem (20), (21). It is clear that condition (17) is satisfied in view of the fact that $P_{\Lambda^*} = 0$. Further, we consider condition (18):

$$P_{Q^*_{d_1}}\left(\alpha - W\Lambda^{+}g\right) = 2 + \frac{2}{15\beta}\left(5\pi(2\pi-3) + 15(1-\pi)(\pi-1) + \left(\pi^2-\pi\right)\pi\right)$$

$$= 2 + \frac{2}{15\beta}\left(\pi^3 - 6\pi^2 + 15\pi - 15\right) = 0.$$

Thus, condition (18) is also satisfied. By Theorem 2, the boundary-value problem (20), (21) possesses a unique solution $x \in L_2[-1, 1]$. In the analyzed case, this solution takes the form

$$x(t) = \Phi(t)\Lambda^{-1}g = \frac{\sqrt{10}}{300\beta}\left(2\sqrt{2},\ 2\sqrt{6}t,\ \sqrt{10}(3t^2-1)\right)\begin{pmatrix} 45\sqrt{5}\beta \\ 15\sqrt{15}\beta \\ 0 \end{pmatrix} = 3(t+1). \qquad (24)$$

Thus, we have established that the Noetherian boundary-value problem for the integral equation with unbounded kernel (20), (21) is uniquely solvable and constructed the solution of this problem (24).

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