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On the functional properties of weak (p, q)-quasiconformal homeomorphisms

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The paper is dedicated to the 100th anniversary of G. D. Suvorov

Abstract. We study the functional properties of weak (p, q)-quasiconformal homeomorphisms such as Liouville-type theorems, the global integrability, and the Hölder continuity. The proof of Liouville-type theorems is based on the duality property of weak (p, q)-quasiconformal homeomorphisms.

Keywords. Quasiconformal mappings, Sobolev spaces.

1. Introduction

Various generalizations of quasiconformal mappings such as mappings of the Dirichlet class [23] or mappings quasiconformal in mean, (see, e.g., [15, 16, 21, 25]) play an important role in the geometric function theory. In the present article, we study the functional properties of generalized quasiconformal mappings which are connected with composition operators on Sobolev spaces [7, 26, 32]. Recall that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is called a weak (p, q)-quasiconformal homeomorphism, $1 \leq q \leq p \leq \infty$, if $\varphi \in W^1_{q,\mathrm{loc}}(\Omega)$, has finite distortion and

$$K_{p,q}(\varphi; \Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty, \ 1/q - 1/p = 1/\kappa \ (\kappa = \infty, if p = q),$$

where the p-dilatation of a mapping φ at a point x is defined as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x, \varphi|^{\frac{1}{p}}, \ x \in \Omega\}.$$

In the case $p = n$, we have the usual conformal dilatation. If $p = q = n$, this class coincides with quasiconformal mappings. In the case where $p \neq n$, the p-dilatation arose in [4]. The weak (p, q)-quasiconformal homeomorphisms are a natural generalization of (quasi)conformal mappings and have applications in the theory of elliptic operators and in elasticity theory. These applications are based on the composition operators on Sobolev spaces generated by weak (p, q)-quasiconformal homeomorphisms. Note that, for $p = q = n$, a class of weak (p, q)-quasiconformal homeomorphisms coincides with the usual quasiconformal mappings. In the case where $p = n$ and $q = n - 1$, these mappings are mappings of integrable distortions that were considered in [12, 14]. Weak quasiconformal homeomorphisms allow a capacitary description and, on this way, are closely connected with the so-called Q-homeomorphisms [7]. The studies of Q-homeomorphisms are based on the capacitary (moduli) distortion of these classes and are intensively developed for the last decades (see, e.g., [15, 22]).

In the theory of weak (p, q)-quasiconformal homeomorphisms, the significant role is played by the composition duality property [26]. Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p, q)-quasiconformal homeomorphism, $n - 1 < q \leq p < \infty$, that induces the bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ n - 1 < q \leq p < \infty.$$

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Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ is a weak (q', p')-quasiconformal homeomorphism, where $p' = p/(p - n + 1)$, $q' = q/(q - n + 1)$.

Using this composition duality, we obtain some self-improvement-type theorems for weak quasiconformal homeomorphisms and Liouville-type theorems.

The classical Liouville theorem states that there exists no conformal mapping $\varphi : \mathbb{R}^2 \to \widetilde{\Omega}$ onto any bounded domain $\widetilde{\Omega} \subset \mathbb{R}^2$, and, in the space $\mathbb{R}^n$, $n \geq 3$, the class of conformal mappings coincides with the Möbius group of transformations. Below, we will prove, in particular, that there exists no weak (p, q)-quasiconformal homeomorphism, $n - 1 < q < p \leq n$, $\varphi : \mathbb{R}^n \to \widetilde{\Omega}$ onto any domain of finite measure $\widetilde{\Omega} \subset \mathbb{R}^n$, $n \geq 2$.

Note that, for $p = n$, the Liouville-type theorem was proved in [9]. In capacitary terms, the Liouville-type theorems for the mappings of a bounded (p, q)-distortion were obtained in [28].

The global L_p-integrability of the weak derivatives of quasiconformal mappings and their Hölder continuity represent an interesting part of the quasiconformal mapping theory [1,5,18]. In the second part of the present paper, we prove the property of global integrability of the weak derivatives of weak (p, q)-quasiconformal mappings and obtain a self-improvement-type theorem.

2. Composition operators on Sobolev spaces

2.1. Sobolev spaces

Let E be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. The Lebesgue space $L_p(E)$, $1 \leq p \leq \infty$, is defined as a Banach space of p-summable functions $f : E \to \mathbb{R}$ equipped with the standard norm.

If Ω is an open subset of $\mathbb{R}^n$, the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|.$$

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

Sobolev spaces are Banach spaces of equivalence classes [20]. To clarify the notion of equivalence classes, we use the non-linear p-capacity associated with Sobolev spaces. Recall the notion of the p-capacity of a set $E \subset \Omega$ [20]. Let Ω be a domain in $\mathbb{R}^n$, and let a compact $F \subset \Omega$. The p-capacity of the compact F is defined by

$$\operatorname{cap}_p(F; \Omega) = \inf\{\|f | L^1_p(\Omega)\|^p,$$

where the infimum is taken over all continuous functions with a compact support $f \in L^1_p(\Omega)$ such that $f \geq 1$ on F. In a similar way, we can define the p-capacity of open sets.

For an arbitrary set $E \subset \Omega$, we define an inner p-capacity as

$$\underline{\operatorname{cap}}_p(E; \Omega) = \sup\{\operatorname{cap}_p(e; \Omega),\ e \subset E \subset \Omega,\ e \text{ is a compact set}\},$$

and an outer p-capacity as

$$\overline{\operatorname{cap}}_p(E; \Omega) = \inf\{\operatorname{cap}_p(U; \Omega),\ E \subset U \subset \Omega,\ U \text{ is an open set}\}.$$

A set $E \subset \Omega$ is called p-capacity measurable, if $\underline{\operatorname{cap}}_p(E; \Omega) = \overline{\operatorname{cap}}_p(E; \Omega)$. The value

$$\operatorname{cap}_p(E; \Omega) = \underline{\operatorname{cap}}_p(E; \Omega) = \overline{\operatorname{cap}}_p(E; \Omega)$$

is called the p-capacity of the set $E \subset \Omega$.

The notion of p-capacity allows us to refine the notion of Sobolev functions. Let a function $f \in L^1_p(\Omega)$. Then the refined function

$$\tilde{f}(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\ dy$$

is defined quasieverywhere, i.e., up to a set of p-capacity zero, and it is absolutely continuous on almost all lines [20]. This refined function $\tilde{f} \in L^1_p(\Omega)$ is called a unique quasicontinuous representation (*a canonical representation*) of the function $f \in L^1_p(\Omega)$. Recall that a function $\tilde{f}$ is called quasicontinuous, if, for any $\varepsilon > 0$, there is an open set U_ε such that the p-capacity of U_ε is less than ε, and the function $\tilde{f}$ is continuous on the set $\Omega \setminus U_\varepsilon$ (see, e.g., [11, 20]). In what follows, we will use the quasicontinuous (refined) functions only.

Note that the first weak derivatives of the function f coincide almost everywhere with the usual partial derivatives (see, e.g., [20]).

2.2. Composition operators and the composition duality property

Let $\varphi : \Omega \to \mathbb{R}^n$ be a weakly differentiable mapping. Then the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x,\varphi)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ is the operator norm of $D\varphi(x)$, i.e., $|D\varphi(x)| = \max\{D\varphi(x) \cdot h : h \in \mathbb{R}^n, |h| = 1\}$. Recall that a weakly differentiable mapping $\varphi : \Omega \to \mathbb{R}^n$ is a mapping of finite distortion, if $D\varphi(x) = 0$ for almost all $x \in Z = \{x \in \Omega : J(x,\varphi)\} = 0\}$ [30].

Let us recall also the change of variables formula for the Lebesgue integral [3, 10]. Suppose that, for a mapping $\varphi : \Omega \to \mathbb{R}^n$, there exists a collection of closed sets $\{A_k\}_1^\infty$, $A_k \subset A_{k+1} \subset \Omega$ for which restrictions $\varphi|_{A_k}$ are Lipschitz mappings on the sets A_k and

$$\left| \Omega \setminus \sum_{k=1}^{\infty} A_k \right| = 0.$$

Then there exists a measurable set $S \subset \Omega$, $|S| = 0$ such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^n$ has the Luzin N-property and the change of variables formula

$$\int\limits_{E} f \circ \varphi(x) |J(x,\varphi)|\ dx = \int\limits_{\mathbb{R}^n \setminus \varphi(S)} f(y) N_f(E,y)\ dy \tag{2.1}$$

holds for every measurable set $E \subset \Omega$ and every nonnegative measurable function $f : \mathbb{R}^n \to \mathbb{R}$. Here, $N_f(y,E)$ is the multiplicity function defined as the number of preimages of y under f in E.

Note that Sobolev mappings of the class $W^1_{1,\mathrm{loc}}(\Omega)$ satisfy the conditions of the change of variable formula [10]. So, the change of variable formula (2.1) holds for Sobolev mappings.

If the mapping φ possesses the Luzin N-property (the image of a set of measure zero has measure zero), then $|\varphi(S)| = 0$, and the second integral can be rewritten as the integral on $\mathbb{R}^n$. Note that Sobolev homeomorphisms of the class $L^1_p(\Omega)$, $p \geq n$, possess the Luzin N-property.

Let Ω and $\widetilde{\Omega}$ be domains in $\mathbb{R}^n$, $n \geq 2$. We say that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if, for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasieverywhere in Ω, and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leq K_{p,q}(\Omega)\|f \mid L^1_p(\widetilde{\Omega})\|.$$

The problem of composition operators on Sobolev spaces arose firstly in work [19] where the subareal mappings were introduced, and in Reshennyak's problem (1969) related to quasiconformal mappings [29]. In connection with the geometric function theory, we define the p-dilatation of a mapping φ at a point x as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x, \varphi|^{\frac{1}{p}},\ x \in \Omega\}.$$

If $p = n$, we have the usual conformal dilatation. For $p \neq n$, the p-dilatation arose in [4].

The geometric theory of composition operators on Sobolev spaces is based on the measure property of composition operators introduced in [26] (in the limit case $p = \infty$, it was introduced in [27]).

Theorem 2.1. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induce a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1 \leq q < p \leq \infty.$$

Then

$$\Phi(\widetilde{A}) = \sup_{f \in L^1_p(\widetilde{A}) \cap C_0(\widetilde{A})} \left(\frac{\|\varphi^*(f) \mid L^1_q(\Omega)\|}{\|f \mid L^1_p(\widetilde{A})\|} \right)^{\kappa},$$

(where $1/q - 1/p = 1/\kappa$) is a bounded monotone countably additive set function defined on open bounded subsets $\widetilde{A} \subset \widetilde{\Omega}$.

The following theorem allows one to refine this function Φ as a measure generated by the p-dilatation K_p.

Theorem 2.2. *A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ 1 \leq q \leq p \leq \infty,$$

if and only if $\varphi \in W^1_{1,\mathrm{loc}}(\Omega)$ has finite distortion, and

$$K_{p,q}(\varphi; \Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,$$

where $1/q - 1/p = 1/\kappa$.

This theorem was proved in the case where $1 \leq q = p < \infty$ in [7] and in the case where $1 \leq q < p < \infty$ in [26] (see also [32]); the case $p = \infty$ was considered in [9]. Homeomorphisms that satisfy the conditions of Theorem 2.2 are called weak (p, q)-quasiconformal homeomorphisms [7, 31] and are a natural generalization of quasiconformal mappings ($p = q = n$).

In the geometric theory of composition operators on Sobolev spaces, the significant role is played by the following composition duality property [26]:

Theorem 2.3. *Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, induce a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega),\ \ n - 1 < q \leq p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}),\ \ n - 1 < p' \leq q' < \infty,$$

where $p' = p/(p - n + 1)$ and $q' = q/(q - n + 1)$.

For readers' convenience, we recall a short highlight of the proof [26]. On the first step, we need to check that the inverse mapping $\varphi^{-1} \in W^1_{1,\mathrm{loc}}(\widetilde{\Omega})$ [26, Theorem 3]. Because $\varphi^{-1} \in W^1_{1,\mathrm{loc}}(\widetilde{\Omega})$, we have [24] (see also [2,8,13])

$$|D\varphi^{-1}(y)| = \begin{cases} \left(\frac{|\operatorname{adj} D\varphi|(x)}{|J(x,\varphi)|}\right)_{x=\varphi^{-1}(y)} & \text{if} \quad x \in \Omega \setminus (S \cup Z)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$|D\varphi^{-1}(y)| \leq \frac{|D\varphi(x)|^{n-1}}{|J(x,\varphi)|}$$

for almost all $x \in \Omega \setminus (S \cup Z)$, $y = \varphi(x) \in \Omega' \setminus \varphi(S \cup Z)$, and

$$|D\varphi^{-1}(y)| = 0 \text{ for almost all } y \in \varphi(S).$$

Now, taking into account that

$$\frac{q'p'}{q'-p'} = \frac{pq}{(p-q)(n-1)},$$

we obtain

$$\int\limits_{\Omega'} \left(\frac{|D\varphi^{-1}(y)|^{q'}}{|J(y,\varphi^{-1})|}\right)^{p'/(q'-p')} dy \leq \int\limits_{\Omega} \left(\frac{|D\varphi(x)|^{p}}{|J(x,\varphi)|}\right)^{q/(p-q)} dx$$

(in the case $p = q$, we have $p' = q'$ and L_∞-norms instead of integrals). By Theorem 2.2, we have a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \;\; n-1 < p' \leq q' < \infty.$$

Remark 2.4. *For $n = 2$, we have $p' = p/(p-1)$, $q' = q/(q-1)$ and $p'' = p$, $q'' = q$. Hence, the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, induces a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 < q \leq p < \infty,$$

if and only if the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ induces a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \;\; 1 < p' \leq q' < \infty.$$

In the case $n \neq 2$, we have

$$p'' = \left(p'\right)' = \frac{p}{(n-1)^2 - p(n-2)} \neq p, \;\; \text{if } p' > n-1,$$

$$q'' = \left(q'\right)' = \frac{q}{(n-1)^2 - q(n-2)} \neq q, \;\; \text{if } q' > n-1,$$

and this case is more complicated.

Using this composition duality property, we obtain the following self-improvement-type proposition.

Theorem 2.5. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $n-1 < q \leq p < n$. Then φ induces a bounded composition operator*

$$\varphi^* : L^1_r(\widetilde{\Omega}) \to L^1_s(\Omega)$$

for all $s \leq r$ such that $q'' \leq s \leq q$ and $p'' \leq r \leq p$.

Proof. Because $\varphi : \Omega \to \widetilde{\Omega}$ is a weak (p,q)-quasiconformal homeomorphism, the composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; n-1 < q \leq p < \infty,$$

is bounded.

In the case $p < n$ and $q < n$, we have

$$p'' = \big(p'\big)' = \frac{p}{(n-1)^2 - p(n-2)} < p$$

and

$$q'' = \big(q'\big)' = \frac{q}{(n-1)^2 - q(n-2)} < q.$$

So, by the composition duality property, we have that this weak (p,q)-quasiconformal homeomorphism generates also a bounded composition operator

$$\varphi^* : L^1_{p''}(\widetilde{\Omega}) \to L^1_{q''}(\Omega), \;\; 1 < q'' < q \leq p'' < p < \infty.$$

Using the Marcinkiewicz interpolation theorem [6], we obtain that

$$\varphi^* : L^1_r(\widetilde{\Omega}) \to L^1_s(\Omega)$$

is bounded for all $s \leq r$ such that $q'' \leq s \leq q$ and $p'' \leq r \leq p$. □

3. Liouville-type theorems for weak (p,q)-quasiconformal homeomorphisms

The composition duality property allows us to obtain Liouville-type theorems for weak (p,q)-quasiconformal homeomorphisms.

Theorem 3.1. *Let $n < p < \infty$. Suppose that there exists a weak (p,n)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\widetilde{\Omega}| < \infty$.*

Proof. Because $\varphi : \Omega \to \widetilde{\Omega}$ is a weak (p,n)-quasiconformal homeomorphism, the composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_n(\Omega), \;\; n < p < \infty,$$

is bounded. By the duality property, the inverse composition operator

$$\big(\varphi^{-1}\big)^* : L^1_n(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \;\; p' < n,$$

is bounded as well. Hence, for any function $f \in L^1_p(\widetilde{\Omega})$, the inequality

$$\|f \mid L^1_{p'}(\widetilde{\Omega})\| \leq \| \big(\varphi^{-1}\big)^* \| \|\varphi^*(f) \mid L^1_n(\Omega)\| \leq \| \big(\varphi^{-1}\big)^* \| \|\varphi^*\| \|f \mid L^1_p(\widetilde{\Omega})\|$$

holds. This means that the embedding

$$L^1_p(\widetilde{\Omega}) \hookrightarrow L^1_{p'}(\widetilde{\Omega}), \;\; n-1 < p' < p < \infty,$$

holds. Hence, $|\widetilde{\Omega}| < \infty$. □

This theorem immediately yields

Corollary 3.2. *For any $n < p < \infty$ and any domain $\Omega \subset \mathbb{R}^n$, a weak (p,n)-quasiconformal homeomorphism $\varphi : \Omega \to \mathbb{R}^n$ does not exist.*

Remark 3.3. *This corollary can be formulated in the strong form: for any domain $\Omega \subset \mathbb{R}^n$ and any $n < p < \infty$, a weak (p,n)-quasiconformal homeomorphism φ from Ω onto any domain with unbounded volume does not exist.*

In the case $n < q < p$, we have an additional assumption of finiteness of a measure of Ω.

Theorem 3.4. *Let $n < q < p \leq \infty$ and $|\Omega| < \infty$. Suppose that there exists a weak (p,q)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\widetilde{\Omega}| < \infty$.*

Proof. Because $\varphi : \Omega \to \widetilde{\Omega}$ is a weak (p,q)-quasiconformal homeomorphism, the composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n < q < p < \infty,$$

is bounded. By the duality property the inverse composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \ \ n-1 < p' < q' < n,$$

is also bounded. Because $|\Omega| < \infty$, the embedding

$$L^1_q(\Omega) \hookrightarrow L^1_{q'}(\Omega)$$

holds. Hence, the embedding

$$L^1_p(\widetilde{\Omega}) \hookrightarrow L^1_{p'}(\widetilde{\Omega}), \ n-1 < p' < p < \infty,$$

holds. Therefore, $|\widetilde{\Omega}| < \infty$. □

From this theorem, we immediately get

Corollary 3.5. *For any domain $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$, and any $n < p < \infty$, a weak (p,q)-quasiconformal homeomorphism $\varphi : \Omega \to \mathbb{R}^n$ does not exist.*

Remark 3.6. *This corollary can be formulated in the strong form: for any domain $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$ and any $n < p < \infty$, a weak (p,n)-quasiconformal homeomorphism φ from Ω onto any domain with unbounded volume does not exist.*

In the case $n-1 < q < p \leq n$, by using the dual composition property, we obtain dual Liouville-type theorems for weak (p,q)-quasiconformal homeomorphisms.

Theorem 3.7. *Let $n-1 < q < n$. Suppose that there exists a weak (n,q)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\Omega| < \infty$.*

This theorem yields

Corollary 3.8. *For any $n-1 < q < n$ and any domain $\widetilde{\Omega}$, a weak (n,q)-quasiconformal homeomorphism $\varphi : \mathbb{R}^n \to \widetilde{\Omega}$ does not exist.*

In the case $n-1 < q < p < n$, we have an additional assumption of finiteness of a measure of $\widetilde{\Omega}$.

Theorem 3.9. *Let $n-1 < q < p < n$ and $|\widetilde{\Omega}| < \infty$. Suppose that there exists a weak (p,q)-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then $|\Omega| < \infty$.*

From this theorem, we get

Corollary 3.10. *For any domain $\widetilde{\Omega}$ such that $|\widetilde{\Omega}| < \infty$ and for any $n-1 < q < p < n$, a weak (p,q)-quasiconformal homeomorphism $\varphi : \mathbb{R}^n \to \widetilde{\Omega}$ does not exist.*

4. Composition operators and the integrability of derivatives

The global L_p-integrability of the weak derivatives of quasiconformal mappings and their Hölder continuity represent an interesting part of the quasiconformal mapping theory [1, 5, 18]. In the next theorem, we consider the property of global integrability of the weak derivatives of weak (p,q)-quasiconformal mappings.

Theorem 4.1. *Let the homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ induce a bounded composition operator*

$$\varphi^{*}: L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),\ 1\leq q\leq p\leq\infty.$$

If $p\neq n$, then $|D\varphi|^{\frac{p-n}{p}}\in L_{\kappa}(\Omega)$, where $1/q-1/p=1/\kappa$.

Proof. The case $p=q$ was proved in [7] and the case $p=\infty$ was considered in [9].

Let $n<p<\infty$. We denote $Z=\{x\in\Omega: J(x,\varphi)=0\}$. Because φ generates a bounded composition operator

$$\varphi^{*}: L^1_p(\widetilde{\Omega})\to L^1_q(\Omega),$$

φ is a mapping of finite distortion by Theorem 2.2, and $D\varphi(x)=0$ for almost all $x\in Z$. Using Theorem 2.2 and Hadamard's inequality

$$|J(x,\varphi)|\leq|D\varphi(x)|^n,\ \text{for almost all } x\in\Omega\setminus Z,$$

we have

$$\||D\varphi|^{\frac{p-n}{p}}\mid L_{\kappa}(\Omega)\|=\left(\int\limits_{\Omega}|D\varphi(x)|^{\frac{p-n}{p}\frac{pq}{p-q}}\,dx\right)^{\frac{p-q}{pq}}=\left(\int\limits_{\Omega\setminus Z}|D\varphi(x)|^{\frac{p-n}{p}\frac{pq}{p-q}}\,dx\right)^{\frac{p-q}{pq}}$$

$$=\left(\int\limits_{\Omega\setminus Z}\left(|D\varphi(x)|^{p-n}\right)^{\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}\leq\left(\int\limits_{\Omega\setminus Z}\left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}=\|K_p\mid L_{\kappa}(\Omega)\|<\infty.$$

Let $1\leq q<p<n$. Because $Z=\{x\in\Omega: J(x,\varphi)=0\}$, we have $|\varphi(Z)|=0$ by the change of variables formula for weakly differentiable mappings [10]. Since the mapping φ possesses the Luzin N^{-1} property (preimage of a set of a measure zero has measure zero) in the case $1\leq q<p<n$ [31,32], we have $|Z|=0$ and $|J(x,\varphi)|\neq 0$ a.e. in Ω. Hence, by Hadamard's inequality,

$$\||D\varphi|^{\frac{p-n}{p}}\mid L_{\kappa}(\Omega)\|=\left(\int\limits_{\Omega}|D\varphi(x)|^{\frac{p-n}{p}\frac{pq}{p-q}}\,dx\right)^{\frac{p-q}{pq}}=\left(\int\limits_{\Omega\setminus Z}\left(|D\varphi(x)|^{p-n}\right)^{\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}$$

$$\leq\left(\int\limits_{\Omega\setminus Z}\left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}=\|K_p\mid L_{\kappa}(\Omega)\|<\infty.$$

□

Remark 4.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, and let $\varphi:\Omega\to\widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $p\neq n$. Then $|D\varphi|^{\frac{p-n}{p}}\in L_{\alpha}(\Omega)$ for any $\alpha\leq\kappa$.*

In [31], it was proved that the weak (p,q)-quasiconformal homeomorphisms, $n < q < p < \infty$, are locally Hölder-continuous with the Hölder exponent $\alpha = p(q-n)/q(p-n)$. As a consequence of Theorem 4.1, we obtain the property of global Hölder continuity for a weak (p,q)-quasiconformal homeomorphism in the case of continuous embedding domains. We call a domain $\Omega \subset \mathbb{R}^n$ a Hölder-continuous embedding domain, if the embedding operator of the Sobolev space to the space of continuous functions

$$W^1_p(\Omega) \hookrightarrow C(\Omega), \ p > n,$$

is bounded. Examples of such domains are domains with Lipschitz boundaries or domains with the uniform interior cone condition (see, e.g., [6]).

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a Hölder-continuous embedding domain, and let the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induce a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n < q \leq p \leq \infty.$$

Then φ belongs to the Hölder space $H^\alpha(\Omega)$, $\alpha = p(q-n)/q(p-n)$.

Proof. By Theorem 4.1, a mapping $\varphi \in L^1_s(\Omega)$ for

$$s = \frac{p-n}{p}\frac{pq}{p-q} = \frac{(p-n)q}{p-q}.$$

In the case $n < q < p \leq \infty$, we have

$$\frac{(p-n)q}{p-q} > n.$$

Using the Sobolev theorems of embedding into the spaces of Hölder-continuous functions [20], we obtain that φ belongs to $H^\alpha(\Omega)$, $\alpha = p(q-n)/q(p-n)$. □

Corollary 4.4. *Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a continuous embedding domain, and let the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ induce a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \ \ n-1 < q < p < n.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to the Hölder space $H^\alpha(\widetilde{\Omega})$, $\alpha = q'(p'-n)/p'(q'-n)$.

Proof. By the duality theorem [26,31], the inverse mapping generates a bounded composition operator

$$(\varphi^{-1})^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}),$$

where $q' = q/(q-n+1)$, $p' = p/(p-n+1)$, $n < p' \leq q' \leq \infty$.

By Corollary 4.3, we obtain that the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to the Hölder space $H^\alpha(\widetilde{\Omega})$, $\alpha = q'(p'-n)/p'(q'-n)$. □

The global integrability of derivatives allows us to obtain a theorem on the second-type self-improvement for composition operators on Sobolev spaces. Namely, if φ is a weak (p,q)-quasiconformal homeomorphism, then φ is also a weak (r,s)-quasiconformal mapping under some restrictions on r and s that depend on p and q.

Theorem 4.5. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a weak (p,q)-quasiconformal homeomorphism, $1 < q < p < \infty$. Then φ is a weak (r,s)-quasiconformal homeomorphism for all $1 < s < r < \infty$ such that $p/q \leq r/s$ and*

$$\frac{rs-ps}{rq-ps} = \frac{(p-n)}{p-q}.$$

Proof. By Theorem 2.2, it is sufficient to check that

$$K_{r,s}(\varphi;\Omega) = \|K_r \mid L_{\kappa'}(\Omega)\| < \infty,$$

where $1/s - 1/r = 1/\kappa'$.

Because φ is the weak (p,q)-quasiconformal mapping, φ is a mapping of finite distortion. Denote $Z = \{x \in \Omega : J(x,\varphi) = 0\}$. Then

$$\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^r}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} dx = \int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p |D\varphi(x)|^{r-p}}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} dx$$
$$= \int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} \left(|D\varphi(x)|^{r-p}\right)^{\frac{s}{r-s}} dx.$$

By the conditions of the theorem, we have $s/(r-s) \leq q/(p-q)$. Hence, using the Hölder inequality, we obtain

$$\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^r}{|J(x,\varphi)|}\right)^{\frac{s}{r-s}} dx \quad\leq\quad \left(\int\limits_{\Omega\setminus Z} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{p}{p-q}} dx\right)^{\frac{s(p-q)}{q(r-s)}} \left(\int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{qs(r-p)}{qr-ps}} dx\right)^{\frac{qr-ps}{q(r-s)}}.$$

Using the equality

$$\frac{qs(r-p)}{qr-ps} = \frac{q(p-n)}{p-q}$$

and Theorem 4.1, we have

$$\int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{qs(r-p)}{qr-ps}} dx = \int\limits_{\Omega\setminus Z} |D\varphi(x)|^{\frac{q(p-n)}{p-q}} dx < \infty.$$

Hence,

$$K_{r,s}(\varphi;\Omega) = \|K_r \mid L_{\kappa'}(\Omega)\| < \infty,$$

where $1/s - 1/r = 1/\kappa'$. $\square$

Remark 4.6. *Recall that, for bounded domains, $|D\varphi|^{\frac{p-n}{p}} \in L_\alpha(\Omega)$ for any $\alpha \leq \frac{q(p-n)}{p-q}$. Therefore, for bounded domains, the second condition of the previous theorem is*

$$\frac{qs(r-p)}{qr-ps} \leq \frac{q(p-n)}{p-q}.$$

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