DOI 10.1007/s10958-020-04738-9

Journal of Mathematical Sciences, Vol. 246, No. 2, April, 2020

ALGEBRAS OF LIPSCHITZ-ANALYTIC MAPS

M. V. Martsinkiv

UDC 517.98

We study the class of Lipschitz-analytic maps. The relationships between the classes of Lipschitz, Lipschitz-polynomial, and Lipschitz-analytic maps are clarified. We consider the algebras of Lipschitz-analytic functions and, in particular, establish some estimates for the bounds of the set of characters and the properties of these algebras.

Introduction

We consider nonempty metric spaces X and Y and a Lipschitz map $f: X \to Y$. In the metric space X, we fix a point θ_X. The space of all Lipschitz maps from the metric space X with a fixed point θ_X into the metric space Y with a fixed point θ_Y that map θ_X into θ_Y is denoted by $\mathrm{Lip}_0(X,Y)$. If Y is a linear space, then $\theta_Y = 0$. It is known that any metric space X is normed with respect to the norm

$$\alpha(x) = \rho(\theta_X, x),$$

where the function $\alpha(x)$ satisfies the condition

$$|\alpha(x_1) - \alpha(x_2)| \le \rho(x_1, x_2) \le \alpha(x_1) + \alpha(x_2)$$

for any elements $x_1, x_2 \in X$. This space (X, α) is called a *space with marked point* or a *normed set*. In [2], Pestov proved that, for any metric space X with a marked point θ_X and the norm $\alpha(x)$, there exists a unique (to within isometric isomorphisms) Banach space $B(X)$ such that the metric space X can be embedded in the Banach space $B(X)$, every map $f(x) \in \mathrm{Lip}_0(X,E)$ can be extended to a linear operator

$$\tilde{f}(x):\ B(X) \to E$$

for any normed space E, and moreover, $\|\tilde{f}\| = L_f$. The space $B(X)$ is called a *free Banach space*. Note that, in view of the construction of the space $B(X)$, the elements of the form $\sum_{k=1} \lambda_k \underline{x_k}$, where $\underline{x_k}$ are elements of the linear span of the space X, are dense in $B(X)$. The general theory of Lipschitz maps can be found in the monographs [7, 16]. The theory of Lipschitz maps based on the use of free Banach spaces and the properties of these spaces are described in [3, 4, 8, 10, 12].

Stefanyk Pre-Carpathian National University, Ivano-Frankivs'k, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 60, No. 3, pp. 138–144, August–October, 2017. Original article submitted October 19, 2016.

 1072-3374/20/2462–0284

A class $\mathcal{F}(X,Y)$ of nonlinear mappings from X into Y *admits global linearization* if there exist a linear space $W(X)$ and an injective mapping $U_{\mathcal{F}(X,Y)}\colon X \to W(X)$ such that, for any $F \in \mathcal{F}(X,Y)$, there exists a linear operator $L_F \in \mathcal{L}(W(X), Y)$ for which the diagram

$$\begin{array}{ccc} X & \overset{U_{\mathcal{F}(X,Y)}}{\to} & W(X) \\ {\scriptstyle F}\downarrow & & \swarrow {\scriptstyle L_F} \\ Y & & \end{array}$$

is commutative. In the case of Lipschitz maps, the free Banach space $B(X)$ and a map $\upsilon\colon X \to B(X)$, $\upsilon(x) = \underline{x}$, specify the linearization of nonlinear functions from the class $\mathrm{Lip}_0(X,E)$.

In the present work, we continue our investigation of the Lipschitz maps performed with the use of free Banach spaces. In particular, we consider a notion broader than the notion of Lipschitz maps, namely, the notion of Lipschitz-analytic maps.

Consider metric spaces X and Y. By $\mathbb{P}({}^n X,Y)$ we denote a set of maps from X into Y such that, for any map $F \in \mathbb{P}({}^n X,Y)$, there exists an n-homogeneous continuous polynomial $P_F \in \mathcal{P}({}^n B(X), B(Y))$ for which $P_F(\underline{x}) = \underline{F(x)}$ for any $x \in X$. In other words, $F \in \mathbb{P}({}^n X,Y)$ if the diagram

$$\begin{array}{ccc} X & \overset{F}{\to} & Y \\ \upsilon\downarrow & & \downarrow\upsilon \\ B(X) & \overset{P_F}{\to} & B(Y) \end{array}$$

is commutative for some $P_F \in \mathcal{P}\big({}^n B(X), B(Y)\big)$.

The elements from the class $\mathbb{P}({}^n X,Y)$ are called *n-homogeneous Lipschitz-polynomial maps* from the space X into the space Y. Some properties of maps from this class are described in [1].

Let X be a normed set and let E be a normed space. The map $F\colon X \to E$ is called *Lipschitz-analytic* if there exists a map $\tilde{F}\colon B(X) \to E$, $\tilde{F} \in H(B(X),E)$ such that $F(x) = \tilde{F}(\underline{x})$. If $\tilde{F} \in H(B(X),E)$, i.e., $\tilde{F}$ is a map of bounded type (bounded on bounded sets), then we say that F is a *Lipschitz-analytic map of bounded type*. The set of all Lipschitz-analytic maps (resp., of bounded type) from X into E is denoted by $\mathbb{H}(X,E)$ (resp., $\mathbb{H}_b(X,E)$).

Main Results

Proposition 1. *Let X be a discrete normed set. Then* $\mathbb{H}_b(X,E) = \mathrm{Lip}(X,E)$.

Proof. It is clear that $\mathbb{H}_b(X,E) \supset \mathrm{Lip}(X,E)$. It is sufficient to prove that $\mathrm{Lip}_0(X,E)$ coincides with the subspace of mappings f from $\mathbb{H}_b(X,E)$ for which $f(\theta) = 0$. Let $f \in \mathbb{H}_b(X,E)$ and $f(\theta) = 0$.

Then the restriction of f to $\underline{X}$ is a bounded mapping. In [8], it was proved that the restriction of f to $\underline{X}$ belongs to the class $\mathrm{Lip}_0(\underline{X}, E)$. Thus, $F = f \circ v \in \mathrm{Lip}_0(X, E)$ and $\tilde{F} = f$.

The following example shows that $\mathbb{H}(X, E) \neq \mathrm{Lip}(X, S)$ even in the case of a discrete normed set:

Example 1. Let $X = \mathbb{Z}_+$ with discrete metric. The space $\underline{X}$ can be identified with the standard basis $\{e_n\} \subset \ell_1$, $e_n = \underline{n}$. In this case, $B(X) = \ell_1$. On $B(X)$, we define an analytic function

$$f\left(\sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{\infty} (2a_n)^n .$$

It is known (see, e.g., [13]) that $f \in H(\ell_1) = H(\ell_1, \mathbb{C})$ but $f \notin H_b(\ell_1)$. On the other hand, $f(e_n) = 2^n$, i.e., f is an unbounded function on X. Therefore, $f \notin \mathrm{Lip}_0(X)$ and, thus, $\mathbb{H}(X) \neq \mathrm{Lip}(X)$.

Every linear operator in a Banach space is a Lipschitz map. However, the polynomial operators are not Lipschitz (if the degree of a polynomial is greater than 1) but are Lipschitz-polynomial. Similarly, the class of Lipschitz-analytic maps in a Banach space differs from the class of Lipschitz-polynomial maps. Thus, in the general case,

$$\mathrm{Lip}(X, E) \subset \mathbb{P}(X, E) \subset \mathbb{H}_b(X, E) \subset H(X, E).$$

Theorem 1. *Let X be a normed set.*

(1°) There exist a complex locally convex topological vector space G_X and an embedding $\tau_G : X \to G_X$ such that $G'_X = H(B(X))$ and, for any Lipschitz-analytic function $f \in \mathbb{H}(X)$, there exists a linear functional $\varphi_f : G_X \to \mathbb{C}$ such that $f(x) = \varphi_f(\tau_G(x))$ for every $x \in X$.

(2°) There exist a complex locally convex topological vector space Q_x obtained as the inductive limit of Banach spaces and an embedding $\tau_Q : X \to Q_x$ such that $Q'_X = H_b(B(X))$ and, for any complex normed space E and a map $F \in \mathbb{H}(X, E)$, there exists a linear operator $A_f : Q_x \to E$ such that $A_F(x) = F(\tau_Q(x))$ for any $x \in X$.

Proof. (**1°**) According to [14], for any complex Banach space U, there exist a locally convex space $G(U)$ and an embedding $\delta : U \to G(U)$ such that $G(U)' = H(U)$ and, for any function $g \in H(U)$, one can find a linear functional $\psi_g : G(U) \to \mathbb{C}$ for which the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbb{C} \\ \delta\downarrow & & \nearrow\psi_g \\ & G(U) & \end{array}$$

is commutative.

Assume that $U = B(X)$ and $G_X := G(U) = G(B(X))$. For any function $\tilde{f} \in \mathbb{H}(X)$, there exists a function $\tilde{f} \in H(B(X))$ such that $f(x) = \tilde{f}(v(x))$. We set

$$\tau_G(x) := \delta(v(x)) = \delta(\underline{x}) .$$

Thus, $\varphi_f := g_{\tilde{f}}$.

(**2°**) In [11], it was shown that, for any complex Banach space U, there exist a locally convex LB*-space* (i.e., the inductive limit of Banach spaces) $Q(U)$ and an embedding $e\colon U \to Q(U)$ such that $(Q(U))' = H_b(U)$ and, for any normed space E and every mapping $J \in H_b(U,E)$, one can find a linear operator $\mathcal{L}_j\colon Q(U) \to E$ for which the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ {}^{e}\downarrow & & \nearrow \mathcal{L}_j \\ & Q(U) & \end{array}$$

is commutative.

Setting $U = B(X)$, we get

$$Q_x := Q(B(X)), \quad \tau_Q = e(v(x)), \text{ and } \quad A_F \equiv \mathcal{L}_{\tilde{F}},$$

where $\tilde{F}$ is an analytic map from $H_b\big(B(X),E\big)$ such that $F(x) = \tilde{F}(\underline{x})$ for any $x \in X$.

The theorem is proved.

The space X is called a *Fréchet algebra* if X is a Fréchet space, i.e., a locally convex metrizable space with an associative operation of multiplication. A topology in the space X can be defined by a countable system of seminorms p_i, i.e., $p_i(xy) \le p_i(x)p_i(y)$ for any $x, y \in X$, $1 \le i \le \infty$.

Let X be a normed set. We consider the space of Lipschitz-analytic functions in $\mathbb{H}_b(X) = \mathbb{H}(X,\mathbb{C})$. Note that $\mathbb{H}_b(X)$ is an algebra with respect to pointwise multiplication. For every function $f \in \mathbb{H}(X)$, by $\tilde{f}$ we denote a function from $H_b(B(X))$ such that $\tilde{f}(\underline{x}) = f(x)$. According to the definition of $\mathbb{H}_b(X)$, this function exists. However, generally speaking, it is not unique.

Proposition 2. *The mapping* $j\colon \tilde{f} \mapsto f$ *is a continuous homomorphism of algebras.*

Proof. The proof follows from the fact that j is an operator of restriction to $\underline{X} \subset B(X)$.

Corollary 1. *The set of elements* $\ker j$ *is an ideal of the algebra* $H_b(B(X))$ *and the algebra* $\mathbb{H}_b(X)$ *is isomorphic to* $H_b(B(X))/\ker j$.

Proof. The proof follows from the general theory of topological algebras.

This corollary, in particular, implies that, in view of the fact that $H_b(B(X))$ is a Fréchet algebra and the set $\ker j$ is a closed ideal, $\mathbb{H}_b(X)$ is also a Fréchet algebra. Since the set of characters $\mathcal{M}(A) \subset A^*$, we consider a $*$-weak topology induced from A^* on $\mathcal{M}(A)$.

Corollary 2. *The set of characters* $\mathcal{M}(\mathbb{H}_b(X))$ *of the algebra* $\mathbb{H}_b(X)$ *is a subset of* $\mathcal{M}(B(X))$ *formed by the characters* $\varphi \in \mathcal{M}(B(X))$ *such that* $\varphi(f) = 0$ *for every* $f \in \ker j$.

We recall that the *Stone–Čech compactification* βX of a metric space X is defined as a compact topological space that densely contains X with the property that every function f continuously bounded on X can be extended to a continuous function $\overline{f}$ on βX.

Proposition 3. *Let* X *be a normed discrete set. Then there exists a homeomorphism* $x \mapsto \varphi_x$ *from* βX into $\mathcal{M}(\mathbb{H}_b(X))$ *such that* $\varphi_x(f) = \overline{f}(x)$ *for any* $f \in \mathbb{H}_b(X)$.

Proof. It is clear that φ_x is a character for every $x \in X$. Note that, for a discrete set X, $H_b(X)$ coincides with the algebra of Lipschitz functions (with respect to pointwise multiplication) $\mathrm{Lip}_0(X)$; moreover, this algebra is isomorphic to the algebra $\ell_\infty(X)$ [8], which is, in this case, an algebra of all bounded continuous functions on X. Therefore, the set of characters of this algebra coincides, to within a homeomorphism $x \mapsto \varphi_x$, with the Stone–Čech compactification βX of the set X [9].

For the Banach space Z, by $H_{bc}(Z)$ we denote the smallest closed subalgebra in $H_b(Z)$ that contains all linear functionals on Z, their products, and their sums. By $H_{bw}(Z)$ we denote the subalgebra formed by the functions from $H_b(Z)$ that are weakly continuous on bounded sets. It is clear that $H_{bc} \subset H_{bw}(Z)$. For the normed set X, by $\mathbb{H}_{bc}(X)$ and $\mathbb{H}_{bw}(X)$ we denote the restrictions to $\underline{X} \subset B(X)$ of $H_{bc}(B(X))$ and $H_{bw}(B(X))$, respectively.

Example 2. Let X be a sequence from ℓ_1 of the form

$$X = \{me_m\}_{m=1}^{\infty} \cup \{0\}$$

with a metric induced from ℓ_1. In [1], it was shown that $B(X) = \ell_1$. Let $P \in \mathbb{P}(^n X)$ and let $\tilde{P} \in \mathcal{P}(^n B(X))$ be the corresponding n-homogeneous polynomial on $B(X)$ such that $\tilde{P}(\underline{x}) = P(x)$, $x \in X$. It is clear that $\tilde{P}$ is not unique.

The equality

$$P(\underline{m}) = \tilde{P}(me_m) = m^n \tilde{P}(e_m)$$

specifies the value of $\tilde{P}$ on the basis vectors e_m. In other words, both $\tilde{Q}$ and $\tilde{P} \in \mathcal{P}(^n B(X))$ specify the same element $P \in \mathbb{P}(^n X)$ $\big($we write $\tilde{Q} \sim \tilde{P}$ or $[\tilde{Q}] \sim [\tilde{P}]\big)$ if $\tilde{P}(e_m) = \tilde{Q}(e_m)$ and both polynomials $\tilde{P}$ and $\tilde{Q}$ are n-homogeneous on $B(X) = \ell_1$.

In particular, the diagonal polynomial

$$\widetilde{P_0}\left(\sum_m c_m e_m\right) := \sum_m c_m^n P(e_m)$$

specifies P and two different diagonal polynomials specify different elements from $\mathbb{P}(^n X)$. Thus, the space $\mathbb{P}(^n X)$ is isomorphic to the space of diagonal n-homogeneous polynomials on ℓ_1, i.e., polynomials of the form

$$\tilde{P}\left(\sum_{m=1}^{\infty} c_m e_m\right) = \sum_{m=1}^{\infty} c_m^n \bar{P}(e_m).$$

If $\tilde{P} \in \mathcal{P}(^n X)$ and $\tilde{Q} \in \mathcal{P}(^n X)$, then the pointwise product of the algebra $H_b(\ell_1)$ under the mapping $\tilde{P} \mapsto [\tilde{P}]$ turns into the product

$$[\tilde{P}][\tilde{Q}] = [\tilde{R}], \qquad \tilde{R}\left(\sum_m c_m e_m\right) = \sum_m c_m^{n+k} \tilde{P}(e_m)\tilde{Q}(e_m).$$

Let $f \in \mathbb{H}_b(X)$. Then there exists a function $f \in H_b(\ell_1)$ such that $f(\underline{x}) = f(x)$, $x \in X$, and $\tilde{f}$ takes the diagonal form, i.e.,

$$\tilde{f}(u) = \sum_{m=0}^{\infty} \tilde{P}_m(u) = \sum_{m=0}^{\infty} \tilde{P}_m\left(\sum_{n=1}^{\infty} c_n e_n\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_n^m \tilde{P}_m(e_n) = \sum_{m=0}^{\infty} c_n^m \tilde{P}_m(e_n) = \sum_{n=1}^{\infty} f_n(c_n),$$

where f_n are entire functions of one variable,

$$f_n(t) = \sum_{m=0}^{\infty} t^m \tilde{P}_m(e_n),$$

and $\{f_n(1)\} = \{\tilde{f}(e_n)\}$ is a bounded sequence. Therefore, for any ultrafilter U on the set of natural numbers, the functional $\varphi_U(f) = \lim_U \tilde{f}(e_n)$ is a character of the algebra $\mathbb{H}_b(X)$, i.e., $\mathcal{M}(\mathbb{H}_b(X)) \supset \beta\mathbb{N}$. On the other hand, if $\{b_n\}$ is a bounded sequence, then $\lim_U \tilde{f}(b_n e_n)$ is a also a character. Thus, $\mathcal{M}(\mathbb{H}_b(X))$ does not coincide with $\beta\mathbb{N}$.

Hence, the set of characters of the algebra $\mathbb{H}_b(X)$ can be complicated even for simple spaces X.

We now recall that any continuous n-linear mapping $B\colon X\times\ldots\times X\to\mathbb{C}$ can be extended to a continuous n-linear mapping $\tilde{B}\colon X''\times\ldots\times X''\to\mathbb{C}$ as follows:

$$\tilde{B}(x_1'',\ldots,x_n'') = \lim_{\alpha_1}\ldots\lim_{\alpha_n} B\big(x_{\alpha_1},\ldots,x_{\alpha_n}\big).$$

Here, for any k, (x_{α_k}) are the nets in X convergent in the $*$-weak topology to x_k''.

Let $P\in\mathcal{P}(^nB(X))$ and let B be an n-linear mapping associated with P. Then the *Aron–Berner extension* $\tilde{P}$ of a polynomial P is denoted by $\tilde{P}:=\tilde{B}(x,\ldots,x)$ [5]. The generalizations of the Aron–Berner extension were described in [15]. A Banach space X possesses a *property of bounded approximation* if, for any compact set $K\subset X$ and any $\varepsilon>0$, there exists a bounded operator with finite image $T\colon X\to X$ such that $Tx-x<\varepsilon$ for all $x\in K$.

The following assertion establishes certain bounds for the set $\mathcal{M}\big(\mathbb{H}_{bc}(X)\big)$:

Theorem 2. *Let X be a complex Banach space.*

(1°) Every character of the algebra $\mathbb{H}_{bc}(X)$ has the form $\varphi_z(f)=\tilde{f}_{AB}(z)$, where $f\in H_b(X)$, $\tilde{f}_{AB}$ is the Aron–Berner extension of the function $\tilde{f}\in H_{bc}(B(X))$ to an element of the space $(B(X))''$ and $\tilde{f}(\underline{x})=f(x)$. In this sense, $\mathcal{M}(\mathbb{H}_{bc}(X))\subset(B(X))''$. Moreover, $\mathcal{M}(\mathbb{H}_{bc}(X))\supset\underline{X}$.

(2°) If X has a property of bounded approximation, then $\mathcal{M}(\mathbb{H}_{bw}(X))=\mathcal{M}(\mathbb{H}_{bc}(X))$.

Proof. (**1°**) It is well known (see [2]) that $\mathcal{M}(\mathbb{H}_{bc}(E))=E''$ for any Banach space E. Since $\mathbb{H}_{bc}(X)$ is the quotient algebra of $H_{bc}(B(X))$, we conclude that $\mathcal{M}(\mathbb{H}_{bc}(X))$ is a closed subset in $H_{bc}(B(X))=(B(X))''$. Since the functional $\delta_x\colon f\mapsto f(x)$, $x\in X$, is a character, we get $\mathcal{M}(\mathbb{H}_{bc}(E))\supset\underline{X}$.

(**2°**) According to [6], if the Banach space E has the property of bounded approximation, then

$$H_{bc}(E) = H_{bw}(E).$$

In [12], Godefroy and Kalton showed that if a Banach space X has a property of bounded approximation, then $B(X)$ also has the same property. Combining these results, we complete the proof of the theorem.

Hence, the algebras of Lipschitz-analytic maps obtained as a generalization of Lipschitz functions have common properties with the algebras of Lipschitz map but are characterized by a more complicated structure. Their construction is also more complicated.

REFERENCES

1. M. V. Dubei and A. V. Zagorodnyuk, "Linearization of Lipschitz-polynomial and Lipschitz-analytic mappings," *Karpat. Mat. Publ.*, **3**, No. 1 40–48 (2011).
2. V. G. Pestov, "Free Banach spaces and representation of topological groups," *Funkts. Anal. Prilozh.*, **20**, No. 1 81–82 (1986); ***English translation:*** *Funct. Anal. Appl.*, **20**, No. 1, 70–72 (1986), https://doi.org/10.1007/BF01077324.

3. D. A. Raikov, “Free locally convex subspaces of uniform spaces,” *Mat. Sbornik*, **63**, No. 4 582–590 (1964).
4. R. F. Arens and J. Eells, jr., “On embedding uniform and topological spaces,” *Pacific J. Math.*, **6**, No. 3, 397–403 (1956).
5. R. M. Aron and P. D. Berner, “A Hahn–Banach extension theorem for analytic mappings,” *Bull. Soc. Math. France*, **106**, 3–24 (1978).
6. R. M. Aron and J. B. Prolla, “Polynomial approximation of differentiable functions on Banach spaces,” *J. Reine Angew. Math.*, **1980**, No. 313, 195–216 (1980), https://doi.org/10.1515/crll.1980.313.195.
7. Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. 1, American Mathematical Society, Providence, RI (2000).
8. M. Dubei, E. D. Tymchatyn, and A. Zagorodnyuk, “Free Banach spaces and extension of Lipschitz maps,” *Topology*, **48**, Nos. 2-4, 203–212 (2009), https://doi.org/10.1016/j.top.2009.11.020.
9. R. Engelking, *General Topology*, PWN, Warsaw (1977).
10. J. Flood, *Free Topological Vector Spaces*, PhD Thesis, Australian National University, Canberra (1975).
11. P. Galindo, D. Garcia, and M. Maestre, “Holomorphic mappings of bounded type,” *J. Math. Anal. Appl.*, **166**, No. 1, 236–246 (1992), https://doi.org/10.1016/0022–247X(92)90339-F.
12. G. Godefroy and N. J. Kalton, “Lipschitz-free Banach spaces,” *Studia Math.*, **159**, No. 1, 121–141 (2003), https://doi.org/10.4064/sm159-1-6.
13. J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland, Amsterdam (1986).
14. J. Mujica and L. Nachbin, “Linearization of holomorphic mappings on locally convex spaces,” *J. Math. Pure Appl.*, **71**, 543–560 (1992).
15. O. Taras and A. Zagorodnyuk, “A generalization of the Arens extension for Banach algebras,” *Indagat. Math. New Ser.*, **26**, No. 2, 324–328 (2015), https://doi.org/10.1016/j.indag.2014.10.003.
16. N. Weaver, *Lipschitz Algebras*, World Scientific, Singapore (1999), https://doi.org/10.1142/4100.