DOI 10.1007/s10958-020-04745-w
Journal of Mathematical Sciences, Vol. 246, No. 3, April, 2020

VIBRATIONS OF TWO-LAYER IDEAL LIQUID IN A RIGID CYLINDRICAL VESSEL WITH ELASTIC BASES

Yu. M. Kononov and Yu. O. Dzhukha

UDC 533.6.013.42

We analyze the frequency equation of natural vibrations of a heavy ideal two-layer incompressible liquid placed in a rigid circular cylindrical vessel with elastic bases in the form of fixed thin circular plates. We consider the cases of axially symmetric (longitudinal) and asymmetric (transverse) vibrations of the liquid and plates and different limiting cases (degeneration of the plates into membranes, absolutely rigid plates, and the absence of the top plate). It is shown that the frequency spectrum of coupled asymmetric vibrations of the elastic bases and two-layer ideal liquid consists of three sets of frequencies corresponding to the vibrations of the top and bottom elastic bases and to the vibrations of the inner interface of liquids. The frequency spectrum of coupled axially symmetric vibrations consists of the same three sets as for the asymmetric vibrations and an additional frequency of vibration of the liquid column as a single whole. As an example, we perform the analytic and numerical investigations of the frequency spectrum of a homogeneous liquid with free surface and elastic bottom in the form of a membrane.

Introduction

The problem of transverse and longitudinal vibrations of ideal liquids in rigid vessels with elastic bases is not well studied. Numerous works are devoted to the analyses of transverse (asymmetric) vibrations of ideal liquids in vessels with absolutely rigid bottom base (see, e.g., [1, 2, 3]). In this case, the center of mass of the mechanical system shifts in the transverse direction.

Note that the works devoted to the study of longitudinal (symmetric) vibrations of ideal liquids in vessels with absolutely rigid bottom base are, in fact, absent, possibly due to the fact the center-of-mass of the system does not shift in this case and, as indicated in the monograph [1], "in the case where the walls of the cavity are not deformable and the liquid is incompressible, the plate may take only asymmetric forms orthogonal to a constant."

Indeed, in some special cases, asymmetric vibrations are absent, e.g., in the case of weightlessness or in the case where the plate has no mass [1]. The influence of overloading on the axially symmetric vibrations of a circular membrane placed on the free surface of liquid in a rigid circular cylindrical vessel was studied in [4]. For the first time, the problem of axially symmetric vibrations of an ideal liquid with free surface placed in a rigid circular cylindrical vessel with plane elastic base was considered in [5, 6, 7]. In [8], this problem was generalized to the case of cylindrical vessel with cross section of any shape. In [9], the problem was studied for the case of axially symmetric vibrations of a two-layer ideal liquid with free surface. A similar problem for a system of immiscible liquids was studied by the operator methods in [10].

The experimental investigations of dynamical processes in a rigid vessel with elastic bottom partially filled with liquid were carried out in [11]. The most general statement of the problem of longitudinal vibrations of the ideal liquid in a rigid cylindrical vessel with plane elastic bases was presented in [12]. The solvability of the problem of vibrations of the ideal liquid in a rigid vessel with elastic bases was proved by the methods of functional analysis in [13]. The asymptotic formulas for eigenfrequencies were also presented in the same work.

Stus Donetsk National University, 600-Richchya Str., 21, Vinnytsya, 21021, Ukraine; e-mail: kononov.yuriy.nikitovich@gmail.com, yu.djukha@donnu.edu.ua

Translated from Neliniini Kolyvannya, Vol. 21, No. 4, pp. 496–513, October–December, 2018. Original article submitted September 6, 2018; revision submitted November 2, 2018.

Coupled axisymmetric vibrations of the elastic bases and ideal liquid in circular and coaxial cylindrical vessels were investigated in [14, 15]. In [16], this problem was generalized to the case of a two-layer liquid. The problem of axisymmetric vibrations of a two-layer ideal liquid in a circular cylindrical vessel connected with the problem of capillary phase separators was considered in the linear statement in [17]–[20]. Numerous works are devoted to the investigation of vibrations of the free surface of an ideal liquid in a circular vessel with elastic bottom both in the linear and nonlinear statements (see, e.g., [21]–[24]). The longitudinal (symmetric) and transverse (asymmetric) vibrations of plates or membranes on the free surface of ideal liquid in a circular vessel with absolutely rigid lower base were investigated in detail in [25]. The case of asymmetric vibrations of a circular plate on the liquid surface was considered in [26]. We also especially mention a very interesting paper [27] devoted to the analysis of problems similar to those considered in the present work in which the authors studied vibrations of a homogeneous ideal liquid in a cylindrical vessel with identical elastic bases in the form of circular plates. The analytic method proposed in this paper is based on the expansions in Fourier–Bessel series and the Rayleigh–Riesz method. Note that the structure of the spectrum is, as a rule, not analytically investigated in the cited works.

1. Statement of the Problem

We consider coupled vibrations of elastic bases and heavy two-layer ideal incompressible liquids with densities ρ_1 and ρ_2, which completely fill a direct circular cylindrical vessel of radius a with rigid lateral surface up to depths h_1 and h_2, respectively. The bases of the vessel have the form of fixed circular isotropic plates with flexural stiffnesses D_i. They are subjected to the action of the tensile forces T_i in the middle plane $i = 1, 2$. The subscripts $i = 1$ and $i = 2$ correspond to the top base and top liquid with density ρ_1 and to the bottom base and bottom liquid with density ρ_2, respectively. We use a cylindrical coordinate system $Or\theta z$ whose $Or\theta$ plane coincides with the interface of liquids and the Oz-axis is directed along the axis of the cylinder in the direction opposite to the direction of the gravitational acceleration $\vec{g}$ (Fig. 1). We consider the problem in the linear statement under the assumptions that the motion of liquid is potential and that the coupled vibrations of the plates and liquid are not separated. The equations of motion of the analyzed mechanical system take the form [16]

$$k_{01}\frac{\partial^2 W_1}{\partial t^2} + D_1\Delta_2^2 W_1 - T_1\Delta_2 W_1 + \rho_1 g W_1 = \rho_1\left(Q_1 - \left.\frac{\partial \Phi_1}{\partial t}\right|_{z=h_1} - gh_1\right) - gk_{01}, \tag{1}$$

$$k_{02}\frac{\partial^2 W_2}{\partial t^2} + D_2\Delta_2^2 W_2 - T_2\Delta_2 W_2 - \rho_2 g W_2 = -\rho_2\left(Q_2 - \left.\frac{\partial \Phi_2}{\partial t}\right|_{z=-h_2} + gh_2\right) - gk_{02}, \tag{2}$$

$$\Delta\Phi_i = 0$$

with boundary conditions

$$\left.\frac{\partial \Phi_i}{\partial r}\right|_{\gamma} = 0, \quad i = 1, 2, \qquad \left.\frac{\partial \Phi_1}{\partial z}\right|_{z=h_1} = \frac{\partial W_1}{\partial t}, \qquad \left.\frac{\partial \Phi_2}{\partial z}\right|_{z=-h_2} = \frac{\partial W_2}{\partial t},$$
$$\rho_1\left(Q_1 - \left.\frac{\partial \Phi_1}{\partial t}\right|_{z=0} - g\zeta\right) = \rho_2\left(Q_2 - \left.\frac{\partial \Phi_2}{\partial t}\right|_{z=0} - g\zeta\right), \qquad \left.\frac{\partial \Phi_1}{\partial z}\right|_{z=0} = \left.\frac{\partial \Phi_2}{\partial z}\right|_{z=0} = \frac{\partial \zeta}{\partial t}, \tag{3}$$

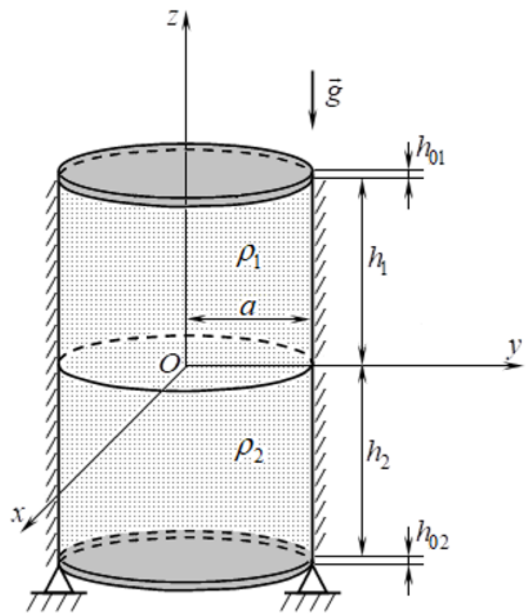

Fig. 1. Statement of the problem.

$$W_i|_\gamma = 0, \qquad \left.\frac{\partial W_i}{\partial r}\right|_\gamma = 0, \quad i = 1, 2, \tag{4}$$

$$\frac{1}{S}\int_S W_1 dS = \frac{1}{S}\int_S \zeta dS = \frac{1}{S}\int_S W_2 dS. \tag{5}$$

Here, $k_{0i} = \rho_{0i} h_{0i}$; W_i, ρ_{0i}, and h_{0i} are, respectively, the deflection, density, and thickness of the ith plate; Φ_i is the velocity potential of the ith liquid; $z = \zeta(r, \theta, t)$ is the interface of liquids (inner surface); Q_i are arbitrary functions of time;

$$\Delta_2 \quad \text{and} \quad \Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}$$

are the two- and three-dimensional Laplace operators, respectively; S is a circular domain, and γ is its contour $r = a$.

2. Procedure of Solution

We represent the functions Φ_i and ζ in the form of generalized Fourier series in the eigenfunctions $\psi_{nm}(r, \theta)$ [14, 16]:

$$\Phi_i(r, \theta, z, t) = \sum_{m=0}^{\infty} \Bigg\{ [a_{0i}(t) + a_{1i}(t) z]\delta_{m0} + \sum_{n=1}^{\infty} \Big[A_{inm}(t)\, e^{k_{nm} z} + B_{inm}(t)\, e^{-k_{nm} z} \Big] \psi_{nm}(r, \theta) \Bigg\}, \tag{6}$$

$$\zeta(r,\theta,t) = \sum_{m=0}^{\infty}\left[\zeta_0(t)\delta_{m0} + \sum_{n=1}^{\infty}\zeta_{nm}(t)\psi_{nm}(r,\theta)\right], \tag{7}$$

where δ_{m0} is the Kronecker symbol. The functions $\psi_{nm}(r,\theta)$, together with an arbitrary constant, form a complete system of functions with weight r in the domain S and have the form

$$\psi_{nm}(r,\theta) = Y_{nm}(k_{nm}r)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix},$$

where

$$Y_{nm}(k_{nm}r) = \frac{J_m(k_{nm}r)}{J_m(k_{nm}a)}, \quad k_{nm} = \frac{\xi_{nm}}{a},$$

ξ_{nm} is the nth root of the equation $J'_m(\xi_{nm}) = 0$ and J_m is the Bessel function of the first kind.

Substituting expressions (6) and (7) in the boundary conditions (3) and (5), in view of the orthogonality of the functions $\psi_{nm}(r,\theta)$, we obtain

$$A_{1nm} = \frac{\dot W_{1nm} - \dot\zeta_{nm}e^{-\kappa_{1nm}}}{2k_{nm}\sinh\kappa_{1nm}}, \qquad B_{1nm} = \frac{\dot W_{1nm} - \dot\zeta_{nm}e^{\kappa_{1nm}}}{2k_{nm}\sinh\kappa_{1nm}},$$

$$A_{2nm} = -\frac{\dot W_{2nm} - \dot\zeta_{nm}e^{\kappa_{2nm}}}{2k_{nm}\sinh\kappa_{2nm}}, \qquad B_{2nm} = -\frac{\dot W_{2nm} - \dot\zeta_{nm}e^{-\kappa_{2nm}}}{2k_{nm}\sinh\kappa_{2nm}}, \quad \kappa_{inm} = k_{nm}h_i,$$

$$a_{11} = a_{12} = \dot W_{10} = \dot W_{20} = \dot\zeta_0, \qquad \rho_1\left(Q_1 - \dot a_{01} - g\zeta_0\right) = \rho_2\left(Q_2 - \dot a_{02} - g\zeta_0\right),$$

$$W_{inm} = \frac{1}{N_{nm}^2}\int_S W_i\psi_{nm}dS, \qquad \zeta_{nm} = \frac{1}{N_{nm}^2}\int_S \zeta\psi_{nm}dS, \qquad W_{i0} = \frac{1}{S}\int_S W_i dS,$$

$$\zeta_0 = \frac{1}{S}\int_S \zeta dS, \qquad N_{nm}^2 = \int_S \psi_{nm}^2 dS = \frac{\pi a^2}{2}\left(1 - \frac{m^2}{\xi_{nm}^2}\right)\begin{cases}2, & m = 0,\\ 1, & m \neq 0.\end{cases}$$

Equations (1) and (2) and the equation of the interface of liquids (inner surface) take the form

$$\begin{aligned} k_{01}\frac{\partial^2 W_1}{\partial t^2} &+ D_1\Delta_2^2 W_1 - T_1\Delta_2 W_1 + \rho_1 g W_1 = -gk_{01} \\ &+ \rho_1\sum_{m=0}^{\infty}\left\{\left[Q_1 - \dot a_{01} - h_1\left(\ddot\zeta_0 + g\right)\right]\delta_{m0} - \sum_{n=1}^{\infty}\frac{\ddot W_{1nm}\cosh\kappa_{1nm} - \ddot\zeta_{nm}}{k_{nm}\sinh\kappa_{1nm}}\psi_{nm}\right\}, \end{aligned} \tag{8}$$

$$\begin{aligned} k_{02}\frac{\partial^2 W_2}{\partial t^2} &+ D_2\Delta_2^2 W_2 - T_2\Delta_2 W_2 - \rho_2 g W_2 \\ &= -gk_{02} - \rho_2\sum_{m=0}^{\infty}\left\{\left[Q_2 - \dot a_{02} + h_2\left(\ddot\zeta_0 + g\right)\right]\delta_{m0} + \sum_{n=1}^{\infty}\frac{\ddot W_{2nm}\cosh\kappa_{2nm} - \ddot\zeta_{nm}}{k_{nm}\sinh\kappa_{2nm}}\psi_{nm}\right\}, \end{aligned} \tag{9}$$

$$\ddot{\zeta}_{nm} + \sigma_{nm}^2 \zeta_{nm} - \frac{1}{a_{nm}} \left(b_{1nm} \ddot{W}_{1nm} + b_{2nm} \ddot{W}_{2nm} \right) = 0,$$

where

$$\sigma_{nm}^2 = \frac{g k_{nm} \Delta\rho}{a_{nm}}$$

is the squared vibration frequency of the inner surface of rigid bases,

$$a_{nm} = \rho_1 \coth \kappa_{1nm} + \rho_2 \coth \kappa_{2nm}, \quad \Delta\rho = \rho_2 - \rho_1, \quad \text{and} \quad b_{inm} = \frac{\rho_i}{\sinh \kappa_{inm}}.$$

3. Static Boundary-Value Problem and the Frequency Equation of Coupled Vibrations of the Elastic Bases and Liquid

Consider a problem of natural coupled vibrations of the elastic plates and the liquid. To this end, we represent the deflections of plates as the sum of static and dynamic deflections and set

$$W_i(r, \theta, t) = \mathrm{e}^{i\omega t} w_i(r, \theta) + W_i^{st}(r), \quad \rho_1 \left(Q_1 - \dot{a}_{01} \right) = Q \mathrm{e}^{i\omega t} + g(\rho_1 h_1 + k_{01}) + C,$$

and $\zeta_0(t) = w \mathrm{e}^{i\omega t}$. Separating the angular coordinate, we represent Eqs. (8) and (9) and the boundary conditions (4) and (5) in the form

$$D_i \Delta_{2m}^2 w_{im} - T_i \Delta_{2m} w_{im} - \left[k_{0i} \omega^2 + (-1)^i \rho_i g \right] w_{im}$$
$$= \rho_i \omega^2 \sum_{n=1}^{\infty} \tilde{w}_{inm} Y_{nm}(k_{nm} r)$$
$$+ \left\{ (-1)^{i+1} Q + \left[\rho_i h_i \omega^2 + (\delta_{i1} - 1)\, g \Delta\rho \right] w \right\} \delta_{m0}, \quad i = 1, 2, \tag{10}$$

$$w_{im}|_\gamma = 0, \qquad \left. \frac{dw_{im}}{dr} \right|_\gamma = 0, \quad i = 1, 2, \tag{11}$$

$$w = \frac{2}{a^2} \int_0^a r w_{10} dr = \frac{2}{a^2} \int_0^a r w_{20} dr, \tag{12}$$

where $W_i^{st}(r)$ is a static deflection of the plates,

$$\Delta_{2m} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2},$$

and

$$\tilde{w}_{1nm} = \frac{w_{1nm} \left(\cosh \kappa_{1nm} - \omega^2 \tilde{b}_{1nm} \right) - \omega^2 \tilde{b}_{2nm} w_{2nm}}{k_{nm} \sinh \kappa_{1nm}},$$

$$\tilde{w}_{2nm} = \frac{w_{2nm}\left(\cosh\kappa_{2nm} - \omega^2\tilde{b}_{2nm}\right) - \omega^2\tilde{b}_{1nm}w_{1nm}}{k_{nm}\sinh\kappa_{2nm}}, \qquad \tilde{b}_{inm} = \frac{b_{inm}}{a_{nm}\left(\omega^2 - \sigma_{nm}^2\right)}, \tag{13}$$

$$w_{inm} = \frac{\pi\,(1+\delta_{m0})}{N_{nm}^2}\int\limits_0^a r w_{im} Y_{nm} dr.$$

The solution of the static problem is reduced to the following boundary-value problem:

$$\begin{gathered} D_1\Delta_2^2 W_1^{st} - T_1\Delta_2 W_1^{st} + \rho_1 g W_1^{st} = C - g\left(\rho_1 h_1 + k_{01}\right), \\ D_2\Delta_2^2 W_2^{st} - T_2\Delta_2 W_2^{st} - \rho_2 g W_2^{st} = -C - g\left(\rho_2 h_2 + k_{02}\right), \\ W_i^{st}\big|_\gamma = 0, \qquad \frac{\partial W_i^{st}}{\partial r}\bigg|_\gamma = 0, \quad i = 1, 2, \\ \int\limits_0^a r W_1^{st} dr = \int\limits_0^a r W_2^{st} dr, \end{gathered} \tag{14}$$

where the functions W_i^{st} and constant C are unknown. In view of the physical symmetry, in the boundary-value problem (14), we set $m = 0$.

For the asymmetric vibrations ($m \neq 0$), due to the presence of the functions $\sin(m\theta)$ and $\cos(m\theta)$, Eqs. (5) are true. For the axially symmetric ($m = 0$) vibrations, it is necessary to take into account Eq. (12).

Thus, it is necessary to mention an essential difference between the axially symmetric (longitudinal) and asymmetric (transverse) vibrations of the plates and liquid. In the case of longitudinal vibrations, it is necessary to find the constants w and Q caused by the appearance of vibrations of the liquid column as a single whole. In the case of transverse vibrations, it is not necessary to determine these constants.

We seek the solution of Eq. (3) as the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation [1, 8, 14, 16]:

$$\begin{aligned} w_{im} = {} & \sum_{k=1}^{2} w_{ikm}^0 A_{ik}^0 + \rho_i\omega^2 \sum_{n=1}^{\infty} \frac{\tilde{w}_{inm}}{d_{inm}} Y_{nm} \\ & + \tilde{k}_{0i}\left\{Q + (-1)^{i+1}\left[\rho_i h_i \omega^2 + (\delta_{i1} - 1)\, g\Delta\rho\right] w\right\}\delta_{m0}, \quad i = 1, 2, \end{aligned} \tag{15}$$

where

$$\tilde{k}_{0i} = \frac{1}{\rho_i g + (-1)^i k_{0i}\omega^2} \qquad \left(\omega^2 \neq \frac{\rho_1 g}{k_{01}}\right),$$

$$d_{inm} = \left(D_i k_{nm}^2 + T_i\right) k_{nm}^2 - \left[k_{0i}\omega^2 + (-1)^i \rho_i g\right] \neq 0,$$

A^0_{ik}, $i,k = 1,2$, w_{inm}, Q, and w are unknown constants.

Substituting (15) in condition (13) and solving the system of two linear equations for w_{1nm} and w_{2nm}, we finally obtain

$$w_{1m} = \sum_{k=1}^{2} \left[\left(w^0_{1km} + \sum_{n=1}^{\infty} a_{11nm} E^0_{1knm} Y_{nm} \right) A^0_{1k} + \left(\sum_{n=1}^{\infty} a_{12nm} E^0_{2knm} Y_{nm} \right) A^0_{2k} \right] + \tilde{k}_{01}(Q + \rho_1 h_1 \omega^2 w)\delta_{m0},$$

$$w_{2m} = \sum_{k=1}^{2} \left[\left(\sum_{n=1}^{\infty} a_{21nm} E^0_{1knm} Y_{nm} \right) A^0_{1k} + \left(w^0_{2km} + \sum_{n=1}^{\infty} a_{22nm} E^0_{2knm} Y_{nm} \right) A^0_{2k} \right] + \tilde{k}_{02}\left[Q - \left(\rho_2 h_2 \omega^2 - g\Delta\rho\right) w\right]\delta_{m0}, \quad (16)$$

where

$$a_{11nm} = \omega^2 \frac{\left[\left(gk_{nm}\Delta\rho - a_{nm}\omega^2\right) a_{1nm} + \omega^2 b^2_{1nm}\right]\left(k_{nm}d_{2nm} - a_{2nm}\omega^2\right) - \omega^4 b^2_{2nm} a_{1nm}}{\Delta_{nm}},$$

$$a_{12nm} = \omega^4 \frac{k_{nm}d_{2nm}b_{1nm}b_{2nm}}{\Delta_{nm}}, \qquad a_{21nm} = \omega^4 \frac{k_{nm}d_{1nm}b_{1nm}b_{2nm}}{\Delta_{nm}},$$

$$a_{22nm} = \omega^2 \frac{\left[\left(gk_{nm}\Delta\rho - a_{nm}\omega^2\right) a_{2nm} + \omega^2 b^2_{2nm}\right]\left(k_{nm}d_{1nm} - a_{1nm}\omega^2\right) - \omega^4 b^2_{1nm} a_{2nm}}{\Delta_{nm}},$$

$$\Delta_{nm} = \left(k_{nm}d_{1nm} - a_{1nm}\omega^2\right)\left(gk_{nm}\Delta\rho - a_{nm}\omega^2\right)\left(k_{nm}d_{2nm} - a_{2nm}\omega^2\right) - \omega^4\left[b^2_{2nm}\left(k_{nm}d_{1nm} - a_{1nm}\omega^2\right) + b^2_{1nm}\left(k_{nm}d_{2nm} - a_{2nm}\omega^2\right)\right], \quad (17)$$

$$a_{1nm} = \rho_1 \coth \kappa_{1nm}, \qquad a_{2nm} = \rho_2 \coth \kappa_{2nm},$$

$$a_{nm} = a_{1nm} + a_{2nm} = \rho_1 \coth \kappa_{1nm} + \rho_2 \coth \kappa_{2nm},$$

$$E^0_{iknm} = \frac{\pi\left(1 + \delta_{m0}\right)}{N^2_{nm}} \int_0^a r w^0_{ikm} Y_{nm} dr. \quad (18)$$

To check the obtained relations (17), we consider the case of homogeneous liquid: $\rho_1 = \rho_2 = \rho$ ($\Delta\rho = 0$), $h_1 + h_2 = h$, and $\kappa_{1nm} + \kappa_{2nm} = \kappa_{nm}$. Moreover,

$$\Delta_{nm} = -a_{nm}\omega^2 \left[\left(k_{nm}d_{1nm} - \omega^2 \rho \coth \kappa_{nm}\right)\left(k_{nm}d_{2nm} - \omega^2 \rho \coth \kappa_{nm}\right) - \omega^4 \frac{\rho^2}{\sinh^2 \kappa_{nm}} \right],$$

$$a_{nm} = \rho \frac{\sinh\left(\kappa_{nm}\right)}{\sinh\left(\kappa_{1nm}\right)\sinh\left(\kappa_{2nm}\right)}.$$

Thus, for $T_2 = \infty$, the coefficient a_{11nm} takes the form

$$a_{11nm} = \frac{\omega^2 \rho}{k_{nm} d_{1nm} \tanh(\kappa_{nm}) - \omega^2 \rho}$$

and coincides with the similar relation from [1].

4. Frequency Equation of Coupled Asymmetric (Transverse) Vibrations of a Two-Layer Liquid and Elastic Bases

By using the conditions of fastening of the plates (11), we get the following frequency equation of asymmetric natural coupled vibrations ($m \neq 0$) of the two-layer liquid and elastic bases:

$$\left| \|C_{qr}\|_{q,r=1}^{4} \right| = 0, \tag{19}$$

where

$$C_{1,k} = B_{1km} + \sum_{n=1}^{\infty} a_{11nm} E_{1knm}^{0} = \sum_{n=1}^{\infty} \tilde{a}_{11nm} E_{1knm}^{0},$$

$$C_{1,k+2} = \sum_{n=1}^{\infty} a_{12nm} E_{2knm}^{0}, \qquad C_{2,k} = C_{1km}^{0}, \qquad C_{2,k+2} = 0, \qquad C_{3,k} = \sum_{n=1}^{\infty} a_{21nm} E_{1knm}^{0},$$

$$C_{3,k+2} = B_{2km} + \sum_{n=1}^{\infty} a_{22nm} E_{2knm}^{0} = \sum_{n=1}^{\infty} \tilde{a}_{22nm} E_{2knm}^{0},$$

$$C_{4,k} = 0, \qquad C_{4,k+2} = C_{2km}^{0}, \quad k = 1, 2.$$

Here,

$$B_{ikm} = w_{ikm}^{0}\big|_{\gamma}, \qquad C_{ikm}^{0} = \left.\frac{dw_{ikm}^{0}}{dr}\right|_{\gamma}, \qquad 1 + a_{iinm} = \tilde{a}_{iinm}.$$

It can be expected that the frequency spectrum of coupled asymmetric vibrations of the two-layer ideal liquid and elastic bases consists of three sets of frequencies corresponding to the vibrations of the top and bottom elastic bases and the vibrations of the interface of liquids. At the same time, for the homogeneous liquid $\rho_1 = \rho_2$, the frequency spectrum consists of two sets of frequencies corresponding to the vibrations of elastic bases.

5. Frequency Equation of Coupled Axially Symmetric (Longitudinal) Vibrations of a Two-Layer Liquid and Elastic Bases

In order to deduce the frequency equation of symmetric vibrations ($m = 0$) of the liquid and elastic bases, it is necessary to have two additional equations for the unknown quantities Q and w. To this end, we substitute (16) in (12) and obtain the following system of equations:

$$\tilde{k}_{01} Q + \tilde{k}_1 w = -\sum_{k=1}^{2} \tilde{w}_{1k}^0 A_{1k}^0,$$

$$\tilde{k}_{02} Q + \tilde{k}_2 w = -\sum_{k=1}^{2} \tilde{w}_{2k}^0 A_{2k}^0.$$

Solving this system for Q and w, we get

$$\begin{aligned} Q &= \frac{1}{\Delta}\left(-\tilde{k}_2 \sum_{k=1}^{2} \tilde{w}_{1k}^0 A_{1k}^0 + \tilde{k}_1 \sum_{k=1}^{2} \tilde{w}_{2k}^0 A_{2k}^0\right), \\ w &= \frac{1}{\Delta}\left(\tilde{k}_{02} \sum_{k=1}^{2} \tilde{w}_{1k}^0 A_{1k}^0 - \tilde{k}_{01} \sum_{k=1}^{2} \tilde{w}_{2k}^0 A_{2k}^0\right), \end{aligned} \tag{20}$$

where

$$\tilde{k}_1 = \tilde{k}_{01}\rho_1 h_1 \omega^2 - 1, \quad \tilde{k}_2 = \tilde{k}_{02}\left(g\Delta\rho - \rho_2 h_2 \omega^2\right) - 1, \quad \Delta = \tilde{k}_{01}\tilde{k}_2 - \tilde{k}_{02}\tilde{k}_1,$$

and

$$\tilde{w}_{ik}^0 = \frac{2}{a^2}\int_0^a r w_{ik}^0 \, dr. \tag{21}$$

Here and in what follows, we omit the index $m = 0$ in all relations written for the case of symmetric vibrations. Substituting (20) in (16), we finally obtain

$$\begin{aligned} w_1 = \sum_{k=1}^{2}\Bigg[&\left(w_{1k}^0 + \alpha_1 \tilde{w}_{1k}^0 + \sum_{n=1}^{\infty} a_{11n} E_{1kn}^0 Y_n\right) A_{1k}^0 \\ &+ \left(\alpha_2 \tilde{w}_{2k}^0 + \sum_{n=1}^{\infty} a_{12n} E_{2kn}^0 Y_n\right) A_{2k}^0\Bigg], \\ w_2 = \sum_{k=1}^{2}\Bigg[&\left(\beta_1 \tilde{w}_{1k}^0 + \sum_{n=1}^{\infty} a_{21n} E_{1kn}^0 Y_n\right) A_{1k}^0 \\ &+ \left(w_{2k}^0 + \beta_2 \tilde{w}_{2k}^0 + \sum_{n=1}^{\infty} a_{22n} E_{2kn}^0 Y_n\right) A_{2k}^0\Bigg], \end{aligned} \tag{22}$$

where

$$\alpha_1 = \frac{\tilde{k}_{01} k_2^*}{\Delta}, \qquad \alpha_2 = -\frac{\tilde{k}_{01}}{\Delta}, \qquad \beta_1 = \frac{\tilde{k}_{02}}{\Delta}, \qquad \beta_2 = -\frac{\tilde{k}_{02} k_1^*}{\Delta},$$

$$k_i^* = (-1)^{i+1} \tilde{k}_{0i} \left[g\Delta\rho - (\rho_1 h_1 + \rho_2 h_2)\omega^2 \right] + 1.$$

The determinant Δ can be rewritten in the form

$$\Delta = k_1^* \tilde{k}_{02} - \tilde{k}_{01} = \tilde{k}_{02} - k_2^* \tilde{k}_{01}.$$

Hence, the vibrations of elastic plates are described by relations (22).

By using the conditions of fastening of the plates (11), we get the following frequency equation of coupled axisymmetric vibrations of the two-layer liquid and elastic bases:

$$\left| \| C_{qr} \|_{q,r=1}^{4} \right| = 0, \tag{23}$$

where

$$C_{1,k} = B_{1k} + \alpha_1 \tilde{w}_{1k}^0 + \sum_{n=1}^{\infty} a_{11n} E_{1kn}^0 = \alpha_1 \tilde{w}_{1k}^0 + \sum_{n=1}^{\infty} \tilde{a}_{11n} E_{1kn}^0,$$

$$C_{1,k+2} = \alpha_2 \tilde{w}_{2k}^0 + \sum_{n=1}^{\infty} a_{12n} E_{2kn}^0,$$

$$C_{2,k} = C_{1k}^0, \qquad C_{2,k+2} = 0, \qquad C_{3,k} = \beta_1 \tilde{w}_{1k}^0 + \sum_{n=1}^{\infty} a_{21n} E_{1kn}^0, \tag{24}$$

$$C_{3,k+2} = B_{2k} + \beta_2 \tilde{w}_{2k}^0 + \sum_{n=1}^{\infty} a_{22n} E_{2kn}^0 = \beta_2 \tilde{w}_{2k}^0 + \sum_{n=1}^{\infty} \tilde{a}_{22n} E_{2kn}^0,$$

$$C_{4,k} = 0, \qquad C_{4,k+2} = C_{2k}^0, \quad k = 1, 2.$$

Here,

$$B_{ik} = w_{ik}^0 \big|_{\gamma}, \qquad C_{ik}^0 = \left. \frac{d w_{ik}^0}{dr} \right|_{\gamma}.$$

It can be expected that the frequency spectrum of coupled axially symmetric vibrations is also formed by three sets of frequencies, as in the case of asymmetric vibrations, and an additional frequency of vibrations of the liquid column as a single whole. We prove this fact (both analytically and numerically) for a homogeneous liquid with free surface and elastic bottom in the form of a membrane.

In [14], for a homogeneous liquid $\rho_1 = \rho_2$, the authors deduced an equation in the form of a determinant of the fifth order similar to Eq. (23). The indicated equations coincide if the determinant of the fifth order is reduced to a determinant of the fourth order. We also note that the equations presented in [14] have a singularity in the case of identical mass characteristics of the plates $k_{01} = k_{02}$. In Eq. (23), this singularity is absent.

6. Special Cases of the Frequency Equation of Natural Coupled Vibrations of the Liquid and Elastic Bases

The deduced equations (19) and (23) are fairly general and cover a series of special cases, which are also of independent interest.

Weightlessness $(g = 0)$**.** In this case, the frequency equations (19) and (23) become symmetric with respect to the indices 1 and 2, which is physically meaningful and confirms the validity of the deduced equations.

Homogeneous liquid $(\rho_1 = \rho_2 = \rho$ ***and*** $h_1 + h_2 = h)$**.** In this case, Eqs. (19) and (23) do not change, whereas relations (17) and the coefficients α_i and β_i can be rewritten in the form

$$a_{11nm} = \omega^2 \rho \frac{k_{nm} \coth \kappa_{nm} d_{2nm} - \omega^2 \rho}{\Delta_{nm}}, \qquad a_{12nm} = -\omega^2 \frac{k_{nm} b_{nm} d_{2nm}}{\Delta_{nm}},$$

$$a_{21nm} = -\omega^2 \frac{k_{nm} b_{nm} d_{1nm}}{\Delta_{nm}}, \qquad a_{22nm} = \omega^2 \rho \frac{k_{nm} \coth \kappa_{nm} d_{1nm} - \omega^2 \rho}{\Delta_{nm}},$$

$$\Delta_{nm} = \left(k_{nm} d_{1nm} - \omega^2 \rho \coth \kappa_{nm}\right)\left(k_{nm} d_{2nm} - \omega^2 \rho \coth \kappa_{nm}\right) - \omega^4 b_{nm}^2$$

$$= k_{nm}^2 d_{1nm} d_{2nm} - \omega^2 \rho k_{nm} \coth \kappa_{nm} (d_{1nm} + d_{2nm}) + \omega^4 \rho^2,$$

$$b_{nm} = \frac{\rho}{\sinh \kappa_{nm}}, \qquad \alpha_1 = -\frac{k_{02} + \rho\left(h + g/\omega^2\right)}{k_{12}}, \qquad \alpha_2 = \frac{k_{02} + \rho g/\omega^2}{k_{12}},$$

$$\beta_1 = \frac{k_{01} - \rho g/\omega^2}{k_{12}}, \qquad \beta_2 = -\frac{k_{01} + \rho\left(h - g/\omega^2\right)}{k_{12}}, \quad k_{12} = k_{01} + k_{02} + \rho h. \tag{25}$$

Top plate degenerates into a membrane $(D_1 = 0)$**.** In this case, it is necessary to delete the second row and second column in the determinants from Eqs. (19) and (23) and set $D_1 = 0$ in relations (17).

Bottom plate degenerates into a membrane $(D_2 = 0)$**.** As in the previous case, it is necessary to delete the fourth row and fourth column in the determinants from Eqs. (19) and (23) and set $D_2 = 0$ in relations (17).

Top and bottom plates degenerate into membranes $(D_1 = D_2 = 0)$**.** In this case, it is necessary to delete the second and fourth rows and the second and fourth columns in the determinants from Eqs. (19) and (23) and set $D_1 = D_2 = 0$ in relations (17).

The frequency equation (19) now takes the form

$$\left(B_{11m} + \sum_{n=1}^{\infty} a_{11nm} E_{11nm}^0\right)\left(B_{21m} + \sum_{n=1}^{\infty} a_{22nm} E_{21nm}^0\right) - \left(\sum_{n=1}^{\infty} a_{12nm} E_{21nm}^0\right)\left(\sum_{n=1}^{\infty} a_{21nm} E_{11nm}^0\right) = 0 \tag{26}$$

or

$$\left(\sum_{n=1}^{\infty}\tilde{a}_{11nm}E^0_{11nm}\right)\left(\sum_{n=1}^{\infty}\tilde{a}_{22nm}E^0_{21nm}\right)$$
$$-\left(\sum_{n=1}^{\infty}a_{12nm}E^0_{21nm}\right)\left(\sum_{n=1}^{\infty}a_{21nm}E^0_{11nm}\right)=0. \qquad (27)$$

Note that Eq. (26) is more convenient for numerical analyses. At the same time, Eq. (27) is more convenient for analytic investigations.

Bottom (or top) plate is absolutely rigid ($T_2 = \infty$ or $T_1 = \infty$). If the bottom or top plate is absolutely rigid, then $w_2 \equiv 0$ ($\tilde{w}^0_{2k} \equiv 0$) or $w_1 \equiv 0$ ($\tilde{w}^0_{1k} \equiv 0$), respectively. Passing to the limit in Eqs. (17) as $T_2 \to \infty$ or $T_1 \to \infty$, respectively, we obtain $a_{21nm} \to 0$ and $a_{22nm} \to 0$ or $a_{11nm} \to 0$ and $a_{12nm} \to 0$.

The equations of asymmetric (19) and symmetric vibrations (23) take the form

$$\left|\|C_{qr}\|^{1,2}_{q,r=1,2}\right| = 0 \qquad (28)$$

for $T_2 = \infty$ and

$$\left|\|C_{qr}\|^{3,4}_{q,r=3,4}\right| = 0 \qquad (29)$$

for $T_1 = \infty$.

In the case of symmetric vibrations, it is necessary to set $\alpha_1 = -1$ for $T_2 = \infty$ and $\beta_2 = -1$ for $T_1 = \infty$ in (24), which directly follows from the values of these coefficients in relations (22) for $k_{02} = \infty$ and $k_{01} = \infty$, respectively.

For the homogeneous liquid ($\rho_1 = \rho_2$), Eq. (28) coincides with a similar equation in [1].

Liquid with free surface ($k_{01} = 0$, $T_1 = 0$, and $D_1 = 0$). We consider this case in more detail because it is very simple but preserves all principal properties of the analyzed problem. This case is realized in the absence of the top plate. It is necessary to delete the first and second rows and the first and second columns in the determinants of Eqs. (19) and (23) and set $k_{01} = 0$, $T_1 = 0$, and $D_1 = 0$ in relations (17).

In the case where the bottom plate is degenerated into a membrane ($D_2 = 0$), Eqs. (19) and (23) can be rewritten in the form

$$B_{21m} + \sum_{n=1}^{\infty} a_{22nm}E^0_{21nm} = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty}\tilde{a}_{22nm}E^0_{21nm} = 0 \qquad (30)$$

and

$$B_{21} - \frac{\tilde{k}_{02}k_1^*}{\Delta}\,\tilde{w}^0_{21} + \sum_{n=1}^{\infty} a_{22n}E^0_{21n} = 0 \qquad (31)$$

or

$$\sum_{n=1}^{\infty}\tilde{a}_{22n}E^0_{21n} = \frac{\tilde{k}_{02}k_1^*}{\Delta}\,\tilde{w}^0_{21}, \qquad (32)$$

respectively.

Equation (32) coincides with a similar equation in [9] and, in the case of homogeneous liquid, with a similar equation in [8]. However, the last work contains several errors and the symbols g and h are interchanged in the final relations.

The frequency spectrum of Eqs. (30) is formed by three sets of frequencies corresponding to the vibrations of free and inner surfaces of the liquid and to the vibrations of elastic bottom. This follows from the second equation in (30) even for $n = 1$. If we take into account one term of the series, then this equation is reduced to a cubic equation with respect to ω^2. At the same time, for a single-layer liquid, the frequency spectrum is formed by two sets of frequencies.

The frequency spectrum of Eqs. (31) and (32) is formed by four sets of frequencies corresponding to the vibrations of the free and inner surfaces of the liquid, vibrations of the elastic bottom, and vibrations of the liquid column as a single whole. This follows from Eq. (32) even for $n = 1$. If we take into account one term of the series, then this equation is reduced to a quartic equation with respect to ω^2. At the same time, for a single-layer liquid, the frequency spectrum is formed by three sets of frequencies.

We now show this in more detail by analyzing Eq. (32) for a homogeneous liquid ($\rho_1 = \rho_2 = \rho$). Thus, it follows from relations (18), (21), and (25) that

$$E^0_{21n} = \frac{2\mu_2 J_1(\mu_2)}{\mu_2^2 - \xi_n^2}, \qquad \tilde{w}^0_{21} = \frac{2J_1(\mu_2)}{\mu_2}, \qquad \mu_2^2 = \frac{k_{02}\omega^2 + \rho g}{T_2} a^2,$$

$$1 + a_{22n} = -\frac{k_n d_{2n}\left(\omega^2 - \omega_n^2\right)}{\Delta_n},$$

$$\Delta_n = \rho\omega^2\left(\omega^2 \tanh\kappa_n - g k_n\right) - \left(\omega^2 - \omega_n^2\right) k_n d_{2n},$$

$$d_{2n} = T_2 k_n^2 - \rho g - k_{02}\omega^2,$$

$\kappa_n = k_n h$, and $\omega_n^2 = g k_n \tanh\kappa_n$ is the squared eigenfrequency of the free surface of ideal liquid over the rigid bottom [8].

Assume that $J_1(\mu_2) \neq 0$. Otherwise, the right-hand side of the equation is equal to zero and, on the left-hand side, we get $d_{2n} = 0$ in the limit as $\mu_2 \to \xi_n$, which contradicts the condition $d_{in} \neq 0$ introduced above.

We now introduce dimensionless variables as follows:

$$x = \frac{\omega^2 a}{g}, \qquad \tilde{k}_0 = \frac{k_{02}}{\rho a}, \qquad \tilde{h} = \frac{h}{a}, \qquad \tilde{T} = \frac{T_2}{\rho g a^2},$$

$$\tilde{\omega}_n^2 = \frac{\omega_n^2 a}{g} = \xi_n \tanh\kappa_n, \qquad \tilde{\Delta}_n = \frac{\Delta_n a^2}{\rho g^2}, \qquad \tilde{d}_{2n} = \frac{d_{2n}}{\rho g}.$$

In the introduced dimensionless variables (the “tilde” sign is omitted), we represent Eq. (32) in the form

$$\sum_{n=1}^{\infty} \frac{(x - \xi_n \tanh\kappa_n)\,\xi_n}{\Delta_n} = \frac{xh - 1}{x(k_0 x + 1)(h + k_0)}, \tag{33}$$

where

$$\Delta_n = x(x \tanh\kappa_n - \xi_n) - \xi_n d_{2n}(x - \xi_n \tanh\kappa_n)$$

and $d_{2n} = T\xi_n^2 - 1 - xk_0$.

We now show that the frequency spectrum of this equation consists of the frequency of vibrations of the liquid column and two sets of frequencies of vibrations of the free surface of liquid and elastic bottom.

Equation (33) immediately implies that, for $k_0 = \infty$, it has a solution $x_{1n} = \xi_n \tanh \kappa_n$, i.e., in the case of a sufficiently massive base $k_0 \gg 1$, we get only the first set of frequencies of vibrations, i.e., the eigenfrequencies of vibrations of ideal liquid.

For the qualitative analysis of Eq. (33), it is reasonable to use the graphic-analytic method and denote the left- and right-hand sides of this equation by $f_1(x)$ and $f_2(x)$, respectively.

The asymptotics are found from the square equation $\Delta_n = 0$, which has two positive roots

$$x_{1n,2n} = \frac{(T\xi_n + k_0 \tanh \kappa_n)\, \xi_n^2 \pm \sqrt{D_n}}{2\,(k_0\xi_n + \tanh \kappa_n)}, \quad x_{2n} > x_{1n},$$

where

$$D_n = \left[T^2\xi_n^4 - 2\,(k_0\xi_n + 2\tanh \kappa_n)\, \xi_n^2 \tanh \kappa_n T + \right.$$
$$\left. + \left(k_0^2\xi_n^2 \tanh \kappa_n + 4k_0\xi_n + 4\tanh \kappa_n\right) \tanh \kappa_n\right] \xi_n^2.$$

Assume that n takes values from 1 to N, $n = \overline{1, N}$. The function $f_1(x)$ monotonically decreases from $-\sum_{n=1}^{\infty} \dfrac{1}{T\xi_n^2 - 1}$ to $-\infty$ in a half segment $[0, x_{11})$, monotonically decreases from $+\infty$ to $-\infty$ in the intervals (x_{1n}, x_{2n}), and decreases from $+\infty$ to 0 in the remaining interval $(x_{2N}, +\infty)$.

The function $f_2(x)$ monotonically increases in the interval $\left(0, \dfrac{1+\sqrt{1+h_0}}{h}\right)$ from $-\infty$ to $\dfrac{h_0^2}{(1+h_0)\left(1+\sqrt{1+h_0}\right)^2}$, monotonically decreases in the interval $\left(\dfrac{1+\sqrt{1+h_0}}{h}, +\infty\right)$ from $\dfrac{h_0^2}{(1+h_0)\left(1+\sqrt{1+h_0}\right)^2}$ to 0, and attains its maximum value $\dfrac{h_0^2}{(1+h_0)\left(1+\sqrt{1+h_0}\right)^2}$, where $h_0 = h/k_0$ and $k_0 \neq 0$, at the point $\dfrac{1+\sqrt{1+h_0}}{h}$.

For $k_0 = 0$, the function $f_2(x)$ monotonically increases in the entire interval $(0, +\infty)$ and converges to 1 as $x \to \infty$. The plot of this function has one point of intersection with the plot of the function $f_2(x)$ in the half segment $[0, x_{11})$ and one point of intersection in each interval (x_{1n}, x_{2n}). In the last interval $(x_{2N}, +\infty)$, the plots of these functions do not intersect (see Fig. 2). The first point of intersection of the plots of the functions $f_1(x)$ and $f_2(x)$ (lowest frequency) always lies in the half segment $[0, x_{11})$ and describes the vibrations of the liquid column.

For the visualization of the results of investigations, the plots of the left-hand and right-hand sides of Eq. (33) obtained with regard for two terms in the series with $n = \overline{1, 2}$ are depicted in Fig. 2 for $h = 1$, $k_0 = 0.1$, and $T = 0.5$.

On the basis of the results of our investigations, by analyzing the typical plots presented in Fig. 2, we can conclude that the high frequencies ($n \gg 1$) of the first and second sets insignificantly differ from the values x_{1n} and x_{2n} and the lowest frequency corresponds to the vibrations of the liquid column (see Tables 1–4).

If $\tanh \kappa_n = 1$, then Eq. (33), Δ_n, the discriminant D_n, and the roots $x_{1n,2n}$ take the following form:

$$\sum_{n=1}^{\infty} \frac{\xi_n}{(x - \xi_n d_{2n})} = \frac{xh - 1}{x\,(k_0 x + 1)\,(h + k_0)}, \tag{34}$$

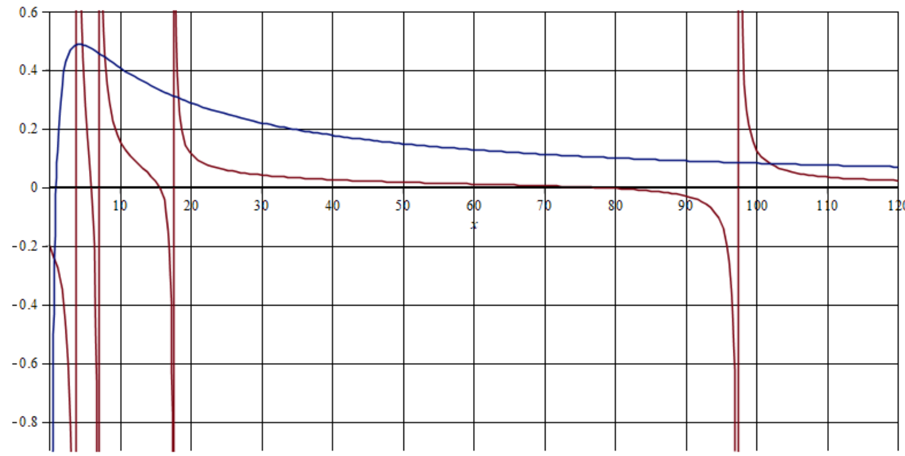

Fig. 2. Plots of the left- and right-hand sides of the frequency equation (33).

$$\Delta_n = (x - \xi_n)\left[x\,(\xi_n k_0 + 1) - \left(T\xi_n^2 - 1\right)\xi_n\right],$$

$$D_n = \left(T\xi_n^2 - k_0\xi_n - 2\right)^2 \xi_n^2, \qquad x_{1n} = \xi_n, \quad x_{2n} = \frac{\left(T\xi_n^2 - 1\right)\xi_n}{k_0\xi_n + 1}.$$

Since the eigenvalues ξ_n form an infinitely increasing number sequence $\xi \approx 3.8317$, we conclude that $\tanh\kappa_n \approx 1$ for $h \geq 1$. Thus, we can use the simplified equation (34) and the obtained first and second approximate sets of frequencies.

It is worth noting that the main properties of Eq. (33) and the set of frequencies become clear even for $n = 1$. As the number of analyzed terms in the series $n > 1$ increases, the indicated three main frequencies become more correct and we observe the appearance of two new frequencies (see Tables 1–4). For $n = 1$, Eq. (33) takes the form

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \tag{35}$$

where

$$a_0 = h\tanh\kappa_1 - \xi_1 k_0^2, \qquad a_1 = -\left[\left(T\xi_1^2 + 1\right)\xi_1 h + 2\xi_1 k_0 + \left(1 - \xi_1^2 k_0^2\right)\tanh\kappa_1\right],$$

$$a_2 = \left[(1 + \xi_1 h\tanh\kappa_1)\,\xi_1 T + 2k_0\tanh\kappa_1\right]\xi_1^2, \qquad a_3 = -\left(T\xi_1^2 - 1\right)\xi_1^2\tanh\kappa_1.$$

For the alternation of signs of the coefficients of cubic equation $a_0 > 0$, $a_1 < 0$, $a_2 > 0$, and $a_3 < 0$, it is sufficient to demand that

$$k_0 < \sqrt{\frac{h\tanh\kappa_1}{\xi_1}} \approx 0.5109\sqrt{h\tanh\kappa_1}, \qquad k_0 < \frac{1}{\xi_1} \approx 0.2610, \qquad T > \frac{1}{\xi_1^2} \approx 0.06811. \tag{36}$$

Table 1. $h = 1$, $k_0 = 0.1$, $T = 1$, and $D = 0$

$n = 1$			$n = \overline{1.2}$			$n = \overline{1.3}$		
0.9175	3.8276	59.4837	0.8971	3.8276	54.5494	0.8879	3.8276	53.3272
				7.0156	638.0322		7.0156	297.5138
							10.1735	

Table 2. $h = 1$, $k_0 = 0.1$, $T = 0.5$, and $D = 0$

$n = 1$			$n = \overline{1.2}$			$n = \overline{1.3}$		
0.8352	3.8271	30.2887	0.8019	3.8271	26.969	0.7874	3.827	26.2091
				7.0156	327.6779		7.0156	147.3403
							10.1735	

Table 3. $h = 0.5$, $k_0 = 0.1$, $T = 0.5$, and $D = 0$

$n = 1$			$n = \overline{1.2}$			$n = \overline{1.3}$		
1.608	3.6345	34.6318	1.5377	3.6324	29.4953	1.5075	3.6316	28.4802
				7.0019	486.3209		7.0019	150.7179
							10.1727	

Table 4. $h = 1$, $k_0 = 0.05$, $T = 0.5$, and $D = 0$

$n = 1$			$n = \overline{1.2}$			$n = \overline{1.3}$		
0.8474	3.8271	28.9854	0.8156	3.827	27.7381	0.8017	3.827	27.3574
				7.0156	198.1447		7.0156	166.6982
							10.1735	1146.5336

It is also worth noting that the parameters T, h, and k_0 must satisfy the condition of linearity of the static deflection

$$\left(\frac{1}{J_0(\eta)} - 1\right)(h + k_0) < A,$$

where

$$\eta = \sqrt{\frac{1}{T}}$$

and $A < 0.1$.

Expanding the discriminant of Eq. (35) in a series in T, we conclude that the leading coefficient of T^4 is positive and equal to $(1 - \xi_1 h \tanh \kappa_1)^2 h^2 \xi_1^{12}$.

Table 5. $h = 1$, $k_0 = 1$, $T = 0$, and $D = 1$

$n = 1$			$n = \overline{1.2}$			$n = \overline{1.3}$		
62.772	3.8281	1313.866	62.772	3.8281	1230.289	62.764	3.8281	1229.657
		7407.853		7.0156	7166.632		7.0156	6789.014
							10.173	

Thus, for sufficiently large $T > 1$, three real roots of the equation always exist. In view of inequalities (36) and the Descartes rule of signs for the number of positive roots, we have three positive roots.

7. Numerical Investigation of the Frequency Equations (33) and (23)

The data in presented Tables 1–5 illustrate the dependences of the squared dimensionless frequencies of axially symmetric vibrations of the liquid column (first column) and the first and second sets (second and third columns) on the number of terms in the series of the frequency equation (33) (Tables 1–4) and of the frequency equation (23) (Table 5) for a homogeneous liquid with free surface [14]. The numerical analyses were performed for the following values of the parameters: $h = 0.5; 1.0$, $k_0 = 0.05; 0.1$, $T = 0.5; 1.0$, and $D = 0$ (Tables 1–4) and $h = 1.0$, $k_0 = 1.0$, $T = 0$, and $D = D_2 \big/ \rho g a^4 = 1$ (Table 5).

It follows from Tables 1–4 that in the case of a membrane, the lowest frequency of vibrations is observed for the liquid column. This frequency increases as the level of tension increases and as the depth of filling (or the mass of the membrane) decreases. The increase in the number of terms in the series is accompanied by an insignificant decrease in the frequency of vibrations of liquid column. According to the results presented in Table 5, in the case of a plate, the lowest frequency of vibrations is no longer observed for the liquid column (as in the case of a membrane). On the contrary, the frequency of vibrations of the liquid column lies between the lowest frequencies of the first and second sets.

8. Conclusions

In the present paper, we study the frequency equation of natural vibrations of a two-layer heavy ideal incompressible liquid in a rigid circular cylindrical vessel with elastic bases in the form of thin circular fastened plates. We consider both axisymmetric (longitudinal) and asymmetric (transverse) vibrations of the liquid and plates and various limit cases of degeneration of plates into membranes or absolutely rigid plates, and also the case where the top plate is absent. It is shown that the frequency spectrum of coupled asymmetric vibrations of the elastic bases and two-layer ideal liquid is formed by three sets of frequencies corresponding to the vibrations of the top and lower bottom bases and vibrations of the interface of liquids. At the same time, the frequency spectrum of coupled axisymmetric vibrations contains the same three sets as for the asymmetric vibrations and an additional frequency of vibrations of the liquid column as a single whole.

We have generalized the results obtained in [14] to the case of two-layer liquid and asymmetric vibrations, deduced simpler frequency equations, and eliminated the singularity in the denominator of the coefficients of frequency equation in the case of identical mass characteristics of the plates. The results of the work [16] are generalized to the case of asymmetric vibrations. The analytic investigations of the frequency spectrum are performed for both asymmetric and axially symmetric vibrations. As an example, we studied, both analytically and numerically, the frequency spectrum of homogeneous liquid with free surface and elastic bottom in the form of a membrane, obtained simpler coefficients of the frequency equations, and revealed the coincidence of the frequency equations for the two-layer liquid with similar equations for the case of homogeneous liquid [14]. In numerous special cases, the obtained frequency equations coincide with the well-known equations.

The numerical results confirm the results of analytic investigations of the structure of frequency spectrum and, for a special case of homogeneous liquid with free surface and elastic bottom, show that the frequency spectrum of coupled symmetric vibrations of the elastic bottom and liquid is formed by the frequency of vibrations of the liquid column as a single whole and two sets of frequencies corresponding to the vibrations of the free liquid surface and elastic bottom. It is also shown that the frequency of vibrations of the liquid column is minimum in the case of a membrane and lies between the lowest frequencies of the first and second sets in the case of a plate. Note that, under the conditions of weightlessness, symmetric vibrations are absent if one of the plates or membranes is perfectly rigid. The series in the frequency equations converge sufficiently rapidly and, as a rule, two or three terms of these series are sufficient to guarantee the accuracy acceptable for practical purposes. The analysis of mass characteristics significantly increases the computation time required for the solution of the frequency equations.

To determine the influence of vibrations of the elastic bases on the vibrations of the interface, it is necessary to perform additional analytic and numerical investigations based on the derived frequency equation. The obtained results and their generalizations can be used in finding the natural modes of vibrations. They can be also applied in the problems of forced vibrations in "solid–liquid–elastic-bases" systems and in the problems of stability of specific hydroelastic systems. The present paper makes it possible to perform subsequent generalizations of the posed problem to the case of nonlinear vibrations.

In our subsequent works, we plan to compare the results of computations performed according to the proposed method with the data obtained by the other authors. Thus, in particular, it can be shown that the values of eigenfrequencies of asymmetric vibrations coincide with the corresponding values presented in [1].

The present work was performed within the framework of the Program of Fundamental Research of the Ukrainian Ministry of Education and Science (Project No. 0116U002522).

REFERENCES

1. L. V. Dokuchaev, *Nonlinear Dynamics of Aircrafts with Deformed Elements* [in Russian], Mashinostroenie, Moscow (1987).
2. V. A. Trotsenko and R. I. Bogun, "Transverse vibrations of a fluid in a long cylindrical container with membrane or elastic plate on the free surface," *Nelin. Kolyv.,* **12**, No. 3, 379–404 (2009); ***English translation:*** *Nonlin. Oscillat.,* **12**, No. 3, 392–416 (2009).
3. R. I. Bohun and V. A. Trotsenko, "Free oscillations of a fluid in a cylindrical container with arbitrary axisymmetric bottom and elastic elements on the free surface of the fluid," *Nelin. Kolyvannya,* **13**, No. 4, 461–482 (2010); ***English translation:*** *Nonlin. Oscillat.,* **13**, No. 4, 493–514 (2011).
4. Yu. N. Kononov and Yu. A. Dzhukha, "Influence of overloading on the axially symmetric vibrations of a membrane placed on the free liquid surface in a cylindrical vessel," in: *Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences,* **14**, No. 2 (2017), pp. 32–41.
5. A. A. Pozhalostin, "Free vibrations of liquid in a rigid circular cylindrical vessel with elastic plane bottom," *Izv. Vyssh. Uchebn. Zaved., Ser. Aviats. Tekh.,* No. 4, 25–32 (1963).
6. M. P. Petrenko, "Natural vibrations of a liquid with free surface and the elastic bottom of a cylindrical cavity," *Prikl. Mekh.,* **5**, No. 6, 44–50 (1969).
7. M. P. Petrenko, "Forced vibrations of a liquid and the elastic bottom of a cylindrical tank," *Prikl. Mekh.,* **6**, No. 6, 127–131 (1970).
8. A. Yu. Karnaukh and N. K. Didok, "Natural vibrations of the elastic bottom of a cylindrical vessel and liquid with free surface," in: *Proc. of the Institute of Applied Mathematics and Mechanics, Ukrainian National Academy of Sciences,* **20** (2010), pp. 102–108.
9. Yu. N. Kononov, V. P. Shevchenko, and Yu. A. Dzhukha, "Axially symmetric vibrations of a two-layer ideal liquid with free surface in a rigid cylindrical vessel with elastic bottom," *Visn. Don. Nats. Univ., Ser. A,* No. 1-2, 116–125 (2015).
10. Ngo Zui Kan, "Motion of immiscible liquids in a vessel with plane elastic bottom," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela,* No. 5, 151–163 (1979).
11. V. D. Lakiza, "Investigation of dynamic processes in a rigid cylindrical vessel with elastic bottom partially filled with liquid," *Prikl. Mekh.,* **42**, No. 11, 114–120 (2006).
12. N. K. Didok and Yu. N. Kononov, "Dynamics and stability of vibrations of a cylindrical vessel with ideal liquid and elastic bases," in: *Proc. of the Institute of Applied Mathematics and Mechanics, Ukrainian National Academy of Sciences,* **27** (2013), pp. 122–131.

13. N. D. Kopachevskii, S. N. Krein, and Ngo Zui Kan, *Operator Methods in Linear Hydrodynamics: Evolutionary and Spectral Problems* [in Russian], Nauka, Moscow (1989).
14. Yu. N. Kononov, V. F. Rusakov, and Yu. A. Dzhukha, "Axisymmetric vibrations of elastic bases and ideal liquid in a rigid cylindrical vessel," *Visn. Zaporiz. Nats. Univ., Ser. Fiz.-Mat. Nauk,* No. 2, 105–114 (2015).
15. Yu. N. Kononov and Yu. A. Dzhukha, "Axisymmetric vibrations of the elastic bases and ideal liquid in a rigid ring cylindrical vessel," *Visn. Zaporiz. Nats. Univ., Ser. Fiz.-Mat. Nauk,* No. 1, 103–115 (2016).
16. Yu. N. Kononov, V. P. Shevchenko, and Yu. O. Dzhukha, "Axisymmetric vibrations of elastic ring bases and a two-layer ideal liquid in a rigid ring cylindrical vessel," *Mat. Met. Fiz.-Mekh. Polya,* **60**, No. 1, 85–95 (2017).
17. D. A. Goncharov, "Dynamics of two-layer liquid separated by an elastic membrane with regard for the forces of surface tension ," Nauka Obrazov., No. 11 (2013). http://technomag.bmstu.ru/doc/619258.html.
18. A. A. Pozhalostin and D. A. Goncharov, "Free axisymmetric vibrations of a two-layer liquid with rigid separator between layers in the presence of the forces of surface tension ," *Inzh. Zh. Nauka Innovats.,* No. 12 (2013); http://engjournal.ru/ catalog/eng/teormach/1147.html.
19. D. A. Goncharov and A. A. Pozhalostin, "Symmetric vibrations of a liquid in a vessel with a separator and an elastic bottom," *J. Phys.: Conf. Series,* Vol. 991 (2018). http://iopscience.iop.org/article/10.1088/1742-6596/991/1/012027/pdf.
20. A. A. Pozhalostin and D. A. Goncharov, "Free axisymmetric oscillations of a two-layer liquid with an elastic separator between layers," *Russ. Aeronaut.,* **58**, No. 1, 37–41 (2015).
21. P. Tong, "Liquid motion in a circular cylindrical container with a flexible bottom," *AIAA J.,* **5**, No. 10, 1842–1848 (1967).
22. H. F. Bauer, S. Chang, and J. T. S. Wang, "Nonlinear liquid motion in a longitudinally excited container with elastic bottom," *AIAA J.,* **9**, No. 12, 2333–2339 (1971).
23. M. Chiba, "Nonlinear hydroelastic vibration of a cylindrical tank with an elastic bottom, containing liquid. Part II: Linear axisymmetric vibration analysis," *J. Fluids Structures,* **7**, No. 1, 57–73 (1993).
24. M. Chiba, "Axisymmetric free hydroelastic vibration of a flexural bottom plate in a cylindrical tank supported on an elastic foundation," *J. Sound Vibrat.,* **169**, No. 3, 387–394 (1994).
25. H. F. Bauer, "Coupled frequencies of a liquid in a circular cylindrical container with elastic liquid surface cover," *J. Sound Vibrat.,* **180**, No. 5, 689–704 (1995).
26. S. Tariverdilo, M. Shahmardani, J. Mirzapour, and R. Shabani, "Asymmetric free vibration of circular plate in contact with incompressible fluid," *Appl. Math. Model.,* **37**, No. 1-2, 228–239 (2013).
27. K.-H. Jeong, "Free vibration of two identical circular plates coupled with bounded fluid," *J. Sound Vibrat.,* **260**, No. 4, 653–670 (2003).