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THE RAY METHOD AND IDENTIFICATION PROBLEMS FOR EQUATIONS OF THE THEORY OF ELASTICITY

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Using algebraic-analytic methods, we establish numerous connections, in finite and infinite variants, between amplitudes, coefficients, and source functions of dynamical systems of elasticity theory. Bibliography: 18 *titles.*

The general identification problem consists in finding an object from its features and is connected with pattern recognition and inverse problems. In identification problems for dynamical systems of ordinary differential equations, it is preferable to have explicit formulas for solutions, which should be specified. For multidimensional identification problems it is also desirable to have representations of solutions and coefficients of partial differential equations that contain arbitrary functions of one or many variables [1]–[5]. Applying analytic and constructive methods, it is possible not only to establish the existence of solutions to identification problems, but often to construct or approximate the solution [6]–[10]. Therefore, the range of problems related to the search of new representations of solutions and coefficients of equations in mathematical physics, their reproduction, and construction of multidimensional counterparts of classical algebraic-differential transformations are of great interest in problems of identification.

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The ray method has been known since the 50s of the last century and is a powerful tool for solving wave problems [11]–[14]. We propose a new method for studying identification problems for system of equations of the theory of elasticity. The method is based on the ray decomposition of solutions in the case where the coefficients of the system depend not only on the spatial variable, but also on the time variable, which is of practical interest. Using the algebraic-analytic methods, we establish new numerous connections, in finite and infinite cases, between amplitudes, coefficients, and source functions of the dynamical systems of the theory of elasticity (cf. also [10, 15, 16]).

1 Ray Decomposition

Assume that $w = (w^1, w^2, w^3)$ is a vector-valued function of $x \in D \subset \mathbb{R}^3$, $t > 0$, D is a domain, $\lambda = \lambda(x,t)$ and $\mu = \mu(x,t)$ are the Lamé coefficients, and $\rho = \rho(x,t) > 0$ is the density depending on x and t,

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$

where $\alpha_i = \alpha_i(x,t)$ are some functions.

We consider the system of equations of the theory of elasticity [17, 18]

$$\frac{\partial^2 w}{\partial t^2} = \frac{\mu}{\rho}\Delta w + \frac{\lambda+\mu}{\rho}\operatorname{grad}\operatorname{div} w + \frac{1}{\rho}\operatorname{div} w \operatorname{grad}\lambda + \frac{1}{\rho}Aw \operatorname{grad}\mu + \frac{\alpha}{\rho}\frac{\partial w}{\partial t} + \frac{1}{\rho}R(x,t), \tag{1}$$

where

$$Aw = \begin{pmatrix} 2\dfrac{\partial w^1}{\partial x_1} & \dfrac{\partial w^1}{\partial x_2} + \dfrac{\partial w^2}{\partial x_1} & \dfrac{\partial w^1}{\partial x_3} + \dfrac{\partial w^3}{\partial x_1} \\ \dfrac{\partial w^2}{\partial x_1} + \dfrac{\partial w^1}{\partial x_2} & 2\dfrac{\partial w^2}{\partial x_2} & \dfrac{\partial w^2}{\partial x_3} + \dfrac{\partial w^3}{\partial x_2} \\ \dfrac{\partial w^3}{\partial x_1} + \dfrac{\partial w^1}{\partial x_3} & \dfrac{\partial w^3}{\partial x_2} + \dfrac{\partial w^2}{\partial x_3} & 2\dfrac{\partial w^3}{\partial x_3} \end{pmatrix}.$$

We look for a solution to the system (1) in the form of a ray series

$$w(x,t) = \sum_{k=0}^{\infty} w_k(x,t) f_k(t, \tau(x,t)), \tag{2}$$

assuming that all functions in the decomposition are sufficiently many times differentiable and possess the required properties. The following algebraic theorem holds.

Theorem 1. *Assume that the following conditions hold*:

1) $w_k = (w_k^1, w_k^2, w_k^3)(x,t)$ *are vector-valued functions of the variables* (x,t), $k = 0, 1, 2, \ldots$, *and* $w_{-1} = w_{-2} = \ldots = 0$,
2) $\tau(x,t)$ *is a function of the variables* (x,t),
3) $f_k(t,y)$ *are arbitrary functions of the variables* (t,y), $y \in \mathbb{R}$, *connected by the identities*

$$\frac{\partial f_k}{\partial y} = f_{k-1}(t,y), \quad k \in \mathbb{Z}, \tag{3}$$

4) *the functions* $S_k^i(x,t)$, $P_k^i(x,t)$, $Q_k^i(x,t)$, $k=0,1,2,\dots$, $i=1,2,3$, *are defined by*

$$S_k^i = \frac{\partial^2 w_k^i}{\partial t^2} - \frac{\mu}{\rho}\Delta w_k^i - \frac{\lambda+\mu}{\rho}\frac{\partial}{\partial x_i}(\operatorname{div} w_k)$$
$$- \frac{1}{\rho}\Big[\operatorname{div} w_k \frac{\partial \lambda}{\partial x_i} + \big(\operatorname{grad} w_k^i, \operatorname{grad}\mu\big) + \Big(\frac{\partial w_k}{\partial x_i}, \operatorname{grad}\mu\Big) + \alpha_i \frac{\partial w_k^i}{\partial t}\Big], \tag{4}$$

$$P_k^i = 2\frac{\partial w_k^i}{\partial t}\frac{\partial \tau}{\partial t} + w_k^i \frac{\partial^2 \tau}{\partial t^2} - \frac{\mu}{\rho}\Big[2\big(\operatorname{grad} w_k^i, \operatorname{grad}\tau\big) + w_k^i \Delta\tau\Big]$$
$$- \frac{\lambda+\mu}{\rho}\Big[\operatorname{div} w_k \frac{\partial\tau}{\partial x_i} + \Big(\frac{\partial w_k}{\partial x_i}, \operatorname{grad}\tau\Big) + \Big(w_k, \operatorname{grad}\frac{\partial\tau}{\partial x_i}\Big)\Big]$$
$$- \frac{1}{\rho}\Big[(w_k, \operatorname{grad}\tau)\frac{\partial\lambda}{\partial x_i} + w_k^i(\operatorname{grad}\tau, \operatorname{grad}\mu) + \frac{\partial\tau}{\partial x_i}(w_k, \operatorname{grad}\mu) + \alpha_i w_k^i \frac{\partial\tau}{\partial t}\Big], \tag{5}$$

$$Q_k^i = w_k^i\Big[\Big(\frac{\partial\tau}{\partial t}\Big)^2 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2\Big] - \frac{\lambda+\mu}{\rho}(w_k, \operatorname{grad}\tau)\frac{\partial\tau}{\partial x_i}. \tag{6}$$

If the vector-valued source function $R=(R_1,R_2,R_3)$ *admits the representation*

$$\frac{1}{\rho}R_i(x,t) = \sum_{k=0}^{\infty}\Big[2\frac{\partial w_k^i}{\partial t}\frac{\partial f_k}{\partial t} + 2w_k^i\frac{\partial f_{k-1}}{\partial t}\frac{\partial\tau}{\partial t} + w_k^i\frac{\partial^2 f_k}{\partial t^2} - \frac{\alpha_i}{\rho}w_k^i\frac{\partial f_k}{\partial t}\Big] + \sum_{k=0}^{\infty} B_k^i f_{k-2}, \tag{7}$$

where $i=1,2,3$, *and*

$$B_k^i = Q_k^i + S_{k-2}^i + P_{k-1}^i, \quad k=0,1,\dots, \tag{8}$$

where $S_{-2}=S_{-1}=P_{-1}=0$, $k=0,1,2,\dots$, $i=1,2,3$, *then the vector-valued function* $w(x,t)$ *represented as a formal ray series* (2) *is a solution to the system* (1).

Proof. We find the derivatives of the function w represented in the form (2):

$$\frac{\partial w}{\partial t} = \sum_{k=0}^{\infty}\Big[\frac{\partial w_k}{\partial t}f_k + w_k\frac{\partial f_k}{\partial t} + w_k f_{k-1}\frac{\partial\tau}{\partial t}\Big],$$

$$\frac{\partial^2 w}{\partial t^2} = \sum_{k=0}^{\infty}\Big[\frac{\partial^2 w_k}{\partial t^2}f_k + 2\frac{\partial w_k}{\partial t}\frac{\partial f_k}{\partial t} + 2\frac{\partial w_k}{\partial t}f_{k-1}\frac{\partial\tau}{\partial t} + 2w_k\frac{\partial f_{k-1}}{\partial t}\frac{\partial\tau}{\partial t}$$
$$+ w_k\frac{\partial^2 f_k}{\partial t^2} + w_k f_{k-2}\Big(\frac{\partial\tau}{\partial t}\Big)^2 + w_k f_{k-1}\frac{\partial^2\tau}{\partial t^2}\Big],$$

$$\frac{\partial w}{\partial x_i} = \sum_{k=0}^{\infty}\Big[\frac{\partial w_k}{\partial x_i}f_k + w_k f_{k-1}\frac{\partial\tau}{\partial x_i}\Big],$$

$$\frac{\partial^2 w}{\partial x_i\partial x_j} = \sum_{k=0}^{\infty}\Big[\frac{\partial^2 w_k}{\partial x_i\partial x_j}f_k + \Big(\frac{\partial w_k}{\partial x_i}\frac{\partial\tau}{\partial x_j} + \frac{\partial w_k}{\partial x_j}\frac{\partial\tau}{\partial x_i} + w_k\frac{\partial^2\tau}{\partial x_i\partial x_j}\Big)f_{k-1} + w_k\frac{\partial\tau}{\partial x_i}\frac{\partial\tau}{\partial x_j}f_{k-2}\Big].$$

We substitute the found derivatives into the system (1). Using the notation (4)–(6) and representation (7), we obtain the equality

$$\sum_{k=0}^{\infty}S_k^i f_k + \sum_{k=0}^{\infty}P_k^i f_{k-1} + \sum_{k=0}^{\infty}Q_k^i f_{k-2} - \sum_{k=0}^{\infty}B_k^i f_{k-2} = 0. \tag{9}$$

Since $w_{-1} = 0$, we have $S_{-1} = 0$ and

$$\sum_{k=0}^{\infty} S_k^i f_k = \sum_{k=0}^{\infty} S_{k-1}^i f_{k-1}.$$

Therefore, the equality (9) can be written as

$$\sum_{k=0}^{\infty} \left(S_{k-1}^i + P_k^i\right) f_{k-1} + \sum_{k=0}^{\infty} \left(Q_k^i - B_k^i\right) f_{k-2} = 0.$$

Since f_k are arbitrary, we obtain (8). The theorem is proved. $\square$

If $w_k = w_k(x)$, $f_k = f_k(t - \tau(x))$, the Lamé coefficients, density, and entries of the absorption matrix α depend only on the variables x, then we obtain the classical ray decomposition.

Corollary 1. *Let the Lamé coefficients, density, and entries of the matrix* α *depend only on the variables* x, $\lambda = \lambda(x)$, $\mu = \mu(x)$, $\rho = \rho(x)$, $\alpha_i = \alpha_i(x)$, $i = 1, 2, 3$. *Assume that*

1) $w_k = (w_k^1, w_k^2, w_k^3)(x)$ *are vector-valued functions of the variables* x, $k = 0, 1, 2, \ldots,$ *and* $w_{-1} = w_{-2} = \ldots = 0$,
2) $\tau(x)$ *is a function of the variables* x,
3) $f_k(y)$ *are arbitrary functions of the variable* $y \in \mathbb{R}$, *connected by*
$$\frac{df_k}{dy} = f_{k-1}(y), \quad k \in \mathbb{Z},$$
4) *the functions* $S_k^i(x)$, $P_k^i(x)$, $Q_k^i(x)$, $k = 0, 1, 2, \ldots$, $i = 1, 2, 3$, *are defined by*

$$S_k^i = -\frac{\mu}{\rho}\Delta w_k^i - \frac{\lambda+\mu}{\rho}\frac{\partial}{\partial x_i}(\operatorname{div} w_k) - \frac{1}{\rho}\Big[\operatorname{div} w_k \frac{\partial \lambda}{\partial x_i} + (\operatorname{grad} w_k^i, \operatorname{grad}\mu) + \Big(\frac{\partial w_k}{\partial x_i}, \operatorname{grad}\mu\Big)\Big],$$

$$\begin{aligned} P_k^i &= \frac{\mu}{\rho}[2(\operatorname{grad} w_k^i, \operatorname{grad}\tau) + w_k^i \Delta\tau] \\ &\quad + \frac{\lambda+\mu}{\rho}\Big[\operatorname{div} w_k \frac{\partial\tau}{\partial x_i} + \Big(\frac{\partial w_k}{\partial x_i}, \operatorname{grad}\tau\Big) + \Big(w_k, \operatorname{grad}\frac{\partial\tau}{\partial x_i}\Big)\Big] \\ &\quad + \frac{1}{\rho}\Big[(w_k, \operatorname{grad}\tau)\frac{\partial\lambda}{\partial x_i} + w_k^i(\operatorname{grad}\tau, \operatorname{grad}\mu) + \frac{\partial\tau}{\partial x_i}(w_k, \operatorname{grad}\mu) - \alpha_i w_k^i\Big], \end{aligned}$$

$$Q_k^i = w_k^i\Big[1 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2\Big] - \frac{\lambda+\mu}{\rho}(w_k, \operatorname{grad}\tau)\frac{\partial\tau}{\partial x_i}.$$

If the vector-valued source function $R = (R_1, R_2, R_3)$ *admits the representation*

$$\frac{1}{\rho}R_i(x,t) = \sum_{k=0}^{\infty} B_k^i f_{k-2}(t - \tau(x)), \quad i = 1, 2, 3,$$

where $B_k^i = Q_k^i + S_{k-2}^i + P_{k-1}^i$, $k = 0, 1, \ldots$, $S_{-2} = S_{-1} = P_{-1} = 0$, $k = 0, 1, 2, \ldots$, $i = 1, 2, 3$, *then the vector-valued function* $w(x,t)$ *represented as a formal ray series*

$$w(x,t) = \sum_{k=0}^{\infty} w_k(x) f_k(t - \tau(x)),$$

is a solution to the system (1).

Remark 1. By Theorem 1, the amplitudes $w_k = (w_k^1, w_k^2, w_k^3)(x,t)$, density $\rho = \rho(x,t)$, Lamé coefficients $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, function $\tau(x,t)$, and absorption $\alpha = (\alpha_1, \alpha_2, \alpha_3)(x,t)$ are connected with the source function $R = (R_1, R_2, R_3)(x,t)$ by the recurrent relations (8) for given $f_k(t,y)$. This fact is fundamental in the theory of identifications problems for equations of elasticity and will be used below. We will see that the ray decomposition (2) yields different type waves, in particular, longitudinal, transverse, and surface ones.

2 Systems of Linear Equations Connected with Equations of Elasticity

In this section, we consider the solvability conditions and general solutions to systems of linear algebraic equations connected with the algebraic expression Q_k and ray decompositions.

Theorem 2. *Let $Z = (z_1, z_2, z_3)$ be a set of variables. The system of linear equations*

$$Z\Big[\frac{\mu}{\rho}|\operatorname{grad}\tau|^2 - \Big(\frac{\partial\tau}{\partial t}\Big)^2\Big] + \frac{\lambda+\mu}{\rho}(Z, \operatorname{grad}\tau)\operatorname{grad}\tau = 0 \tag{10}$$

for Z has a nontrivial solution if and only if one of the following identities holds:

$$\frac{\mu}{\rho}|\operatorname{grad}\tau|^2 = \Big(\frac{\partial\tau}{\partial t}\Big)^2, \tag{11}$$

$$\frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2 = \Big(\frac{\partial\tau}{\partial t}\Big)^2. \tag{12}$$

Proof. The linear equations (10) for components of the vector Z are homogeneous. Consequently, the determinant of the matrix

$$P = \begin{pmatrix} \theta + \frac{\lambda+\mu}{\rho}\Big(\frac{\partial\tau}{\partial x_1}\Big)^2 & \frac{\lambda+\mu}{\rho}\frac{\partial\tau}{\partial x_1}\frac{\partial\tau}{\partial x_2} & \frac{\lambda+\mu}{\rho}\frac{\partial\tau}{\partial x_1}\frac{\partial\tau}{\partial x_3} \\ \frac{\lambda+\mu}{\rho}\frac{\partial\tau}{\partial x_1}\frac{\partial\tau}{\partial x_2} & \theta + \frac{\lambda+\mu}{\rho}\Big(\frac{\partial\tau}{\partial x_2}\Big)^2 & \frac{\lambda+\mu}{\rho}\frac{\partial\tau}{\partial x_2}\frac{\partial\tau}{\partial x_3} \\ \frac{\lambda+\mu}{\rho}\frac{\partial\tau}{\partial x_1}\frac{\partial\tau}{\partial x_3} & \frac{\lambda+\mu}{\rho}\frac{\partial\tau}{\partial x_2}\frac{\partial\tau}{\partial x_3} & \theta + \frac{\lambda+\mu}{\rho}\Big(\frac{\partial\tau}{\partial x_3}\Big)^2 \end{pmatrix} \tag{13}$$

with the variables Z, where

$$\theta = \frac{\mu}{\rho}|\operatorname{grad}\tau|^2 - \Big(\frac{\partial\tau}{\partial t}\Big)^2,$$

vanishes. We have $\det P = \lambda_1\lambda_2\lambda_3$, where λ_1, λ_2, λ_3 are eigenvalues of the matrix P. It is easy to see that $\operatorname{rank}[P - \theta E] = 1$. Since P is a symmetric matrix, we have $\lambda_1 = \lambda_2 = \theta$. Since

$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} P,$$

we find

$$2\theta + \lambda_3 = 3\theta + \frac{\lambda+\mu}{\rho}|\operatorname{grad}\tau|^2.$$

In other words,

$$\lambda_3 = \theta + \frac{\lambda+\mu}{\rho}|\operatorname{grad}\tau|^2 = \frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2 - \Big(\frac{\partial\tau}{\partial t}\Big)^2.$$

Therefore,

$$\det P = \lambda_1\lambda_2\lambda_3 = \Big(\frac{\mu}{\rho}|\operatorname{grad}\tau|^2 - \Big(\frac{\partial\tau}{\partial t}\Big)^2\Big)^2\Big(\frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2 - \Big(\frac{\partial\tau}{\partial t}\Big)^2\Big).$$

Thus, the determinant of the matrix vanishes in the case (11) or (12). The theorem is proved. □

Remark 2. 1. The case (11) corresponds to transverse waves for $Z = w_k$. Moreover, the vector $Z = w_k$ is orthogonal to the vector $\operatorname{grad}\tau$, i.e., $(Z, \operatorname{grad}\tau) = 0$.

2. The case (12) corresponds to longitudinal waves. Moreover, the vector $Z = w_k$ is proportional to the vector $\operatorname{grad}\tau$.

The relations (11) and (12) will be used for removing the density ρ.

Remark 3. Under the assumptions of Corollary 1, the relations (11) and (12) take the form $\rho = \mu|\operatorname{grad}\tau|^2$ and $\rho = (\lambda+2\mu)|\operatorname{grad}\tau|^2$ respectively.

We formulate the general result about the formula for the inverse matrix of a special form.

Lemma 1. *Assume that the following conditions hold:*

1) $\mu, \omega \in \mathbb{C}$ *are complex numbers,*
2) $m = (m_1, \ldots, m_n) \in \mathbb{C}^n$ *is an arbitrary fixed vector,*
3) $m\otimes m$ *is an* $(n\times n)$*-matrix such that* $(m\otimes m)_{ij} = m_i m_j$, $i, j = 1, \ldots, n$, *i.e.,* $m\otimes m = m^T m$ *is the product of column* m^T *and row* m.

If the matrix $M = \mu E + \omega m\otimes m$, *where* E *is the identity matrix of order* n, *is nonsingular, then*

$$M^{-1} = \frac{1}{\mu}E - \frac{\omega m\otimes m}{\mu(\mu+\omega|m|^2)}, \quad |m|^2 = m_1^2 + \ldots + m_n^2.$$

Proof. Eigenvalues of the matrix M are equal to μ (of multiplicity $n-1$) and $\mu+\omega|m|^2$ (of multiplicity 1). Since M is nonsingular, we have $\mu, \mu+\omega|m|^2 \neq 0$. A direct verification of the identity $MM^{-1} = E$ and the relation $(m\otimes m)^2 = |m|^2 m\otimes m$ complete the proof. □

We note that the matrix P can be represented as

$$P = \theta E + \gamma \operatorname{grad}\tau\otimes\operatorname{grad}\tau,$$

where $\gamma = (\lambda+\mu)/\rho$. Therefore if the relations (11) and (12) fail, then the matrix P is invertible and

$$P^{-1} = \frac{1}{\theta}E - \frac{\gamma}{\theta(\theta+\gamma|\operatorname{grad}\tau|^2)}\operatorname{grad}\tau\otimes\operatorname{grad}\tau.$$

Hence the following assertion holds.

Lemma 2. 1. *If the matrix* P *defined by* (13) *is nonsingular, then the solution* $Z = (z_1, z_2, z_3)$ *to the system*

$$Z\Big[\frac{\mu}{\rho}|\operatorname{grad}\tau|^2 - \Big(\frac{\partial\tau}{\partial t}\Big)^2\Big] + \gamma(Z, \operatorname{grad}\tau)\operatorname{grad}\tau = Y, \tag{14}$$

where $Y = (y_1, y_2, y_3)$, *is found by*

$$Z = \frac{Y}{\theta} - \frac{\gamma}{\theta(\theta+\gamma|\operatorname{grad}\tau|^2)}\big(Y, \operatorname{grad}\tau\big)\operatorname{grad}\tau.$$

2. *Let* (11) *hold. Then the system* (14) *has a solution if and only if the right-hand side* Y *is proportional to the vector* $\operatorname{grad}\tau$, *i.e.,* $Y = \gamma\beta \operatorname{grad}\tau$, *where* $\beta = \beta(x,t)$ *is some function. In this case, the solution* Z *is found from the relation*

$$(Z, \operatorname{grad}\tau) = \beta.$$

3. *Let* (12) *hold. Then the system* (14) *has a solution if and only the right-hand side* Y *is orthogonal to the vector* $\operatorname{grad}\tau$, *i.e.,* $(Y, \operatorname{grad}\tau) = 0$. *In this case, the solution* Z *is found by*

$$Z = \nu \operatorname{grad}\tau - \frac{1}{\gamma|\operatorname{grad}\tau|^2} Y,$$

where $\nu = \nu(x,t)$ *is an arbitrary function.*

Corollary 2. 1. *If the matrix* P *defined by* (13) *is nonsingular and the vector* Y *is proportional to the vector* $\operatorname{grad}\tau$, *i.e.,* $Y = \beta \operatorname{grad}\tau$ *for some function* $\beta = \beta(x,t)$, *then the solution* Z *to the system* (14) *has the form*

$$Z = \frac{\beta}{\theta + \gamma|\operatorname{grad}\tau|^2} \operatorname{grad}\tau.$$

2. *If the matrix* P *defined by* (13) *is nonsingular and the vector* Y *is orthogonal to the vector* $\operatorname{grad}\tau$, *i.e.,* $(Y, \operatorname{grad}\tau) = 0$, *then the solution* Z *to the system* (14) *has the form*

$$Z = \frac{Y}{\theta}.$$

Remark 4. By the results of Section 2, the ray decomposition in Theorem 1 yields different type waves (not only longitudinal and transverse ones) depending on the matrix P.

3 Case $w = w_0(x,t) f_0(t, \tau(x,t))$

In the case $w = w_0(x,t) f_0(t, \tau(x,t))$, the system (8) consists of the nine equations

$$Q_0^i = B_0^i, \quad P_0^i = B_1^i, \quad S_0^i = B_2^i, \quad i = 1,2,3. \tag{15}$$

Proposition 1. *Assume that the following conditions hold*:

1) *the matrix* P *defined by* (13) *is nonsingular,*
2) $B_0 = \big(B_0^1, B_0^2, B_0^3\big)(x,t)$ *is orthogonal to the vector* $\operatorname{grad}\tau$, $B_0 = -\theta b$, $(b, \operatorname{grad}\tau) = 0$,
3) $B_1 = \big(B_1^1, B_1^2, B_1^3\big)(x,t)$ *and* $B_2 = \big(B_2^1, B_2^2, B_2^3\big)(x,t)$ *are given vector-valued functions,*
4) *the vector-valued function* $b = (b_1, b_2, b_3)(x,t)$ *and scalar functions* $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$, $\tau = \tau(x,t)$ *satisfy the system of equations* $(i = 1,2,3)$

$$\begin{aligned} &2\frac{\partial b_i}{\partial t}\frac{\partial \tau}{\partial t} + b_i \frac{\partial^2 \tau}{\partial t^2} - \frac{\mu}{\rho}\Big[2(\operatorname{grad} b_i, \operatorname{grad}\tau) + b_i \Delta\tau\Big] - \gamma \frac{\partial \tau}{\partial x_i} \operatorname{div} b \\ &\qquad - \frac{1}{\rho}\Big[b_i(\operatorname{grad}\tau, \operatorname{grad}\mu) + \frac{\partial \tau}{\partial x_i}(b, \operatorname{grad}\mu) + \alpha_i b_i \frac{\partial \tau}{\partial t}\Big] = B_1^i, \end{aligned} \tag{16}$$

$$\begin{aligned} &\frac{\partial^2 b_i}{\partial t^2} - \frac{\mu}{\rho}\Delta b_i - \gamma \frac{\partial(\operatorname{div} b)}{\partial x_i} \\ &\qquad - \frac{1}{\rho}\Big[\operatorname{div} b \frac{\partial \lambda}{\partial x_i} + (\operatorname{grad} b_i, \operatorname{grad}\mu) + \Big(\frac{\partial b}{\partial x_i}, \operatorname{grad}\mu\Big) + \alpha_i \frac{\partial b_i}{\partial t}\Big] = B_2^i. \end{aligned} \tag{17}$$

Then the vector-valued function $w = w_0(x,t) f_0(t, \tau(x,t))$ *with* $w_0(x,t) = b(x,t)$ *is a solution to the system* (1).

Remark 5. Since $(b, \operatorname{grad}\tau) = 0$, the vector-valued function b is determined by two arbitrary scalar functions. For example, if $\dfrac{\partial\tau}{\partial x_1} \neq 0$, then

$$b = \beta_1\Big(\frac{\partial\tau}{\partial x_2}, -\frac{\partial\tau}{\partial x_1}, 0\Big) + \beta_2\Big(\frac{\partial\tau}{\partial x_3}, 0, -\frac{\partial\tau}{\partial x_1}\Big),$$

where $\beta_1 = \beta_1(x,t)$ and $\beta_2 = \beta_2(x,t)$ are scalar functions. Hence the system (16), (17) is well defined since it consists of six equations for the six unknown functions $\lambda(x,t)$, $\mu(x,t)$, $\rho(x,t)$, $\tau(x,t)$, $\beta_1(x,t)$, $\beta_2(x,t)$.

Proposition 2. *Assume that the following conditions hold*:

1) *the matrix* P *defined by* (13) *is nonsingular,*
2) *the vector* $B_0 = (B_0^1, B_0^2, B_0^3)(x,t)$ *is proportional to the vector* $\operatorname{grad}\tau$:

$$B_0 = -(\theta + \gamma|\operatorname{grad}\tau|^2)\beta\operatorname{grad}\tau,$$

3) $B_1 = (B_1^1, B_1^2, B_1^3)(x,t)$ *and* $B_2 = (B_2^1, B_2^2, B_2^3)(x,t)$ *are given vector-valued functions,*
4) *the scalar functions* $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$, $\tau = \tau(x,t)$, $\beta = \beta(x,t)$ *satisfy the system of equations* $(i = 1,2,3)$

$$\begin{aligned} &2\frac{\partial}{\partial t}\Big(\beta\frac{\partial\tau}{\partial x_i}\Big)\frac{\partial\tau}{\partial t} + \beta\frac{\partial\tau}{\partial x_i}\frac{\partial^2\tau}{\partial t^2} - \frac{\mu}{\rho}\Big[2\Big(\operatorname{grad}\Big(\beta\frac{\partial\tau}{\partial x_i}\Big), \operatorname{grad}\tau\Big) + \beta\frac{\partial\tau}{\partial x_i}\Delta\tau\Big] \\ &\quad - \gamma\Big[\operatorname{div}(\beta\operatorname{grad}\tau)\frac{\partial\tau}{\partial x_i} + \frac{\partial}{\partial x_i}\Big(\beta|\operatorname{grad}\tau|^2\Big)\Big] \\ &\quad - \frac{1}{\rho}\Big[\beta|\operatorname{grad}\tau|^2\frac{\partial\lambda}{\partial x_i} + 2\beta\frac{\partial\tau}{\partial x_i}(\operatorname{grad}\tau, \operatorname{grad}\mu) + \alpha_i\beta\frac{\partial\tau}{\partial x_i}\frac{\partial\tau}{\partial t}\Big] = B_1^i, \end{aligned} \tag{18}$$

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\Big(\beta\frac{\partial\tau}{\partial x_i}\Big) - \frac{\mu}{\rho}\Delta\Big(\beta\frac{\partial\tau}{\partial x_i}\Big) - \gamma\frac{\partial}{\partial x_i}\operatorname{div}(\beta\operatorname{grad}\tau) \\ &\quad - \frac{1}{\rho}\Big[\operatorname{div}(\beta\operatorname{grad}\tau)\frac{\partial\lambda}{\partial x_i} + \Big(\operatorname{grad}\Big(\beta\frac{\partial\tau}{\partial x_i}\Big), \operatorname{grad}\mu\Big) \\ &\quad + \Big(\frac{\partial}{\partial x_i}(\beta\operatorname{grad}\tau), \operatorname{grad}\mu\Big) + \alpha_i\frac{\partial}{\partial t}\Big(\beta\frac{\partial\tau}{\partial x_i}\Big)\Big] = B_2^i. \end{aligned} \tag{19}$$

Then the vector-valued function $w = w_0(x,t) f_0(t, \tau(x,t))$ *with* $w_0(x,t) = \beta(x,t)\operatorname{grad}\tau$ *is a solution to the system* (1).

Remark 6. The system (18), (19) is overdetermined since it consists of six equations for the five unknown functions $\lambda(x,t)$, $\mu(x,t)$, $\rho(x,t)$, $\tau(x,t)$, $\beta(x,t)$.

If $\alpha_1(x,t) = \alpha_2(x,t) = \alpha_3(x,t) = \alpha(x,t)$, then the function $\alpha(x,t)$ can be added to the system (18), (19), which makes the system determined.

Based on Propositions 1 and 2, it is possible to construct solutions of general form (2) to the system (1) by using the following lemma.

Lemma 3. *Assume that we are given*

1) *functions* $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$, $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1,2,3$,
2) *vector-valued functions* $B_k = (B_k^1, B_k^2, B_k^3)(x,t)$, $k = 0,1,2,\dots$,
3) *vector-valued functions* $w_k = (w_k^1, w_k^2, w_k^3)$, $k = 0,1,2,\dots,m$, *where* $m \geqslant 0$ *is a fixed integer,*
4) *a nonsingular matrix* P.

Then the vector-valued functions $w_k = (w_k^1, w_k^2, w_k^3)$, $k \geqslant m+1$, *are uniquely found from the system* (8).

Proof. The vector-valued functions w_k, $k \geqslant m+1$, are included in the system (8)

$$B_k = Q_k + S_{k-2} + P_{k-1}, \quad k \geqslant m+1.$$

Since w_m and w_{m-1} are known, P_m, S_{m-1}, and S_m are also known. Therefore, from the relation

$$Q_{m+1} = B_{m+1} - P_m - S_{m-1}$$

we find

$$w_{m+1} = P^{-1}(B_{m+1} - P_m - S_{m-1})$$

by formula (14) in Lemma 2. Consequently, we find P_{m+1}. Therefore, from the relation

$$Q_{m+2} = B_{m+2} - P_{m+1} - S_m$$

we find

$$w_{m+2} = P^{-1}(B_{m+2} - P_{m+1} - S_m).$$

To complete the proof of the lemma, we apply induction on $k \geqslant m+1$. □

The following two assertions deal with a singular matrix P, i.e., (11) or (12) holds.

Proposition 3. *Assume that the following conditions hold:*

1) *the identity* (11) *holds,*
2) *the vector* $B_0 = (B_0^1, B_0^2, B_0^3)(x,t)$ *is proportional to the vector* $\operatorname{grad}\tau$,
$$B_0 = -\gamma \frac{\beta}{|\operatorname{grad}\tau|^2}\operatorname{grad}\tau, \quad \beta = \beta(x,t),$$
3) $B_1 = (B_1^1, B_1^2, B_1^3)(x,t)$ *and* $B_2 = (B_2^1, B_2^2, B_2^3)(x,t)$ *are given vector-valued functions,*
4) *the scalar functions* $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\tau = \tau(x,t)$, $\beta = \beta(x,t)$ *and the vector* $w_0 = (w_0^1, w_0^2, w_0^3)(x,t)$ *satisfy the system of equations* ($i = 1,2,3$)

$$(w_0, \operatorname{grad}\tau) = \beta, \tag{20}$$

$$2\frac{\partial w_0^i}{\partial t}\frac{\partial \tau}{\partial t} + w_0^i \frac{\partial^2 \tau}{\partial t^2} - \frac{\mu}{\rho}[2(\operatorname{grad} w_0^i, \operatorname{grad}\tau) + w_0^i \Delta\tau] - \gamma\Big[\operatorname{div} w_k \frac{\partial \tau}{\partial x_i} + \frac{\partial \beta}{\partial x_i}\Big]$$
$$- \frac{1}{\rho}\Big[\frac{\partial \beta}{\partial x_i}\frac{\partial \lambda}{\partial x_i} + w_0^i(\operatorname{grad}\tau, \operatorname{grad}\mu) + \frac{\partial \tau}{\partial x_i}(w_0, \operatorname{grad}\mu) + \alpha_i w_0^i \frac{\partial \tau}{\partial t}\Big] = B_1^i, \tag{21}$$

$$\frac{\partial^2 w_0^i}{\partial t^2} - \frac{\mu}{\rho}\Delta w_0^i - \gamma\frac{\partial}{\partial x_i}(\operatorname{div} w_0)$$

$$-\frac{1}{\rho}\Big[\operatorname{div} w_0 \frac{\partial \lambda}{\partial x_i} + (\operatorname{grad} w_0^i, \operatorname{grad}\mu) + \Big(\frac{\partial w_0}{\partial x_i}, \operatorname{grad}\mu\Big) + \alpha_i \frac{\partial w_0^i}{\partial t}\Big] = B_2^i, \tag{22}$$

Then the vector-valued function $w = w_0(x,t) f_0(t, \tau(x,t))$ is a solution to the system (1).

Remark 7. If, for example, $\dfrac{\partial \tau}{\partial x_1} \neq 0$, then the equality $(w_0, \operatorname{grad}\tau) = \beta$ can be written in the form

$$w_0 = \beta_1 \left(\frac{\partial \tau}{\partial x_2}, -\frac{\partial \tau}{\partial x_1}, 0\right) + \beta_2 \left(\frac{\partial \tau}{\partial x_3}, 0, -\frac{\partial \tau}{\partial x_1}\right) + \beta \frac{\operatorname{grad}\tau}{|\operatorname{grad}\tau|^2},$$

where $\beta_1 = \beta_1(x,t)$ and $\beta_2 = \beta_2(x,t)$ are scalar functions. We can express ρ from (11) and substitute into (21) and (22). Then the system (21), (22) is completely determined since it consists of six equations for the six unknown functions $\lambda(x,t)$, $\mu(x,t)$, $\tau(x,t)$, $\beta_1(x,t)$, $\beta_2(x,t)$, $\beta(x,t)$.

Based on the formulas in Proposition 3, it is possible to construct a solution of general form (2) to the system (1) by using the following lemma.

Lemma 4. *Assume that we are given*

1) *functions $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$, $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1,2,3$, and* (11) *holds,*
2) *vector-valued functions $B_k = (B_k^1, B_k^2, B_k^3)(x,t)$, $k = 0, \dots, m$, where $m \geqslant 0$ is a fixed integer,*
3) *vector-valued functions $w_k = (w_k^1, w_k^2, w_k^3)$, $k = 0, 1, 2, \dots, m$.*

Then the vector-valued functions $w_k = (w_k^1, w_k^2, w_k^3)$, $B_k = (B_k^1, B_k^2, B_k^3)$, $k \geqslant m+1$, can be successively found from the system (7) *by using formulas in Lemma* 2.

Proof. We consider the relation (7) with $k = m+1$

$$Q_{m+1} = B_{m+1} - P_m - S_{m-1}.$$

Here, the vector-valued functions P_m and S_{m-1} are known. By assertion 3 of Lemma 2, we have

$$B_{m+1} = P_m + S_{m-1} - \gamma \beta_{m+1} \operatorname{grad}\tau$$

for some function $\beta_{m+1} = \beta_{m+1}(x,t)$. In this case, the vector-valued function w_{m+1} is also found from the relation

$$(w_{m+1}, \operatorname{grad}\tau) = \beta_{m+1}.$$

Then we apply induction on $k \geqslant m+1$. If w_k, B_k, $m+1 \leqslant k \leqslant M$ are already constructed, then B_{M+1} is found by

$$B_{M+1} = P_M + S_{M-1} - \gamma \beta_{M+1} \operatorname{grad}\tau$$

for some function $\beta_{M+1} = \beta_{m+1}(x,t)$ and the vector-valued function w_{M+1} is found from the relation

$$(w_{M+1}, \operatorname{grad}\tau) = \beta_{M+1}.$$

We note that w_k, B_k, $k \geqslant m+1$, are found at each step with a certain arbitrariness in three scalar functions. □

Proposition 4. *Assume that the following conditions hold*:

1) *the identity* (12) *holds,*
2) *the vector* $B_0 = (B_0^1, B_0^2, B_0^3)(x,t)$ *is orthogonal to the vector* $\operatorname{grad}\tau$*;*
3) $B_1 = (B_1^1, B_1^2, B_1^3)(x,t)$ *and* $B_2 = (B_2^1, B_2^2, B_2^3)(x,t)$ *are given vector-valued functions,*
4) *the scalar functions* $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t))$ $\tau = \tau(x,t)$, $\nu = \nu(x,t)$ *and the vector*

$$w_0 = (w_0^1, w_0^2, w_0^3) \equiv \frac{1}{\gamma|\operatorname{grad}\tau|^2} B_0 - \nu \operatorname{grad}\tau \tag{23}$$

satisfy the system of equations $(i = 1,2,3)$

$$\begin{gathered} 2\frac{\partial w_0^i}{\partial t}\frac{\partial \tau}{\partial t} + w_0^i \frac{\partial^2 \tau}{\partial t^2} - \frac{\mu}{\rho}[2(\operatorname{grad} w_0^i, \operatorname{grad}\tau) + w_0^i \Delta\tau] \\ - \frac{\lambda+\mu}{\rho}\Big[\operatorname{div} w_0 \frac{\partial \tau}{\partial x_i} + \Big(\frac{\partial w_0}{\partial x_i}, \operatorname{grad}\tau\Big) + \Big(w_0, \operatorname{grad}\frac{\partial \tau}{\partial x_i}\Big)\Big] \\ - \frac{1}{\rho}\Big[(w_0, \operatorname{grad}\tau)\frac{\partial \lambda}{\partial x_i} + w_0^i(\operatorname{grad}\tau, \operatorname{grad}\mu) + \frac{\partial \tau}{\partial x_i}(w_0, \operatorname{grad}\mu) + \alpha_i w_0^i \frac{\partial \tau}{\partial t}\Big] = B_1^i, \end{gathered} \tag{24}$$

$$\begin{gathered} \frac{\partial^2 w_0^i}{\partial t^2} - \frac{\mu}{\rho}\Delta w_0^i - \gamma \frac{\partial}{\partial x_i}(\operatorname{div} w_0) \\ - \frac{1}{\rho}\Big[\operatorname{div} w_0 \frac{\partial \lambda}{\partial x_i} + (\operatorname{grad} w_0^i, \operatorname{grad}\mu) + \Big(\frac{\partial w_0}{\partial x_i}, \operatorname{grad}\mu\Big) + \alpha_i \frac{\partial w_0^i}{\partial t}\Big] = B_2^i. \end{gathered} \tag{25}$$

Then the vector-valued function $w = w_0(x,t) f_0(t, \tau(x,t))$ *is a solution to the system* (1).

Remark 8. We can express ρ from (12) and substitute into (23)–(25). Then, in view of (23) and the relation $(B_0, \operatorname{grad}\tau) = 0$, the system (24), (25) consists of seven equations for the seven unknown functions $\lambda(x,t)$, $\mu(x,t)$, $\tau(x,t)$, $\nu(x,t)$, $B_0 = (B_0^1, B_0^2, B_0^3)(x,t)$.

Based on the formulas in Proposition 4, it is possible to construct solutions of more general form (2) to the system (1) by using the following lemma.

Lemma 5. *Assume that we are given*

1) *functions* $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$, $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1,2,3$, *and the relation* (12) *holds,*
2) *vector-valued functions* $B_k = (B_k^1, B_k^2, B_k^3)(x,t)$, $k = 0, \dots, m$, *where* $m \geqslant 0$ *is a fixed integer,*
3) *vector-valued functions* $w_k = (w_k^1, w_k^2, w_k^3)$, $k = 0,1,2,\dots,m$.

Then the vector-valued functions $w_k = (w_k^1, w_k^2, w_k^3)$, $B_k = (B_k^1, B_k^2, B_k^3)$, $k \geqslant m+1$, *can be successively found from the system* (8) *by using formulas in Lemma* 2.

Proof. We consider the relation in (8) with $k = m+1$

$$Q_{m+1} = B_{m+1} - P_m - S_{m-1}.$$

Here, the vector-valued functions P_m and S_{m-1} are known. By assertion 3) of Lemma 2,

$$(B_{m+1} - P_m - S_{m-1}, \operatorname{grad}\tau) = 0.$$

and, in this case, the vector-valued function w_{m+1} is found by the formula

$$w_{m+1} = \nu_{m+1} \operatorname{grad}\tau + \frac{1}{\gamma|\operatorname{grad}\tau|^2}\big(B_{m+1} - P_m - S_{m-1}\big)$$

for some function $\nu_{m+1} = \nu_{m+1}(x,t)$.

Then we use induction on $k \geqslant m+1$. If w_k, B_k, $m+1 \leqslant k \leqslant M$, are already constructed, then B_{M+1} is found from the relation

$$(B_{M+1} - P_M - S_{M-1}, \operatorname{grad}\tau) = 0$$

and the vector-valued function w_{M+1} is found from the formula

$$w_{M+1} = \nu_{M+1} \operatorname{grad}\tau + \frac{1}{\gamma|\operatorname{grad}\tau|^2}\big(B_{M+1} - P_M - S_{M-1}\big)$$

for some function $\nu_{M+1} = \nu_{M+1}(x,t)$. $\square$

We note that w_k, B_k, $k \geqslant m+1$, are found at each step with a certain arbitrariness in three scalar functions.

The above identification systems of equations with respect to $w_0(x,t)$, $\lambda(x,t)$, $\mu(x,t)$, $\rho(x,t)$, $\tau(x,t)$, $\alpha_i(x,t)$, $i = 1,2,3$, for different type waves should be further studied, and the authors hope to discuss them in forthcoming works.

4 Solutions of General Form

The assertions of this section are based on the fact that if the relation in (8))

$$Q_k = B_k - P_{k-1} - S_{k-2}$$

is resolved with respect to w_k, then the series (2) can be successively constructed by using recurrent formulas expressing w_k in terms of w_{k-1} and w_{k-2}.

Theorem 3. *Assume that the matrix P is nonsingular and the following conditions hold*:

1) $B_k = (B_k^1, B_k^2, B_k^3)$, $k = 0,1,2,\dots,$ *are arbitrary vector-valued functions,*
2) *the vector-valued functions* $w_k = (w_k^1, w_k^2, w_k^3)$, $P_k = (P_k^1, P_k^2, P_k^3)$, $S_k = (S_k^1, S_k^2, S_k^3)$, $k = 0,1,2,\dots,$ *are defined by the recurrent formulas* (4), (5) *and*

$$w_k^i = \frac{\gamma}{\theta(\theta + \gamma|\operatorname{grad}\tau|^2)}(B_k - P_{k-1} - S_{k-2}, \operatorname{grad}\tau)\frac{\partial\tau}{\partial x_i} - \frac{1}{\theta}(B_k^i - P_{k-1}^i - S_{k-2}^i),$$

   where $i = 1,2,3$ *and* $w_k^i = P_k^i = S_k^i = 0$ *for* $k \leqslant -1$.

Then the vector-valued function $w(x,t)$ represented as the ray series (2) *is a solution to the system* (1) *with the vector-valued source function $R = (R_1, R_2, R_3)$ admitting the representation* (7).

Below, we indicate cases of the existence of solutions subject to the conditions $(w, \operatorname{grad}\tau) = 0$ and $w \times \operatorname{grad}\tau = 0$. Here, the symbol $\times$ denotes the vector product in $\mathbb{R}^3$.

4.1. Waves with property $(w, \operatorname{grad}\tau) = 0$**.**

Theorem 4. *Assume that the following conditions hold*:

1) $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$, $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1,2,3$, *are scalar functions such that*

$$\left(\frac{\partial\tau}{\partial t}\right)^2 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2 \neq 0,$$

2) *the vector-valued functions* $w_k = (w_k^1, w_k^2, w_k^3)$, $P_k = (P_k^1, P_k^2, P_k^3)$, $S_k = (S_k^1, S_k^2, S_k^3)$, $B_k = (B_k^1, B_k^2, B_k^3)$, $k = 0,1,2,\dots,$ *are defined by the recurrent formulas*

$$(B_k - P_{k-1} - S_{k-2}, \operatorname{grad}\tau) = 0,$$

$$w_k = \frac{B_k - P_{k-1} - S_{k-2}}{(\frac{\partial\tau}{\partial t})^2 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2},$$

$$P_k^i = 2\frac{\partial w_k^i}{\partial t}\frac{\partial\tau}{\partial t} + w_k^i\frac{\partial^2\tau}{\partial t^2} - \frac{\mu}{\rho}[2(\operatorname{grad} w_k^i, \operatorname{grad}\tau) + w_k^i\Delta\tau]$$
$$- \frac{\lambda+\mu}{\rho}\operatorname{div} w_k\frac{\partial\tau}{\partial x_i} - \frac{1}{\rho}\left[w_k^i(\operatorname{grad}\tau, \operatorname{grad}\mu) + \frac{\partial\tau}{\partial x_i}(w_k, \operatorname{grad}\mu) + \alpha_i w_k^i\frac{\partial\tau}{\partial t}\right], \qquad (26)$$

S_k^i *is expressed by formula* (4), $i = 1,2,3$.

Then the vector-valued function $w(x,t)$ *represented as the formal ray series* (2) *is a solution to the system* (1) *with the source function* (7); *moreover,* $(w, \operatorname{grad}\tau) = 0$.

Remark 9. If, for example, $\dfrac{\partial\tau}{\partial x_1} \neq 0$, then the vector-valued function B_k is represented as

$$B_k = \beta_{k1}\Big(\frac{\partial\tau}{\partial x_2}, -\frac{\partial\tau}{\partial x_1}, 0\Big) + \beta_{k2}\Big(\frac{\partial\tau}{\partial x_3}, 0, -\frac{\partial\tau}{\partial x_1}\Big) + \beta\operatorname{grad}\tau,$$

where $\beta_{k1} = \beta_{k1}(x,t)$ and $\beta_{k2} = \beta_{k2}(x,t)$ are arbitrary scalar functions and the function $\beta_k = \beta_k(x,t)$ is given by

$$\beta_k = \frac{1}{|\operatorname{grad}\tau|^2}(P_{k-1} + S_{k-2}, \operatorname{grad}\tau).$$

Setting $\beta_{k1} = \beta_{k2} = 0$, we arrive at the following assertion.

Corollary 3. *Assume that the following conditions hold*:

1) $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t)$) $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1,2,3$, *are scalar functions such that*

$$\left(\frac{\partial\tau}{\partial t}\right)^2 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2 \neq 0,$$

2) $B_0 = (B_0^1, B_0^2, B_0^3)$ *is a nonzero vector-valued function orthogonal to the vector* $\operatorname{grad}\tau$, $(B_0, \operatorname{grad}\tau) = 0$,

3) *the vector-valued functions* $w_k = (w_k^1, w_k^2, w_k^3)$, $P_k = (P_k^1, P_k^2, P_k^3)$, $S_k = (S_k^1, S_k^2, S_k^3)$, $k = 0, 1, 2, \dots$, *and* $B_k = (B_k^1, B_k^2, B_k^3)$, $k = 1, 2, \dots$, *are defined by the recurrent formulas* (26), (4), *and*

$$B_k = \beta_k \operatorname{grad}\tau, \quad \beta_k = \frac{1}{|\operatorname{grad}\tau|^2}(P_{k-1} + S_{k-2}, \operatorname{grad}\tau), \quad w_k = \frac{\beta_k \operatorname{grad}\tau - P_{k-1} - S_{k-2}}{\left(\frac{\partial\tau}{\partial t}\right)^2 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2}.$$

Then the series (2) *is a solution to the system* (1) *with the source function* (7); *moreover,* $(w, \operatorname{grad}\tau) = 0$.

We note that the term

$$\sum_{k=0}^{\infty} B_k(x,t) f_{k-2}(t, \tau(x,t))$$

in the source function (7) is the sum of the term $B_0(x,t)f_{-2}(t,\tau(x,t))$ orthogonal to the vector $\operatorname{grad}\tau$ and the term

$$\sum_{k=1}^{\infty} B_k(x,t) f_{k-2}(t, \tau(x,t))$$

proportional to the vector $\operatorname{grad}\tau$.

We clarify whether there exist waves such that $(w, \operatorname{grad}\tau) = 0$ provided that (11) holds.

Theorem 5. *Assume that the following conditions hold*:

1) $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t))$ $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1, 2, 3$, *are scalar functions such that*

$$\left(\frac{\partial\tau}{\partial t}\right)^2 - \frac{\mu}{\rho}|\operatorname{grad}\tau|^2 = 0,$$

2) $w_k = (w_k^1, w_k^2, w_k^3)$, $k = 0, 1, 2, \dots$, *are arbitrary vector-valued functions orthogonal to the vector* $\operatorname{grad}\tau$, $(w_k, \operatorname{grad}\tau) = 0$,

3) *the vector-valued functions* $P_k = (P_k^1, P_k^2, P_k^3)$, $S_k = (S_k^1, S_k^2, S_k^3)$, $k = 0, 1, 2, \dots$, *are defined by* (26) *and* (4),

4) *the vector-valued functions* $B_k = (B_k^1, B_k^2, B_k^3)$, $k = 0, 1, 2, \dots$, *are defined by*

$$B_k = P_{k-1} + S_{k-2}.$$

Then the vector-valued function $w(x,t)$ *represented as the formal ray series* (2) *is a solution to the system* (1) *with the source function* (7); *moreover,* $(w, \operatorname{grad}\tau) = 0$.

4.2. Waves with property $w \times \operatorname{grad}\tau = 0$.

Theorem 6. *Assume that the following conditions hold*:

1) $\lambda = \lambda(x,t)$, $\mu = \mu(x,t)$, $\rho = \rho(x,t))$ $\tau = \tau(x,t)$, $\alpha_i = \alpha_i(x,t)$, $i = 1, 2, 3$, *are scalar functions such that*

$$\left(\frac{\partial\tau}{\partial t}\right)^2 - \frac{\lambda + 2\mu}{\rho}|\operatorname{grad}\tau|^2 \neq 0,$$

2) *the vector-valued function* $B_0 = (B_0^1, B_0^2, B_0^3)(x,t)$ *admits the representation*

$$B_0 = \beta_0\Big(\Big(\frac{\partial\tau}{\partial t}\Big)^2 - \frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2\Big)\operatorname{grad}\tau,$$

3) *the vector-valued function* $w_0 = (w_0^1, w_0^2, w_0^3)(x,t)$ *admits the representation*

$$w_0 = \beta_0 \operatorname{grad}\tau,$$

4) *the scalar functions* $\beta_k = \beta_k(x,t)$, *vector-valued functions* $w_k = (w_k^1, w_k^2, w_k^3)(x,t)$, $k = 1,2,\dots$, *and vector-valued functions* $P_k = (P_k^1, P_k^2, P_k^3)(x,t)$, $S_k = (S_k^1, S_k^2, S_k^3)(x,t)$, $Q_k = (Q_k^1, Q_k^2, Q_k^3)(x,t)$, $k = 0,1,2,\dots$, *are defined by* (5), (4), *and*

$$\beta_k = -\frac{(P_{k-1}+S_{k-2}, \operatorname{grad}\tau)}{|\operatorname{grad}\tau|^2\Big(\frac{\partial\tau}{\partial t}\Big)^2 - \frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2}, \quad w_k = \beta_k \operatorname{grad}\tau,$$

$$Q_k = \beta_k\Big(\Big(\frac{\partial\tau}{\partial t}\Big)^2 - \frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2\Big)\operatorname{grad}\tau,$$

5) *the vector-valued functions* $B_k = (B_k^1, B_k^2, B_k^3)(x,t)$, $k = 0,1,2,\dots$, *are defined by*

$$B_k = Q_k + P_{k-1} + S_{k-2}.$$

Then the vector-valued function $w(x,t)$ *represented as the formal ray series* (2) *is a solution to the system* (1) *with the source function* (7); *moreover,* $w \times \operatorname{grad}\tau = 0$.

We note that the term

$$\sum_{k=0}^{\infty} B_k^i f_{k-2}$$

in the source function (7) is orthogonal to the vector $\operatorname{grad}\tau$ since $(B_k, \operatorname{grad}\tau) = 0$, $k = 0,1,2,\dots$.

Remark 10. The representation

$$B_0 = \beta_0\Big(\Big(\frac{\partial\tau}{\partial t}\Big)^2 - \frac{\lambda+2\mu}{\rho}|\operatorname{grad}\tau|^2\Big)\operatorname{grad}\tau$$

in Theorem 6 can be replaced with the relation

$$(B_0, \operatorname{rot} B_0) = 0$$

which implies that the vector B_0 is proportional to the gradient of some function $\tau(x,t)$; moreover, the proportionality coefficient is found up to a factor which is a function of t.

Apparently, it is reasonable to add a vector-valued function $\widetilde{w}(x,t)$ to the ray decomposition, assuming that $\widetilde{w}(x,t)$ is an arbitrary solution to the homogeneous system of elasticity theory (1), so that the ray decomposition (2) is a partial solution to the problem (1).

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