DOI 10.1007/s10958-020-04785-2
Journal of Mathematical Sciences, Vol. 246, No. 6, May, 2020

UNBLOCKED IMPUTATIONS OF FUZZY GAMES. I: EXISTENCE

V. A. Vasil'ev
Sobolev Institute of Mathematics SB RAS
4, Akad. Koptyuga pr., Novosibirsk 630090, Russia
Novosibirsk State University
1, Pirogova St., Novosibirsk 630090, Russia
vasilev@math.nsc.ru

UDC 519.86

We generalize the well-known Scarf theorem on the nonemptiness of the core to the case of generalized fuzzy cooperative games without side payments provided that the set of blocking coalitions is extended by the so-called fuzzy coalitions. The notion of a balanced family is extended to the case of an arbitrary set of fuzzy blocking coalitions, owing to which it is possible to introduce a natural analogue of balancedness of a fuzzy game for the characteristic function with an arbitrary efficiency domain. Based on an appropriate approximation of a fuzzy game by finitely-generated games, together with the seminal combinatorial Scarf lemma on ordinal and admissible bases, we obtain rather general conditions of the existence of unblocked imputations for F-balanced fuzzy cooperative games. Bibliography: 15 *titles.*

The paper is based on the results of [1]. We generalize the well-known Scarf theorem on the nonemptiness of the core [2, 3] to the large class of cooperative games without side payments in the case where the set of blocking coalitions, together with usual ones, includes an arbitrary family of the so-called fuzzy coalitions. To the knowledge of the author, except for [1] and [4], there are no results in the literature on general conditions of the existence of an unblocked imputation in fuzzy cooperative games without side payments. We focus on the case where the above-mentioned family is infinite (in particular, all fuzzy coalitions are blocking). The notion of a balanced cover [5, 6] is extended to the case of an arbitrary family of fuzzy blocking coalitions, which allows us to introduce a natural analogue of the notion of balancedness for generalized cooperative game with any efficiency set of its characteristic function. For an arbitrary finite efficiency set we introduce the notion of a finitely generated game and, using the combinatorial Scarf lemma on ordinal and admissible bases [7], establish a counterpart of the classical Scarf theorem on the nonemptiness of the core. Then, using appropriate approximations of arbitrary finitely generated games, we prove an analogue of the Scarf theorem on the nonemptiness of the core in the general case (provided that the efficiency domain of the characteristic function is finite). It is of interest that the obtained generalization verbatim coincides with the original

Translated from *Sibirskii Zhurnal Chistoi i Prikladnoi Matematiki* **18**, No. 1, 2018, pp. 35-53.

1072-3374/20/2466-0828 © 2020 Springer Science+Business Media, LLC

Scarf theorem, where ordinary coalitions are suitably replaced by more general ones. Finally, based on traditional compactness arguments, it is possible to extend the results of this paper to the case of an infinite efficiency set of the characteristic function of a generalized cooperative game.

1 Definitions and Formulation of the Main Results

We introduce the notation. Let n be an arbitrary natural number. We set $N = \{1, \dots, n\}$ and denote by 2^N the collection of subsets of N. In the traditional game-theoretic terminology [8], elements of the set N are called *players* and elements of the family 2^N are referred to as *coalitions*. In a number of cases, it is convenient to identify coalitions with the corresponding vertices of the unit n-dimensional cube $I^n = \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^N \mid \tau_i \in [0, 1],\ i \in N\}$ (hereinafter, $\mathbb{R}$ denotes the set of real numbers). As usual, for any coalition $S \subseteq N$ we denote by e^S its indicator function:

$$(e^S)_i = \begin{cases} 1, & i \in S, \\ 0, & i \in N \setminus S. \end{cases}$$

These indicator functions e^S (which are obviously vertices of the hypercube I^n) correspond to standard coalitions S which are identified with elements of I^n.

As known, together with vertices (ordinary coalitions), an important role in the description of the Walrasian and Edgeworth distributions is played by other elements of the hypercube I^n (cf., for example, [9]–[12]). We mean the so-called fuzzy coalitions, the collection of which (including all standard coalitions) is denoted by σ_F and is defined by the formula

$$\sigma_F = I^n \setminus \{0\}.$$

Thus, fuzzy coalitions are nonzero elements of the unit n-dimensional cube I^n. Moreover, as was already mentioned, the vertices (the points $\tau \in \sigma_F$ with coordinates only 0 or 1) are naturally identified with standard coalitions $S_\tau = \{i \in N \mid \tau_i = 1\}$. We recall [13] that the quantity of the component τ_i of the fuzzy coalition $\tau = (\tau_1, \dots, \tau_n)$ is interpreted as the level of participation of player i in the coordination of efforts of players of the large coalition N. Respectively, in ordinary coalitions, each player i either fully participates ($\tau_i = 1$) or does not participate at all ($\tau_i = 0$) in this coordination. For τ in σ_F we denote by $N(\tau)$ the *support of the fuzzy coalition* τ:

$$N(\tau) = \{i \in N \mid \tau_i > 0\}.$$

By the definition of σ_F, the supports of fuzzy coalitions are nonempty sets.

Saying informally, the generalizied games considered below are mappings sending each variant of the union $\tau \in \sigma_F$ to some (possibly, empty) set of payoff vectors reachable while formatting the fuzzy coalition τ under consideration.

Definition 1. A *generalized fuzzy cooperative n-person game* is a set-valued mapping

$$\tau \mapsto G(\tau), \quad \tau \in \sigma_F,$$

sending a coalition $\tau \in \sigma_F$ to a subset $G(\tau)$ of the space $\mathbb{R}^{N(\tau)}$. Elements of $G(\tau)$ are called *payoff vectors of the coalition* τ, and the payoff vectors of the coalition e^N are also called the *payoff vectors of the game* G. We define G at zero by setting $G(0) = \varnothing$.

It is clear that the set of "capable" coalitions of the game G providing the nontrivial cooperation effect is of a great interest.

Definition 2. The *efficiency set* of a game G is the collection $e(G)$ of all fuzzy coalitions τ such that $G(\tau) \neq \varnothing$:

$$e(G) = \{\tau \in \sigma_F \mid G(\tau) \neq \varnothing\}.$$

Elements of the set $e(G)$ are called *blocking coalitions.*

In what follows, we often use the following assumption.

Assumption 1. The sets $G(e^{\{1\}}), \dots, G(e^{\{n\}})$ and $G(e^N)$ are nonempty and closed. In particular, the efficiency set of the game G contains all one-element coalitions and the coalition of all players: $\{e^{\{1\}}, \dots, e^{\{n\}}, e^N\} \subseteq e(G)$.

Definition 3. A generalized fuzzy cooperative game G is called *regular* if Assumption 1 is satisfied.

Remark 1. It is clear that the above-mentioned games $\tau \mapsto G(\tau)$, $\tau \in I^n$, naturally generalize the classical cooperative n-person games since each classical game $S \mapsto F(S) \subseteq \mathbb{R}^S$, $S \subseteq N$, is trivially extended to the whole hypercube I^n by the formula

$$F(\tau) = \varnothing, \quad \tau \in \sigma_F^0 = I^n \setminus \sigma_0,$$

where σ_0 is the set of nonzero vertices of the cube I^n. Thus, the classical cooperative n-person games can be identified with a generalized cooperative game G such that $e(G) = \sigma_0$.

For the sake of brevity, throughout the paper, in addition to the notation e^S we sometimes use the symbol S to denote the vertices of the hypercube I^n corresponding to ordinary coalitions $S \subseteq N$ (we often use the same notation for e^N and N). Furthermore, as usual, one-element coalitions $\{i\}$ are sometimes denoted by i. Finally, we use the standard notation: for any vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ in $\mathbb{R}^m$ we set

$$x \geqslant y \Leftrightarrow x_k \geqslant y_k, \quad k = 1, \dots, m,$$

$$x \gg y \Leftrightarrow x_k > y_k, \quad k = 1, \dots, m.$$

We introduce the key notion of this paper; namely, an F-balanced cover, where for covering we can take not only ordinary, but also properly fuzzy coalitions.[1)]

Definition 4. A finite family of fuzzy coalitions $\{\tau^k\}_{k\in K}$ is called an *F-balanced cover* of the set N if there are nonnegative numbers $\{\lambda_k\}_{k\in K}$ such that $\sum\limits_{k\in K} \lambda_k \tau^k = e^N$. As in the classical definition, the numbers λ_k are called *weights* of fuzzy coalitions τ^k.

Remark 2. We indicate a simple geometric interpretation of the F-balancedness of the cover $\mathscr{T} = \{\tau^k\}_{k\in K}$: the family $\mathscr{T}$ forms an F-balanced cover of a coalition N if and only if its conical hull contains the diagonal $\mathscr{D} = \{te^N \big| t > 0\} : \mathscr{D} \subseteq \operatorname{cone} \mathscr{T}$.

To define an F-balanced generalized game G, we begin by introducing an analogue of a G-balanced vector [11] for such a game. Further, as usual, $u_S \in \mathbb{R}^S$ denotes the restriction of the vector $u = (u_1, \dots, u_n) \in \mathbb{R}^N$ on the set $S \subseteq N$, i.e., $(u_S)_i = u_i$, $i \in S$.

---

1) In the classical notion of a balanced cover, it is assumed that all its elements are standard coalitions [5]. In the literature, along with a *balanced cover*, the term a *balanced family* is used (cf., for example, [14]).

Definition 5. Let G be an arbitrary generalized cooperative n-person game. A vector $u \in \mathbb{R}^N$ is called G-*balanced* if there exists an F-balanced cover $\{\tau^k\}_{k \in K}$ of the set N such that $u_{N(\tau^k)} \in G(\tau^k)$ for all $k \in K$.

Definition 6. A game G is said to be F-*balanced* if any G-balanced vector belongs to $G(N)$.

We formulate the central notion of the paper: the core of a generalized cooperative game G.

Definition 7. We say that a coalition $\tau \in e(G)$ *blocks* a payoff vector $u = (u_1, \dots, u_n) \in G(N)$ if there exists a vector $v = (v_i)_{i \in N(\tau)} \in G(\tau)$ such that $v_i > u_i$ for all $i \in N(\tau)$. The collection of all payoff vectors in $G(N)$ that are not blocked by any coalition $\tau \in e(G)$ is denoted by $C(G)$ and is called the *core* of the game G.

We recall (cf., for example, [2, 11]) that the blocking in the classical cooperative game is defined in the same way as in Definition 7 (applied to the blocking coalitions in the family σ_0). Unlike the classical case, the blocking in a generalized game is defined not only for elements of a finite set consisting of $2^n - 1$ standard coalitions, but also for the remaining coalitions in $e(G)$. If necessary to emphasize that we deal with a nonclassical game, the blocking in such a game will be also called F-*blocking.*

Remark 3. We indicate the property of cores of generalized cooperative games which will be useful below. Namely, as in the case of an ordinary cooperative game, if the set $G(N)$ is closed, then the core of the generalized game G is closed. Indeed, assume that the sequence of payoff vectors $\{v^m\}_{m=1}^{\infty} \subseteq C(G)$ converges to some vector v^0. By the embedding $C(G) \subseteq G(N)$ and closeness of $G(N)$, we have $v^0 \in G(N)$. Thus, to prove that v^0 belongs to the core $C(G)$, it remains to check that there is no coalition blocking the payoff vector v^0. Assume the contrary. Let v^0 be blocked by a coalition $\tau \in \sigma_F$. Then $u_i > v_i^0$, $i \in N(\tau)$, for some payoff vector $u \in G(\tau)$. Since $v^0 = \lim\limits_{m \to \infty} v^m$, there exists a sufficiently large natural number m_0 such that $u_i > v_i^m$ for all $i \in N(\tau)$ and $m \geqslant m_0$. Consequently, the coalition τ blocks all payoff vectors v^m with $m \geqslant m_0$. However, by assumption, the vectors v^m are contained in the core $C(G)$ for all $m \geqslant 1$. This contradiction shows that the limit of any converging sequence of vectors in $C(G)$ also belongs to the core of the generalized cooperative game G.

We recall one of the assumptions of the classical Scarf theorem on the nonemptiness of the core [2] which will be used in a generalization of this theorem below. We set

$$u_i^G = \sup\{u_i \in \mathbb{R} \mid u_i \in G(e^i)\}, \quad i \in N. \tag{1}$$

It is clear that for the nonemptiness of the core $C(G)$ it is necessary that for each player $i \in N$ the quantity u_i^G (the *maximal guaranteed payoff* of player i) to be finite; otherwise, the one-element coalition e^i can block any vector in $G(N)$. Therefore, we assume that the following condition is satisfied everywhere below.

Assumption 2. For all players of the game G the quantities u_i^G are finite, i.e., $u_i^G < \infty$ for every $i \in N$.

Remark 4. It is clear that from Assumption 2 we have the inclusions $u_i^G \in G(e^{\{i\}})$, $i \in N$, for any regular game G.

As in the classical Scarf theorem, we introduce the set of individually rational payoff vectors of the large coalition N:

$$\widehat{G}(N) = \{u \in G(N) \mid u \geqslant u^G\},$$

where $u^G = (u_1^G, \dots, u_n^G)$ is a vector in $\mathbb{R}^n$ whose components are defined by (1). In what follows, the term *individual rationalilty* will be also used for the remaining coalitions τ: a vector $u \in G(\tau)$ is called an *individually rational payoff vector* of a coalition τ if $u \geqslant u_{N(\tau)}^G$, where $u_{N(\tau)}^G$ is the restriction of the vector u^G onto the set $N(\tau)$: $u_{N(\tau)}^G = (u_i^G)_{i \in N(\tau)}$. The collection of individually rational payoff vectors of a coalition τ is denoted by $\widehat{G}(\tau)$:

$$\widehat{G}(\tau) = \{u \in G(\tau) \mid u \geqslant u_{N(\tau)}^G\}.$$

We introduce a useful counterpart of another object of the classical theory of cooperative games, namely, the set of imputations of a game with side payments [14]. By *imputations* of a generalized game G we mean elements of the set $I(G)$ of collectively rational payoff vectors in $\widehat{G}(e_N)$:

$$I(G) = \{u \in \widehat{G}(N) \mid \text{there are no } v \in G(N) \text{ such that } u \ll v\}.$$

Remark 5. It is clear that the set $I(G)$ of imputations of a generalized game G, as in the case of usual games, is the set of all payoff vectors of the large coalition e^N of unblocked neither one-element coalitions $e^{\{1\}}, \dots, e^{\{n\}}$ nor coalition of all players e^N. Consequently, for any regular game G we have the embedding $C(G) \subseteq I(G)$. Therefore, in what follows, elements of the core $C(G)$ of a regular game G are referred to as *unblocked imputations*, as in the case of usual games.

We also recall that a set $X \subseteq \mathbb{R}^m$ is said to be *comprehensive from below* if, together with each element x of X the set X contains any element y such that $y \leqslant x$.

Definition 8. We say that a game G is *comprehensive from below* if all sets $G(\tau)$, $\tau \in e(G)$, are comprehensive from below.

We indicate the following useful specification of the notion of F-balancedness for games comprehensive from below. We introduce the necessary notation. Let $\mathscr{T} = \{\tau^k\}_{k \in K}$ be a finite family of fuzzy coalitions, and let $\mathscr{V} = \{v^k\}_{k \in K}$ be a family of payoff vectors reachable by efforts of these coalitions: $v^k \in G(\tau^k)$ for every $k \in K$. We set $N(\mathscr{T}) = \bigcup_{k \in K} N(\tau^k)$ and for every $i \in N(\mathscr{T})$ denote by K_i the set of numbers in K corresponding to the coalitions τ^k, including player i:

$$K_i = \{k \in K \mid i \in N(\tau^k)\}, \quad i \in N(\mathscr{T}).$$

In what follows, we denote by $u^{\mathscr{V}}$ the vector in $\mathbb{R}^{N(\mathscr{T})}$ associated with the family $\mathscr{V}$ by the formula

$$u_i^{\mathscr{V}} = \min\{v_i^k \mid k \in K_i\}, \quad i \in N(\mathscr{T}). \tag{2}$$

The following characterization of F-balancedness is valid for comprehensive from below generalized cooperative games .

Proposition 1. *Let a generalized cooperative game G be comprehensive from below. Then G is F-balanced if and only if for any F-balanced cover $\mathscr{T} = \{\tau^k\}_{k \in K}$ of the set N and for any family of payoff vectors $\mathscr{V} = \{v^k\}_{k \in K}$ such that $v^k \in G(\tau^k)$, $k \in K$, we have $u^{\mathscr{V}} \in G(N)$, where $u^{\mathscr{V}}$ is defined by formula* (2).

Proof. Let G be a comprehensive from below generalized game. We first show that the F-balancedness of G implies the inclusion $u^{\mathscr{V}} \in G(N)$ for any finite collection $\mathscr{V} = \{v^k\}_{k \in K}$ of payoff vectors of the fuzzy coalitions τ^k, $k \in K$, that form an F-balanced cover of the set N.

Indeed, by the construction of the vector $u^{\mathscr{V}}$, we have $u^{\mathscr{V}}_{N(\tau^k)} \leqslant v^k$, $k \in K$; moreover, $v^k \in G(\tau^k)$, $k \in K$, by assumption. Consequently, since the game G is comprehensive from below, we have $u^{\mathscr{V}}_{N(\tau^k)} \in G(\tau^k)$, $k \in K$. Since the family $\mathscr{T} = \{\tau^k\}_{k\in K}$ forms an F-balanced cover of the set N, we have $u^{\mathscr{V}} \in G(N)$, which is required.

Let a comprehensive from below game G be such that $u^{\mathscr{V}} \in G(N)$ for every finite family $\mathscr{V}$ of payoff vectors $v^k \in G(\tau^k)$, $k \in K$, such that the corresponding family of coalitions $\{\tau^k\}_{k\in K}$ forms an F-balanced cover of the set N. We show that G is an F-balanced game. We assume that a vector $v \in \mathbb{R}^N$ is such that $v_{N(\tau^k)} \in G(\tau^k)$, $k \in K$, for some F-balanced cover $\{\tau^k\}_{k\in K}$ of the set N. It is clear that the vector v can be represented as $v = u^{\mathscr{V}}$, where $\mathscr{V} = \{v_{N(\tau^k)}\}_{k\in K}$ (moreover, the fuzzy coalitions τ^k corresponding to the vectors $v^k = v_{N(\tau^k)}$ form an F-balanced cover of the set N by assumption). Since the vector $u^{\mathscr{V}}$ belongs to the set $G(N)$ by assumption, we obtain the required assertion $v = u^{\mathscr{V}} \in G(N)$. □

Remark 6. By the second part of the proof of Proposition 1, the fact that the vector $u^{\mathscr{V}}$ of the set $G(N)$ for any finite family $\mathscr{V}$ of imputations $v^k \in G(\tau^k)$, $k \in K$, such that the corresponding family of coalitions $\{\tau^k\}_{k\in K}$ is an F-balanced cover of the set N is sufficient for the F-balancedness of the game G in the most general case where the game G is not necessarily comprehensive from below.

According to the traditional formulation [2], the classical cooperative game G without side payments is a set-valued mapping $S \mapsto G(S)$, $S \in 2^N$ that associates any coalition $S \subseteq N$ with a nonempty subset $G(S)$ of the space $\mathbb{R}^S$. Consequently, as was already emphasized in Remark 1, the classical cooperative games can be identified with the generalized cooperative games G satisfying the condition $e(G) = \sigma_0$. Taking into account this fact, we can formulate the Scarf theorem on the nonemptiness of the core as follows.

Theorem (Scarf [2])**.** *Let the efficiency set of a generalized cooperative game G coincide with σ_0. Furthermore, if the game G is comprehensive from below and F-balanced, all sets $G(\tau)$ are closed, and the set $\widehat{G}(N)$ is bounded from above, then the core of the game G is nonempty.*

It turns out that the Scarf theorem, which is the key results in the modern theory of cooperative games, can be practically word-by-word generalized to the case of an arbitrary efficiency set. Namely, the analogue of this theorem proved below repeats the formulation of the Scarf theorem in the case of generalized cooperative games with any restrictions on the efficiency set of the game (in addition to the regularity condition). We remain details for further discussion and emphasize that the principal point for obtaining such an analog is proposed in sufficiently natural and "operable" generalization of the classical notions of balanced cover and balanced cooperative game.

Theorem 1. *If a regular generalized fuzzy cooperative game G is comprehensive from below and F-balanced, all the sets $G(\tau)$ are closed, and the set $\widehat{G}(N)$ is bounded from above, then the core of the game G is nonempty.*

The further analysis whether the saturation from below and F-balancedness properties are inherited shows that Theorem 1 can be considerably improved. Namely, the conclusion of Theorem 1 remains valid if the requirement of the closeness of $G(\tau)$ is imposed only on the sets of payoff vectors of one-element coalitions e^i, $i \in N$, and the coalition e^N (in other words, the requirement of the closeness of all sets $G(\tau)$, $\tau \in \sigma_F$, in Theorem 1 is superfluous).

Theorem 2. *If a regular generalized cooperative game G is comprehensive from below and F-balanced and the set $\widehat{G}(N)$ is bounded from above, then the core of the game G is nonempty.*

2 Common Properties of Cores of Regular Games

Proceeding by considering auxiliary constructions necessary for proving Theorems 1 and 2, we establish some useful general facts concerning some properties of regular cooperative games. Let G be an arbitrary regular cooperative game. With G we associate the game G^0 defined by

$$G^0(\tau) = \widehat{G}(\tau) - \mathbb{R}_+^{N(\tau)}, \quad \tau \in e(G). \tag{3}$$

For τ such that $\widehat{G}(\tau) = \varnothing$ we set $G^0(\tau) = \varnothing$.

Remark 7. It is clear that the set $e(G^0)$, in general, is only a part of the efficiency set $e(G)$ of the regular game G. However, in any case, for a nonempty and bounded from above set $\widehat{G}(N)$ the game G^0 is also regular. Indeed, by Remark 4, $\widehat{G}(e^i) = \{u_i^G\}$, $i \in N$, for a regular game G. Consequently, the relation $G^0(e^i) = (-\infty, u_i^G]$ for every $i \in N$, which proves the nonemptiness and closeness of the sets $G^0(e^i)$. The nonemptiness and closeness of $G^0(N)$ follow from the nonemptiness, closeness, and boundedness of $\widehat{G}(N)$. Indeed, in this case, the nonempty set $G^0(N)$, regarded as the algebraic sum of the compact set $\widehat{G}(N)$ and the closed set $-\mathbb{R}_+^N$, is a closed set in view of the known theorem in mathematical analysis.

A close connection between the games G and G^0 is illustrated by the following lemma.

Lemma 1. *For any comprehensive from below regular game G such that $\widehat{G}(N) \neq \varnothing$ the following equality holds*:

$$C(G) = C(G^0);$$

moreover, if the game G is F-balanced and the set $\widehat{G}(N)$ is bounded from above, then G^0 is also a comprehensive from below, F-balanced, and regular game such that

$$\widehat{G^0}(N) = \widehat{G}(N).$$

Proof. It is clear that for any game G the corresponding game G^0 is comprehensive from below by construction. Therefore if the game G itself is comprehensive from below, then $G^0(\tau) \subseteq G(\tau)$ for all coalitions $\tau \in e(G)$. By the first of the following obvious relations

$$C(G) \subseteq \widehat{G}(N), \quad C(G^0) \subseteq \widehat{G}(N), \tag{4}$$

we have $C(G) \subseteq C(G^0)$. On the other hand, the fact that of some payoff vector $u \in \widehat{G}(N)$ is blocked by a coalition $\tau \in e(G)$ in the game G means that $v_i > u_i$, $i \in N(\tau)$, for some $v \in G(\tau)$. By the relation $u \geqslant u^G$, we have the inequality $v \geqslant u_{N(\tau)}^G$ which means that the vector v belongs to $G^0(\tau)$. Consequently, by formula (3) and the relation $v \gg u_{N(\tau)}$, the payoff vector u is blocked by the coalition τ and in the game G^0. Thus, vectors in $\widehat{G}(N)$ that are blocked in the game G are also blocked in the game G^0. By the second relation in (4), we have $C(G^0) \subseteq C(G)$, which together with the inverse embedding proved above, leads to the required equality $C(G) = C(G^0)$.

Now, we assume that a comprehensive from below regular game G has an individually rational payoff vector ($\widehat{G}(N) \neq \varnothing$) and is F-balanced. We first verify that the associated game G^0 is also

F-balanced. For this purpose we consider an arbitrary G^0-balanced vector $v = (v_1, \dots, v_n)$ and show that this vector belongs to the set $G^0(N)$. Indeed, the G^0-balancedness of v implies the existence of an F-balanced cover $\{\tau^k\}_{k\in K}$ of the coalition N for which $v_{N_k} \in G^0(\tau^k)$, $k \in K$, where $N_k = N(\tau^k)$. By the definition of the game G^0, from these inclusions imply the existence of vectors $u^k \in \widehat{G}(\tau^k)$, $k \in K$, such that

$$v_{N_k} \leqslant u^k, \quad k \in K.$$

We denote by $\overline{u} = \bigwedge_{k\in K} u^k$ the vector in $\mathbb{R}^N$ defined by

$$\overline{u}_i = \min_{k\in K_i} u_i^k, \quad i \in N,$$

where $K_i = \{k \in K \mid i \in N_k\}$, $i \in N$. By the definition of $\overline{u}$, it is obvious that $u^k \in \widehat{G}(\tau^k)$, $k \in K$, imply $\overline{u} \geqslant u^G$. On the other hand, by the obvious inequalities $\overline{u}_{N_k} \leqslant u^k$, $k \in K$, and, since the sets $G(\tau^k)$ are comprehensive from below, we have $\overline{u}_{N_k} \in G(\tau^k)$, $k \in K$. Consequently, in view of the F-balancedness of the game G, we have the inclusion $\overline{u} \in G(N)$ which, together with the above-mentioned inequality $\overline{u} \geqslant u^G$, shows that the vector $\overline{u}$ belongs to the set $\widehat{G}(N)$. To complete the proof of the inclusion $v \in G^0(N)$, it remains to note that the inequality $v \leqslant \overline{u}$ holds since $v_{N_k} \leqslant u^k$, $k \in K$. Indeed, since the set is saturated from below, the set $G^0(N)$ is comprehensive from below, the last inequality, together with the inclusion $\overline{u} \in \widehat{G}(N)$ proved above, leads to the required result: v belongs to $G^0(N)$.

As for the regularity and saturation from below of the game G^0, these properties immediately follow from the regularity of G, the condition $\widehat{G}(N) \neq \varnothing$, and the definition of the game G^0. Indeed, it suffices to prove the closeness of the set $G^0(N)$, but this property follows from the nonemptiness and compactness of $\widehat{G}(N)$ (recall that in the case under consideration, in view of the boundedness of $\widehat{G}(N)$, the set $G^0(N)$ is the algebraic sum of the compact set $\widehat{G}(N)$ and the closed set $-\mathbb{R}^N_+$). Finally, the equality $\widehat{G^0}(N) = \widehat{G}(N)$ immediately follows from formula (3) and the definition of an individually rational payoff vector of the coalition N in the games G and G^0. Thus, Lemma 1 is proved. □

We also need the notion of the closure of a generalized cooperative game G.

Definition 9. By the *closure* of a game G we mean the game $\overline{G}$ defined by

$$\overline{G}(\tau) = \mathrm{cl}G(\tau), \quad \tau \in \sigma_F.$$

Hereinafter, for $X \subseteq \mathbb{R}^m$ we denote by $\mathrm{cl}X$ the closure of a set X in $\mathbb{R}^m$.

Lemma 2. *If a set $X \subseteq \mathbb{R}^m$ is comprehensive from below, then its closure $\overline{X} = \mathrm{cl}X$ is also comprehensive from below.*

Proof. Let $x = (x_1, \dots, x_m)$ belong to $\overline{X}$. We consider an arbitrary $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ such that $y \leqslant x$ and show that y belongs to $\overline{X}$. For this purpose we construct a sequence $\{y^r\}_1^\infty$, where $y^r = y - \dfrac{1}{r}e$ and $e = (1, \dots, 1)$ is the vector in $\mathbb{R}^m$ with components equal to 1. We set $\varepsilon_r = \min\{x_k - y_k^r | k = 1, \dots, m\}$, $r \geqslant 1$. Since all ε_r are strictly positive and x belongs to the closure of X, for every $r \geqslant 1$ there exists an element x^r of X such that $\max\{|x_k - x_k^r| | k = 1, \dots, m\} < \varepsilon_r/2$. From the construction of x^r and y^r we get $y^r \leqslant x^r$ for $r \geqslant 1$. Since $x^r \in X$, $r \geqslant 1$, and the set X is saturated from below, we have $y^r \in X$ for all $r \geqslant 1$. By the obvious relation $y = \lim y^r$, the element y belongs to $\overline{X}$, which is required. □

Corollary 1. *If a generalized game G is saturated from below, then its closure $\overline{G}$ is also a saturated from below game.*

Lemma 3. *If a generalized cooperative game G is saturated from below and F-balanced, then its closure $\overline{G} = \mathrm{cl}G$ is also a saturated from below and F-balanced game.*

Proof. The property of saturation from below follows from Corollary 1. Let us prove the F-balancedness property. Let fuzzy coalitions $\{\tau^k\}_{k\in K}$ form a balanced cover of the large coalition N; moreover, for some vector $v \in \mathbb{R}^N$ we have the inclusions $v_{N_k} \in \overline{G}(\tau^k)$, $k \in K$, where, as above, $N_k = N(\tau^k)$, $k \in K$. By the definition of the closure of a game G, for every $r \geqslant 1$ there exist payoff vectors $u^{kr} \in G(\tau^k)$, $k \in K$, such that

$$\max\{|v_i - u_i^{kr}| \big| i \in N_k\} < 1/r, \quad k \in K. \tag{5}$$

Using these payoff vectors, for every $r \geqslant 1$ we construct vectors u^r by the formula

$$u_i^r = \min\{u_i^{kr} \mid k \in K_i\}, \quad i \in N, \tag{6}$$

where $K_i = \{k \in K \mid i \in N_k\}$, $i \in N$. From (6) we immediately obtain the relations $u^r_{N_k} \leqslant u^{kr}$, $k \in K$, $r \geqslant 1$. Since $u^{kr} \in G(\tau^k)$, $k \in K$, and the sets $G(\tau^k)$ are saturated from below, $u^r_{N_k}$ is contained in $G(N_k)$ for every $k \in K$ and $r \geqslant 1$. Based on the above inclusions, from the balancedness of the cover $\{\tau^k\}_{k\in K}$ and F-balancedness of the game G we find that for every $r \geqslant 1$ the vector u^r belongs to the set $G(N)$. To complete the proof of Lemma 3, it remains to verify that $\lim u^r = v$. For this purpose we note that from (5) and (6) it follows that $\|u^r - v\|_\infty < 1/r$, $r \geqslant 1$, where, as usual,

$$\|x\|_\infty = \max\{|x_i| \big| i \in N\}, \quad x \in \mathbb{R}^N.$$

Indeed, from the relations

$$|v_i - u_i^r| = |v_i - \min_{k\in K_i} u_i^{kr}| \leqslant \max\{|v_i - u_i^{kr}| \big| k \in K_i\} < 1/r, \quad i \in N,$$

which are valid in view of (5) and (6), for all $r \geqslant 1$ we find

$$\max\{|v_i - u_i^r| \big| i \in N\} < 1/r, \quad r \geqslant 1,$$

which implies the required relation $\lim u^r = v \in \overline{G}(N)$. Lemma 3 is proved. □

Lemma 4. *The core of the generalized game G is contained in the core of its closure: $C(G) \subseteq C(\overline{G})$. Moreover, if the set $G(N)$ is closed, then the core of the game G coincides with the core of its closure: $C(G) = C(\overline{G})$.*

Proof. Let u belong to the core $C(G)$. Assuming that an imputation u can be blocked in the game $\overline{G}$ by some coalition $\tau \in \sigma_F$, we find that there exists a vector v in $\overline{G}(\tau)$ such that $v_i > u_i$ for all $i \in N(\tau)$. We set $\varepsilon = \min\{v_i - u_i \mid i \in N(\tau)\}$. Since v belongs to the closure of the set $G(\tau)$, there is a vector $\widetilde{v}$ in $G(\tau)$ such that $\|v - \widetilde{v}\|_\infty < \varepsilon/2$. However, it is obvious that the vector $\widetilde{v}$ satisfies the inequality $\widetilde{v}_i > u_i$, $i \in N(\tau)$, which contradicts the assumption $u \in C(G)$. The obtained contradiction proves the embedding $C(G) \subseteq C(\overline{G})$ for any game G.

By the embedding $C(G) \subseteq C(\overline{G})$ proved in the first part of the proof, in order to prove the identity $C(G) = C(\overline{G})$ in the case where $G(N)$ is a closed set, it suffices to verify that $C(\overline{G})$ is contained in $C(G)$. Assume the contrary. Assume that an imputation u in $C(\overline{G})$ is not contained in $C(G)$. Since $C(\overline{G}) \subseteq \overline{G}(N) = G(N)$, from the condition $u \notin C(G)$ it follows that u belongs to the set $G(N) \setminus C(G)$. However, the inclusion $u \in G(N) \setminus C(G)$ means that u is blocked in the game G and, consequently, it is also blocked (by the embeddings $G(\tau) \subseteq \overline{G}(\tau)$, $\tau \in \sigma_F$) in the game $\overline{G}$. This contradicts the assumption $u \in C(\overline{G})$ and thereby completes the proof of Lemma 4. □

The following assertion is a direct consequence of Lemmas 3 and 4.

Corollary 2. *If G is a regular, saturated from below, F-balanced game, then its closure $\overline{G} = \mathrm{cl} G$ is also a regular, saturated from below, F-balanced game. Moreover, the game $\overline{G}$ has the same core*: $C(\overline{G}) = C(G)$.

3 Core of Generalized Finitely Generated Game

To prove Theorem 1, we distinguish two important classes of generalized n-person games.

Definition 10. A regular game G is said to be *F-finite* if its efficiency set $e(G)$ is finite: $|e(G)| < \infty$.

Definition 11. A game G is said to be *finitely generated* if it is F-finite and for any coalition $\tau \in e(G)$ there exists a finite family of vectors $u^{\tau,k} \in \mathbb{R}^{N(\tau)}$, $k \in K(\tau)$, such that

$$G(\tau) = \bigcup_{k \in K(\tau)} \{v \in \mathbb{R}^{N(\tau)} \mid v \leqslant u^{\tau,k}\},$$

where, as everywhere below, $K(\tau) = \{1, \dots, k_\tau\}$, $\tau \in e(G)$. For the sake of brevity the vectors $u^{\tau,1}, \dots, u^{\tau,k_\tau}$ are referred to as *vertices* of the set $G(\tau)$. Furthermore, if $u^{\tau,k} \geqslant u^G_{N(\tau)}$, then the vector $u^{\tau,k}$ is called the *individually rational vertex* of the set $G(\tau)$.

The collection of all individually rational vertices of the set $G(\tau)$ of a finitely generated game G is denoted by

$$E(\tau) = E(G, \tau) = \{k \in K(\tau) \big| u^{\tau,k} \geqslant u^G_{N(\tau)}\}.$$

It is clear that for any finitely generated game G the condition $\widehat{G}(N) \neq \varnothing$ is satisfied if and only if there is at least one individually rational vertex of the set $G(N)$, i.e.,

$$E(G, N) \neq \varnothing. \tag{7}$$

Since the nonemptiness of the set $\widehat{G}(N)$ is necessary for the nonemptiness of the core of a game G, we assume below that the finitely generated games under consideration satisfy the condition (7).

The following property of finitely generated games is useful for analyzing the nonemptiness condition for the cores. Let G be an arbitrary finitely generated cooperative game. In the above notation, with G we associate the game G^0 by

$$G^0(\tau) = \bigcup_{k \in E(\tau)} \big\{v \in \mathbb{R}^{N(\tau)} \bigm| v \leqslant u^{\tau,k}\big\}, \quad \tau \in e(G). \tag{8}$$

Recall that $G^0(\tau) = \varnothing$ for $\tau \notin e(G)$ by definition. In particular, for one-element coalitions

$$G^0(i) = G^0(e^{\{i\}}) = \big\{u_i \in \mathbb{R}^{\{i\}} \bigm| u_i \leqslant u^G_i\big\} \tag{9}$$

for every $i \in N$, where, as above, u_i^G is the maximal guaranteed payoff of the one-element coalition $\{i\}$. Therefore, if the condition (7) is satisfied, then the game G^0 is also regular and, consequently, finitely generated in view of (8).

Based on Lemma 1, we obtain the following result.

Corollary 3. *For any finitely generated game G satisfying the condition $E(G,N) \neq \varnothing$ the following equality holds*: $C(G) = C(G^0)$.

Proof. By the definition of the set $\widehat{G}(N)$, we have

$$\widehat{G}(N) = \bigcup_{k \in E(G,N)} \{v \in \mathbb{R}^N \mid v \leqslant u^{e^N,k}\}.$$

Since $E(G,N) \neq \varnothing$, the set $\widehat{G}(N)$ is nonempty and, consequently, the game G is regular and saturated from below (the last property immediately follows from the definition of a finitely generated game). By Lemma 1, we obtain the required equality $C(G) = C(G^0)$. □

Corollary 4. *If a finitely generated game G is F-balanced, then G^0 is also a finitely generated F-balanced game.*

Proof. If a generalized finitely generated game G is F-balanced, then the F-balancedness of the cover $\{e^1, \ldots, e^n\}$ implies that the vector u^G belongs to $\widehat{G}(N)$. Consequently, the set $\widehat{G}(N)$ is nonempty, which (together with the relation (9) and Definition 11) shows that all the assumptions of Lemma 1 are satisfied. Applying Lemma 1 and taking into account that the game G is F-balanced, we see that the game G^0 is F-balanced. To complete the proof of Corollary 4, it remains to note that the game G^0 is also finitely generated in view of (8). □

Remark 8. In the general case, where the game G is neither finitely generated nor saturated from below, the nonemptiness of the sets $G(e^i)$, $i \in N$, and closeness of the set $G(N)$ imply that the F-balanced game G has an individually rational payoff vector by the large coalition N, i.e., the set $\widehat{G}(N)$ is nonempty. Indeed, since the one-element coalitions $\{e^1, \ldots, e^n\}$ form an F-balanced cover of N, from the definition of the vector u^G and F-balancedness of the game G it follows that for any natural number m there exists a vector $u^m = (u_1^m, \ldots, u_n^m)$ such that $u^m \in G(N) \cap \prod_{i \in N} G(e^i)$ and $\|u^m - u^G\|_\infty < 1/m$. Since the set $G(N)$ is closed, we conclude that $u^G = \lim u^m \in G(N)$. Thus, u^G belongs to the set $G(N)$, which implies the required relation $\widehat{G}(N) \neq \varnothing$.

The proof of Theorem 1 is based on the following analogue of the classical result proved for usual cooperative games in [7] (cf. also [11]).

Proposition 2. *If a finitely generated game G is F-balanced, then $C(G) \neq \varnothing$.*

This generalization is justified, as in the classical case, by using the known combinatorial Scarf lemma [7]. We recall the definitions of two types of basis sets participated in the formulation of this lemma. Let $A = [a_i^j]$ and $C = [c_i^j]$ be arbitrary $(n \times m)$-matrices, where $m > n \geqslant 2$. We set $M = \{1, \ldots, m\}$ and denote by a^j and c^j, $j \in M$, the columns of the matrices A and C respectively. Let b be a nonnegative vector in $\mathbb{R}^n$. We recall one of the main notions of linear programming.

Definition 12. A set $B \subseteq M$ is called an *admissible basis set* of the system of linear equations $Ax = b$ if the vectors $\{a^j\}_{j\in B}$ form a basis for the space $\mathbb{R}^n$ and all the coefficients in the expansion of b in the basis $\{a^j\}_{j\in B}$ are nonnegative.

Throughout the paper, for any set $J \subseteq M$ we denote by C^J a submatrix $[c^j]_{j\in J}$ of the matrix C (the columns are ordered in ascending numbers in J) and by π^J the vector in $\mathbb{R}^n$ which is the row-by-row minimum of the matrices C^J:

$$\pi_i^J = \min\{c_i^j \mid j \in J\}, \quad i = 1, \dots, n.$$

We recall that for the vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ we use the abbreviation $x \gg y \Longleftrightarrow x_i > y_i$, $i = 1, \dots, n$. Finally, $|P|$ denotes the number of elements of a finite set P.

Definition 13. The set $J \subseteq M$ is called an ordinal basis set [2)] of a matrix C if $|J| = n$ and there are no elements $j \in M$ such that $c^j \gg \pi^J$.

To formulate the Scarf lemma, we also need two definitions concerning the form of the matrices A and C. We set

$$M_1 = \{1, \dots, n\}, \quad M_2 = M \setminus M_1 = \{n+1, \dots, m\}.$$

Definition 14 (standard form of the matrix A)**.** We say that a matrix A has the *standard form* if the first n columns of A form the standard basis for the space $\mathbb{R}^n$: $a^j = e^j$ for every $j \in M_1$ (here, e^j is the jth unit vector in the space $\mathbb{R}^n$).

Definition 15 (standard form of the matrix C)**.** We say that a matrix C has the standard (ordinal) form if $c_i^i = \min\limits_{j\in M} c_i^j$ for all $i \in M_1$; moreover, $c_i^j \geqslant \max\limits_{k\in M_2} c_i^k$ for all $i, j \in M_1$ such that $i \neq j$.

It is easy to see that Definition 15 can be formulated in terms of submatrices C^{M_1} and C^{M_2} as follows: a matrix C has the standard form if the diagonal entries of the submatrix C^{M_1} are row-by-row minima of C and each entry of the submatrix C^{M_1} lying outside the diagonal is not less than any entry located in the same row of the submatrix C^{M_2}.

Lemma (Scarf [7])**.** *Assume that matrices A and C have standard form, a vector b belongs to $\mathbb{R}_+^n$, and the set of nonnegative solutions to the system $Ax = b$ is bounded from above. Then there exists an admissible basis of this system which is simultaneously an ordinal basis of the matrix C.*

Using the Scarf lemma, we can prove Proposition 2.

Proof of Proposition 2. Let G be an arbitrary F-balanced finitely generated cooperative game. Since a finitely generated game is regular by definition, the sets $G(e^i)$ are nonempty for all $i \in N$. Since the game G is F-balanced and the family $\{e^1, \dots, e^n\}$ is an F-balanced cover of the coalition N, we find that the vector u^G belongs to the set $G(N)$, which implies $\widehat{G}(N) \neq \varnothing$. Thus, by Corollaries 3 and 4, in order to prove the existence of an unblocked imputation of the game G, we can assume without loss of generality (passing to the game G^0, if necessary) that the vertices of all sets $G(\tau)$ satisfy the individual rationality condition

$$u^{\tau,k} \geqslant u^G_{N(\tau)}, \quad \tau \in e(G), \quad k \in K(\tau). \tag{10}$$

[2)] A *primitive set* in the terminology of [7].

In particular, the set $G(e^i)$ corresponding to one-element coalitions has one vertex ($k_{e^i} = 1$); moreover, it is obvious that $u_i^{e^i,1} = u_i^G$, $i \in N$.

We construct the matrices A_G and C_G associated with G such that these matrices together with the vector $b = e^N$ satisfy all the assumptions of the Scarf lemma. For this purpose we apply (mutatis mutandis) the approach used in [7] (cf. [11]) for constructing similar matrices for standard finitely generated games. We first introduce the columns $a^{(\tau,k)}$ of the matrices A_G by setting

$$a^{(\tau,k)} = \tau, \quad k \in K(\tau), \tag{11}$$

for each coalition $\tau \in e(G)$. Further, to construct columns of the matrix C_G, we use vertices $u^{\tau,k} \in \mathbb{R}^{N(\tau)}$ extended in a suitable way to vectors in the space $\mathbb{R}^N$. Defining the required extensions $c^{(\tau,k)}$, we fix numbers d_i, $i \in N$, such that

$$d_i > \max_{k \in K(\tau)} u_i^{\tau,k} \quad \forall \tau \in e_i(G), \tag{12}$$

where $e_i(G) = \{\tau \in e(G) \mid i \in N(\tau)\}$, $i \in N$, and set

$$c_i^{(\tau,k)} = \begin{cases} u_i^{\tau,k}, & i \in N(\tau), \\ d_i, & i \in N \setminus N(\tau). \end{cases} \tag{13}$$

Thus, the components $c_i^{(\tau,k)}$ of the column $c^{(\tau,k)}$ corresponding to player i which indeed participates in the coalition τ, coincide with the components $u_i^{\tau,k}$ of the vertices $u^{\tau,k}$ of the set $G(\tau)$, whereas the remaining components $c_j^{(\tau,k)}$ coincide with the corresponding numbers $d_j, j \in N \setminus N(\tau)$. Furthermore, by the condition (12) for any player $i{\in}N$ and any coalition τ such that $\tau_i{>}0$ the number d_i exceeds the payoff of this player in any payoff vectors reachable by efforts of this coalition, which directly follows from the structure of the game G:

$$G(\tau) = \bigcup_{k \in K(\tau)} \{u \in \mathbb{R}^{N(\tau)} \mid u \leqslant u^{\tau,k}\}.$$

We fix an ordering of pairs (τ, k), $\tau \in e(G)$, $k \in K(\tau)$, such that the first n pairs form the sequence $(e^1, 1), \ldots, (e^n, 1)$. We compose the $(n \times m)$-matrices A_G and C_G, where $m = \sum\limits_{\tau \in e(G)} k_\tau$, of the constructed columns $a^{(\tau,k)}$ and $c^{(\tau,k)}$ respectively by ordering them in accordance to the above order of pairs (τ, k).

We show that the matrices $A_G = [a_i^{(\tau,k)}]$ and $C_G = [c_i^{(\tau,k)}]$ satisfy all the assumptions of the Scarf lemma. It is clear that, by the ordering of pairs (τ, k), the first n columns of the matrix A_G form the identity matrix: $a^{e^i,1} = e^i$, $i \in N$, in view of (11). Consequently, the matrix A_G has the standard form. By (10), (12), and the above-mentioned relations $u_i^{e^i,1} = u_i^G$, $i \in N$, the columns $c^{(e^i,1)}$ of the matrix C_G corresponding to the pairs $(e^i, 1)$ satisfy the condition $c_i^{e^i,1} = u_i^G \leqslant c_i^{(\tau,k)}$ for every $i \in N$, $\tau \in e(G)$, $k \in K(\tau)$. Furthermore, by formula (13), all entries of the matrix $[c^{(e^1,1)} \ldots c^{(e^n,1)}]$ located outside the diagonal in the ith row are equal to d_i. By (12), this fact means that the entries $c_j^{(e^i,1)}$, $j \in N \setminus i$, lying outside the diagonal are the largest ones in the ith row of the matrix C_G. Consequently, the matrix C_G is also written in the standard form. To complete the verification that the Scarf lemma can be applied to A_G, C_G, and e^N, it remains to note that from (11) it follows that for any nonnegative solution $(x_{(\tau,k)})$ to the system

$$\sum x_{(\tau,k)} a^{(\tau,k)} = e^N$$

we have

$$x_{(\tau,k)} \leqslant 1/\tau_i \quad \forall\, i \in N(\tau), \tau \in e(G), \quad k \in K(\tau).$$

Thus, $x_{(\tau,k)} \leqslant \max_{i\in N(\tau)} 1/\tau_i$ for all $\tau \in e(G)$, $k \in K(\tau)$ and for every nonnegative solution $x = (x_{(\tau,k)})$ to the system $A_G x = e^N$, which means the upper boundedness of the set of such solutions. Summarizing, we conclude that the matrices A_G, C_G and vector $b = e^N$ satisfy all the assumptions of the Scarf lemma.

Applying the Scarf lemma to the constructed matrices and vector e^N, we conclude that there exists an admissible basis set $B = \{(\tau^j, k_j)\}_{j=1}^n$ of the system $A_G x = e^N$ which is simultaneously the ordinal basis of the matrix C_G. By the definition of an admissible basis, there exist nonnegative numbers x_j, $j = 1, \ldots, n$, such that

$$\sum_{j=1}^{n} x_j a^{(\tau^j,k_j)} = e^N$$

for some $\tau^j \in e(G)$ and $k_j \in K(\tau^j)$, $j = 1, \ldots, n$. Because of $a^{(\tau^j,k_j)} = \tau^j$, $j = 1, \ldots, n$, and (11), the family of coalitions $a^{(\tau^1,k_1)} = \tau^1, \ldots, a^{(\tau^n,k_n)} = \tau^n$ is an F-balanced cover of the set N. Using this fact, we show that the row-by-row minimum π^B of the matrix $C_G^B = [c^{(\tau^1,k_1)} \ldots c^{(\tau^n,k_n)}]$ is the payoff vector of the coalition N. Indeed, from the relations

$$\pi_i^B = \min_{j=1,\ldots,n} c_i^{(\tau^j,k_j)}, \quad i \in N,$$

which are valid by the definition of the row-by-row minimum and formula (13), we obtain the inequalities

$$\pi^B_{N(\tau^j)} \leqslant u^{\tau^j,k_j}, \quad j = 1, \ldots, n. \tag{14}$$

By the definition of the game G, the relations (14) mean that the restrictions $\pi^B_{N(\tau^1)}, \ldots, \pi^B_{N(\tau^n)}$ of the vector π^B onto the set $N(\tau^1), \ldots, N(\tau^n)$ belong to $G(\tau^1), \ldots, G(\tau^n)$ respectively. Consequently, by the F-balancedness of the cover $\{\tau^j\}_{j=1}^n$, the vector π^B is G-balanced. By the F-balancedness of the game G, we obtain the required inclusion $\pi^B \in G(N)$.

To complete the proof of Proposition 2, it remains to verify that the vector π^B is not blocked by any coalition in $e(G)$. Assume the contrary, i.e., there exist $\tau \in e(G)$ and $v = (v_i)_{i\in N(\tau)} \in G(\tau)$ such that

$$v_i > \pi_i^B, \quad i \in N(\tau). \tag{15}$$

Since

$$G(\tau) = \bigcup_{k\in K(\tau)} \left\{u \in \mathbb{R}^{N(\tau)} \;\middle|\; u \leqslant u^{\tau,k}\right\}$$

by construction, for a vector v in $G(\tau)$ there exists a vertex $u^{\tau,k}$ such that $v \leqslant u^{\tau,k}$. But this inequality together with (15) means that $u^{\tau,k} \gg \pi^B_{N(\tau)}$. By (13), we find $c_i^{(\tau,k)} = u_i^{\tau,k} > \pi_i^B$ for all $i \in N(\tau)$. We show that the remaining components of the column $c^{(\tau,k)}$ of the matrix C_G are strictly larger than the corresponding components of the vector π^B. For this purpose we note that $\pi^B \in G(N)$ and the structure of $G(N)$ imply the existence of a vertex $u^{e^N,l} \in G(N)$ such that $\pi^B \leqslant u^{e^N,l}$. Therefore, taking into account the relations $d_i > u_i^{e^N,l}$, $i \in N$, following from (12), we arrive at the required assertion $c_j^{(\tau,k)} = d_j > u_j^{e^N,l} \geqslant \pi_j^B$, $j \in N \setminus N(\tau)$. Thus, all the components of the column $c^{(\tau,k)}$ of the matrix C_G are strictly larger than the corresponding components of the row-by-row minimum π^B of the matrix C_G^B. We arrive at a contradiction with the fact that B is the ordinal basis of the matrix C_G, which completes the proof of Proposition 2. □

4 Unblocked Imputations of Regular Games. General Case

We generalize Proposition 2 to the case where the game G is not necessarily finitely generated (but one of the main assumptions of Proposition 2 is preserved; namely, the set $e(G)$ is assumed to be finite). We recall that a game G is F-finite if its efficiency set $e(G)$ is finite.

Proposition 3. *If a regular saturated from below game G is F-finite and F-balanced, all sets $G(\tau)$ are closed, and $\widehat{G}(N)$ is bounded from above, then $C(G) \neq \varnothing$.*

Proof. Since the set $G(N)$ is closed and the game G is regular and F-balanced, from Remark 8 it follows that the set $\widehat{G}(N)$ is nonempty. Since the game G is regular, saturated from below, and F-balanced, from Lemma 1 it follows that $C(G) = C(G^0)$ and the game G^0 satisfies all the assumptions of Proposition 3 (as was already mentioned, the closeness of $G^0(N)$ follows from the compactness of the set $\widehat{G}(N)$ and closeness of the cone $-\mathbb{R}^N_+$). Passing to the proof of the nonemptiness of the core $C(G^0)$, if necessary, we can assume without loss of generality that all sets $\widehat{G}(\tau)$, $\tau \in e(G)$, are nonempty.

We fix a countable everywhere dense set $Q = \{q_1, q_2, \ldots\}$ in $\mathbb{R}$ and, slightly generalizing the constructions used for classical cooperative games in [15], introduce generalized games G_m, $m \geqslant 1$, such that the sets $G_m(\tau)$ are generated by those vectors in $\widehat{G}(\tau)$ whose components belong to the set $Q_m = \{u_1^G, \ldots, u_n^G\} \cup \{q_1, \ldots, q_m\}$. Namely, for every $m \geqslant 1$ we set

$$G_m(\tau) = \begin{cases} \widetilde{G}_m(\tau) - \mathbb{R}_+^{N(\tau)}, & \tau \in e(G), \\ \varnothing, & \tau \in \sigma_F \setminus e(G), \end{cases} \tag{16}$$

where $\widetilde{G}_m(\tau) = \{(v_i)_{i \in N(\tau)} \in \widehat{G}(\tau) \mid v_i \in Q_m, i \in N(\tau)\}$, $\tau \in e(G)$. If the game G is saturated from above and the sets $\widehat{G}(\tau)$, $\tau \in e(G)$, are nonempty, then it is clear that $\widetilde{G}_m(\tau) \neq \varnothing$ for all $m \geqslant 1$ and $\tau \in e(G)$ (in particular, $u^G_{N(\tau)} \in \widetilde{G}_m(\tau)$ for all $m \geqslant 1$ and $\tau \in e(G)$). Therefore, the games G_m are well defined.

We fix $m \geqslant 1$ and verify that the finitely generated game G_m is F-balanced. Let $v = (v_1, \ldots, v_n)$ be a G_m-balanced vector corresponding to the F-balanced covering $\mathscr{T} = \{\tau^1, \ldots, \tau^k\} \subseteq e(G)$ of the set N. By the definition of G_m-balancedness, there exist vectors $u^s \in \widetilde{G}_m(\tau^s)$, $s = 1, \ldots, k$, such that $v_{N(\tau^s)} \leqslant u^s$ for all $s = 1, \ldots, k$. We denote by $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ the vector defined by the formula

$$\overline{u}_i = \min_{s \in K_i} u_i^s, \quad i \in N, \tag{17}$$

where $K_i = \{s \in \{1, \ldots, k\} \mid i \in N(\tau^s)\}$, $i \in N$. It is clear that $\overline{u}_{N(\tau^s)} \leqslant u^s$ for all $s = 1, \ldots, k$. Hence, by the relations $u^s \in \widetilde{G}_m(\tau^s) \subseteq G(\tau^s)$, $s = 1, \ldots, k$, and F-balancedness of the game G, the vector $\overline{u}$ belongs to the set $G(N)$. On the other hand, by the inclusions $u^s \in \widetilde{G}_m(\tau^s) \subseteq \widehat{G}(\tau^s)$ and formula (17), the vector $\overline{u}$ belongs to the set $\widehat{G}(N)$ and all its components belong to the set Q_m. Consequently, $\overline{u} \in \widetilde{G}_m(N)$. To complete the proof of the inclusion $v \in G_m(N)$, it remains to recall that $v_{N(\tau^s)} \leqslant u^s$ for all $s = 1, \ldots, k$, and, consequently (in view of (17)), we have $v \leqslant \overline{u}$. Taking into account that $\overline{u} \in \widetilde{G}_m(N)$, we obtain the required relation $v \in G_m(N)$.

Thus, all finitely generated games G_m defined by formula (16) are F-balanced. By Proposition 2, $C(G_m) \neq \varnothing$ for all $m = 1, \ldots$ We choose imputations $v^m \in C(G_m)$, $m \geqslant 1$. Since these imputations are not unblocked by one-element coalitions, we have $\{v^m\}_{m=1}^{\infty} \subseteq \widehat{G}(N)$. Therefore, by the compactness of $\widehat{G}(N)$, we can assume that the sequence $\{v^m\}_{m=1}^{\infty}$ converges to some vector $v^0 \in \widehat{G}(N)$. To complete the proof of Proposition 3, it remains to verify that v^0 belongs to the core $C(G)$. Assuming the contrary. we find that for some fuzzy coalition $\tau \in e(G)$ there exists a vector $u \in G(\tau)$ such that $u \gg v^0_{N(\tau)}$. Since the set $G(\tau)$ is saturated from below and Q is everywhere dense in $\mathbb{R}$, there is a vector $\overline{v} \in G(\tau)$ such that $u \gg \overline{v} \gg v^0_{N(\tau)}$; moreover, $\overline{v}$ belongs to $\widetilde{G}_m(\tau)$ for all $m > m_1$. Hence the vector $\overline{v}$ belongs to $G_m(\tau)$ for all $m > m_1$. Since $v^0 = \lim v^m$, for sufficiently large $m_2 > m_1$ we have the inequality $\overline{v}_i > v_i^m$ for all $i \in N(\tau)$ and $m \geqslant m_2$. However, this inequality, together with the above-mentioned inclusions $\overline{v} \in G_m(\tau)$, $m > m_1$, contradicts the fact that for every $m \geqslant 1$ the imputation v^m belongs to the core $C(G_m)$. Thus, we have proved that $v^0 \in C(G)$ which means that the core $C(G)$ is nonempty. Proposition 3 is proved. □

Proof of Theorem 1. The case where the efficiency set of a game is finite is covered by Proposition 3. Therefore, it suffices to consider only the case where the set $e(G)$ is infinite. Denote by $\mathscr{F}$ the family of all finite subsets of the efficiency set $e(G)$ of the game G containing the coalition e^N and all one-element coalitions $\{e^i\}$:

$$\mathscr{F} = \big\{ \mathscr{T} \subseteq e(G) \;\big|\; \{e^1, \ldots, e^n, e^N\} \subseteq \mathscr{T}, \quad |\mathscr{T}| < \infty \big\}.$$

Further, for every $\mathscr{T} \in \mathscr{F}$ we denote by $G^{\mathscr{T}}$ the restriction of a game G on the set $\mathscr{T}$:

$$G^{\mathscr{T}}(\tau) = \begin{cases} G(\tau), & \tau \in \mathscr{T}, \\ \varnothing, & \tau \in \sigma_F \setminus \mathscr{T}. \end{cases}$$

From the definition of the core of a cooperative game and the construction of the games $G^{\mathscr{T}}$ we immediately find

$$C(G) = \bigcap_{\mathscr{T} \in \mathscr{F}} C(G^{\mathscr{T}}). \tag{18}$$

Therefore, to prove that the core $C(G)$ is nonempty, it suffices to establish two facts concerning the cores of the games $G^{\mathscr{T}}$:

1) $C(G^{\mathscr{T}}) \neq \varnothing$ for all $\mathscr{T} \in \mathscr{F}$,

2) the cores $\{C(G^{\mathscr{T}})\}_{\mathscr{T} \in \mathscr{F}}$ form the centered family [3)] of compact subsets of the space $\mathbb{R}^n$ (and, consequently, has a nonempty general part).

Indeed, from the above facts, formula (18), and the known compactness criterion expressed in terms of open covers it follows that

$$C(G) = \bigcap_{\mathscr{T} \in \mathscr{F}} C(G^{\mathscr{T}}) \neq \varnothing,$$

[3)] As usual, we say that a family of sets $\{C(G^{\mathscr{T}})\}_{\mathscr{T} \in \mathscr{F}}$ is *centered* if every finite subfamily has nonempty common part, i.e., $\bigcap_{k=1}^{m} C(G^{\mathscr{T}_k}) \neq \varnothing$ for any finite subfamily $\{C(G^{\mathscr{T}_1}), \ldots, C(G^{\mathscr{T}_m})\}$ of the family $\{C(G^{\mathscr{T}})\}_{\mathscr{T} \in \mathscr{F}}$.

which is required. To complete the proof, it remains to recall that the cores $C(G^{\mathscr{T}})$ are closed subsets of the compact set $\widehat{G}(N)$ according to Remark 3. Hence the cores are also compact (the set $\widehat{G}(N)$ is compact under the assumptions of Theorem 1 because it is bounded and closed which follows from the closeness of the set $G(N)$). Furthermore, it is easy to see that the family of cores $\{C(G^{\mathscr{T}})\}_{\mathscr{T}\in\mathscr{F}}$ is centered in view of Proposition 3 which is valid under the assumptions of Theorem 1 related to the games $G^{\mathscr{T}}$, $\mathscr{T}\in\mathscr{F}$, and the obvious formula

$$\bigcap_{k=1}^{m} C(G^{\mathscr{T}_k}) = C(G^{\bigcup_{k=1}^{m}\mathscr{T}_k}),$$

where $\{\mathscr{T}_1,\ldots,\mathscr{T}_m\}$ is an arbitrary finite family in $\mathscr{F}$. □

Proof of Theorem 2. Since $G(N)$ is closed, we have $\widehat{\overline{G}}(N) = \widehat{G}(N)$. By Lemma 3, if the closure of the game G satisfies the assumptions of Theorem 2, then it also satisfies all the assumptions of Theorem 1 (including the upper boundedness of the set $\widehat{\overline{G}}(N)$). Hence the core $C(\overline{G})$ is nonempty by Theorem 1. Using again the closeness of the set $G(N)$ and Lemma 4, we obtain the equality $C(G) = C(\overline{G})$ which implies the required relation $C(G)\neq\varnothing$. □

Acknowledgments

The work is supported by the Russian Foundation for Basic Research (project No. 16-06-00101).

References

1. V. A. Vasil'ev, *An Extension of Scarf Theorem on the Core of an n-person Game* [in Russian], Preprint, No. 283, IM SO RAN USSR, Novosibirsk (2012).
2. H. Scarf, "The core of an n-person game," *Econometrica* **35**, No. 1, 50–69 (1967).
3. Ch. D. Aliprantis, D. J. Brown, O. Burkinshaw, *Existence and Optimality of Competitive Equilibria*, Springer, Berlin etc. (1995).
4. V. A. Vasil'ev, "A fuzzy-core extension of Scarf theorem and related topics," In: *Contributions to Game Theory and Management*, Vol. VIII, pp. 300–314, St.Petersburg State Univ., St. Petersburg (2015).
5. O. N. Bondareva, "Core theory for n-player game" [in Russian], *Vestn. Leningr. Univ.* **13**, No. 3, 141–142 (1962).
6. J. Rosenmüller, *The Theory of Games and Markets*, North-Holland, Amsterdam etc. (1974).
7. T. Hansen and H. Scarf, *On the Applications of a Recent Combinatorial Algorithm*, Cowles Foundation Discussion Paper No. 272 (1969).
8. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, NJ (1953).
9. I. Ekeland, *Éléments d'Économie Mathématique*, Hermann, Paris (1979).

10. V. A. Vasil'ev, "On Edgeworth equilibria for some types of nonclassic markets," *Sib. Adv. Math.* **6**, No. 3, 96–150 (1996).

11. V. A. Vasil'ev, *Mathematical Models of General Economic Equilibrium* [in Russian], Novosibirsk State Univ. Press, Novosibirsk (2009).

12. V. A. Vasil'ev and V. I. Suslov, "On Edgeworth equilibrium in a model of interregional economic relationships" [in Russian], *Sib. Zh. Ind. Mat.* **13**, No. 1, 18–33 (2010).

13. J.-P. Aubin, *Optima and Equilibria. An Introduction to Nonlinear Analysis*, Springer, Berlin etc. (1993).

14. S. L. Pechersky and E. B. Yanovskaya, *Cooperative Games: Solutions and Axioms* [in Russian], SPb-Europ. Univ. Press, St.-Petersb. (2004).

15. L. Billera, "Some theorems on the core of an n-person game without side payments," *SIAM J. Appl. Math.* **18**, No. 3, 567–579 (1970).

Submitted on March 24, 2017