DOI 10.1007/s10958-020-04703-6
Journal of Mathematical Sciences, Vol. 245, No. 3, March, 2020

ON GEOMETRY OF FOLIATIONS OF CODIMENSION 1

A. M. Bayturaev UDC 517.936.8

Abstract. In this paper, we examine geometry and topology of foliations generated by level surfaces of metric functions.

***Keywords and phrases*:** Riemann metric, linear connection, codimension, diffeomorphism, topology, simply connectedness.

***AMS Subject Classification*:** 37C15, 57R25

1. Introduction. Let M be a smooth Riemannian manifold of dimension n and $f : M \to \mathbb{R}^1$ be a differentiable function. Let $p_0 \in M$, $f(p_0) = c_0$, and the set

$$L_{p_0} = \{p \in M^n : L(p) = c_0\}$$

not contain critical points. Then the level set L_{p_0} is a smooth $(n-1)$-dimensional submanifold of M. If we assume that the differentiable function $f : M \to \mathbb{R}^1$ has no critical points, i.e., $|\operatorname{grad} f(p)| > 0$ for every $p \in M$, then the partition of M into connected components of level surfaces of the function f is an $(n-1)$-dimensional foliation (foliation of codimension one).

In the general case, the topology and geometry of foliations generated by level surfaces can be quite complicated.

In [13], the foliation generated by level surfaces of a function $f : M \to \mathbb{R}^1$ defined on a Riemannian manifold M is studied, provided that the magnitude of the gradient vector is constant on each level set. It was also proved in [13] that the level surfaces of such functions generate a Riemannian foliation. Functions considered in [13] are called *metric functions*.

Note that level surfaces of functions of this class are arc-connected sets (see [10]). In [5, 6], the geometry and topology of foliations generated by level surfaces of metric functions were examined. A complete classification of foliations generated by level surfaces in Euclidean space was obtained in [5]; namely, it was proved that for a function of n variables, there exist exactly n types of foliations.

Classification of foliations generated by level surfaces of metric functions defined on an arbitrary Riemannian manifold is a difficult problem. In the Euclidean case, if a metric function does not have critical points, then, as was shown in [5], the level surfaces are hyperplanes. A metric function that does not have critical points on a Riemannian manifold can have compact level surfaces.

It was proved in [11] that a metric function without critical points defined on a complete, simply-connected Riemannian manifold does not have compact level surfaces. It was proved in [10] that level surfaces of a metric function defined in Euclidean space are arc-connected subsets.

Definition 1 (see [9])**.** A foliation F on a Riemannian manifold M is said to be *Riemannian* if each geodesic orthogonal at one point to a leaf of the foliation F remains orthogonal to fibers of F at all its points.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 144, Proceedings of the Conference "Problems of Modern Topology and Its Applications" (May 11–12, 2017), Tashkent, Uzbekistan, 2018.

2. Statement of the problem and main results. In this paper, we study the geometry of Riemannian foliations of codimension 1 on Riemannian manifolds generated by level surfaces of metric functions. As was noted, it was proved in the monograph [13] that if a metric function f has no critical points, then level surfaces of this function generate a Riemannian foliation on the Riemannian manifold M.

In Theorem 1, we study the topology of level surfaces of a metric function and the topology of the manifold on which the metric function is given. We prove that if a metric function without critical points is given on a complete, connected, n-dimensional manifold, then the manifold is diffeomorphic to the direct product of an arbitrary level surface of a function and a straight line. Hence, in particular, it follows that all level surfaces are mutually diffeomorphic.

In Theorem 2, the geometry of foliations generated by level surfaces of metric functions on a complete, connected Riemannian manifold of constant nonnegative sectional curvature is examined. We prove that if a metric function without critical points is defined on a complete, connected Riemannian manifold of constant nonnegative sectional curvature, then level surfaces of the function f generate a totally geodesic foliation whose fibers are mutually isometric. If, in addition, the manifold is simply connected, then it is isometric to the product of an arbitrary level surface and the gradient line of the function.

Let M be a smooth manifold of dimension n and $f : M \to \mathbb{R}^1$ be a metric function without critical points. It was proved in [13] that a foliation generated by level surfaces of a metric function defined on a Riemannian manifold is a Riemannian foliation.

The following theorem characterizes the topology of a manifold on which a metric function is defined.

Theorem 1. *Let M be a smooth, complete, connected manifold of dimension n and $f : M \to \mathbb{R}^1$ be a metric function without critical points. Then the manifold M is diffeomorphic to the direct product $L \times \mathbb{R}^1$, where L is an arbitrary level surface of the function f. In particular, all level surfaces are mutually diffeomorphic.*

Proof. Consider the system of differential equations

$$\dot{x} = X(x), x \in M, \tag{1}$$

where the vector field $X(x)$ is a unit gradient field, i.e.,

$$X(x) = \frac{\operatorname{grad} f(x)}{|\operatorname{grad} f(x)|}.$$

This system is called a *gradient system.* For a point $x_0 \in M$, we denote by $\gamma(t, x_0)$ the solution of this gradient system with the initial condition $\gamma(0) = x_0$. Since the condition $X \in C^1(M, \mathbb{R}^1)$ is fulfilled, a unique trajectory of the system (1) passes through any every point $x_0 \in M$ (see [2, 12]). Note that the curve $t \to \gamma(t, x_0)$ is called a *gradient line* of the function f.

Since $|X(x)| = 1$ and the manifold M is complete, the solution $\gamma(t, x_0)$ is defined for all $t \in (-\infty, +\infty)$ (see [2]).

It is well known that each level surface of the metric function f has only one connected component, i.e., each level surface is an arc-connected set (see [10]).

Let L be an arbitrary level surface of the function f. We denote by U the set of all points y for which there exists a number $t \in \mathbb{R}^1$ and a point $p \in L$ such that $y = \gamma(t, p)$. The set U coincides with the manifold M (see [10]). Therefore, the following mapping appears:

$$F : M \to L \times \mathbb{R}^1,$$

where $F(y) = (p, t)$, $p \in L$, and $t \in \mathbb{R}^1$.

We show that the mapping F is a diffeomorphism. Due to the uniqueness of the solution of the system and the fact that each gradient line intersects the level surface L only once, the mapping F is bijective.

Consider an arbitrary point $y \in M$ for which $F(y) = (p, t)$. Then $\gamma(-t, y) = p$. By the theorem on the smooth dependence of solutions of differential equations on initial data (see [2]), we conclude that the mapping F is differentiable. This implies that the mapping F is a diffeomorphism. □

Corollary 1. *Let M be a smooth, complete, simply-connected manifold of dimension n and $f : M \to \mathbb{R}^1$ be a metric function without critical points. Then an arbitrary level surface L of the function f is simply connected.*

Indeed, since the mapping $F : M \to L \times \mathbb{R}^1$ is a diffeomorphism, the fundamental group of the manifold M is isomorphic to the fundamental group of the level surface L (see [1]). Therefore, the level surface L is simply connected. Note that if $M = \mathbb{R}^n$, then each level surface is a hyperplane (see [5]) and the diffeomorphism $F : M \to L \times \mathbb{R}^1$ is an isometry.

From the results of [5] it follows that if a metric function defined in $\mathbb{R}^n$ does not have critical points, then its level surfaces generate a totally geodesic foliation. The following theorem generalizes this fact for a Riemannian manifold of constant nonnegative sectional curvature.

Theorem 2. *Let M be a smooth, complete, connected Riemannian manifold of constant nonnegative sectional curvature and $f : M \to \mathbb{R}^1$ be a metric function without critical points. Then level surfaces of the function f generate a totally geodesic foliation F on M whose fibers are mutually isometric.*

Recall the definition of a manifold of constant sectional curvature. Let M be a n-dimensional Riemannian manifold with a metric tensor g and T_xM be the tangent space at a point $x \in M$. We denote by $R(X, Y)$ the curvature transformation in T_xM determined by vectors $X, Y \in T_xM$.

The Riemann curvature tensor R for M is a tensor field defined as follows:

$$R\big(X_1, X_2, X_3, X_4\big) = g\big(R(X_3, X_4)X_2, X_1\big), \quad X_i \in T_xM, \quad i = 1, \dots, 4.$$

For each two-dimensional plane π in the tangent space T_xM, the sectional curvature $K(\pi)$ for π is defined as follows:

$$K(\pi)) = R\big(X_1, X_2, X_1, X_2\big) = g\big(R(X_1, X_2)X_2, X_1\big),$$

where X_1, X_2 is an orthonormal basis of π; the sectional curvature $K(\pi)$ is independent of the choice of this basis. The set of values $K(\pi)$ for all planes π in T_xM defines the tensor of a Riemannian curvature at the point x.

If $K(\pi)$ is constant for all planes π in T_xM and all points $x \in M$, then M is called a *space of constant curvature* (see [7]).

The concept of a constant sectional curvature of a Riemannian manifold M has the following simple geometric interpretation. Consider a point $x \in M$ and a two-dimensional plane π passing through it, i.e., the linear span of two noncollinear vectors X_1 and X_2 given at the point x. We draw a geodesic line through x along the direction of each vector of this linear span. The set of these geodesic lines is a two-dimensional surface Φ called a *geodesic surface*. Obviously, the plane π is a tangent plane of the surface Φ at the point x.

The Gaussian curvature at the point x of the surface Φ considered as a two-dimensional Riemannian space coincides with the curvature of the Riemannian manifold M at the same point in the direction of the plane π (see [7]).

Among spaces of constant curvature, we indicate Euclidean space, the Lobachevsky space, and the elliptic space.

Proof of Theorem 2. It is well known that level surfaces of a metric function without critical points are arc-connected sets (see [10]). Therefore, level surfaces form a certain foliation F of codimension 1.

We denote by $L(p)$ the leaf of the foliation F passing through a point p, by $F(p)$ the tangent space of the leaf $L(p)$ at the point p, and by $H(p)$ the orthogonal complement of $F(p)$ in T_pM. The following two subbundles of the tangent bundle TM arise:

$$TF = \{F(p) : p \in M\}, \quad H = \{H(p) : p \in M\};$$

moreover, $TM = TF \oplus H$. Thus, H is a one-dimensional distribution orthogonal to the foliation F.

A piecewise smooth curve $\gamma : [0,1] \to M$ is said to be *horizontal* if $\dot\gamma(t) \in H(\gamma(t))$ for each $t \in [0,1]$. A piecewise smooth curve lying in a leaf of the foliation F is said to be *vertical*.

Let $I = [0,1]$, $v : I \to M$ be a vertical curve, $h : I \to M$ be a horizontal curve, and $h(0) = v(0)$. A piecewise smooth mapping $P : I \times I \to M$, where $t \to P(t,s)$, is a vertical curve for each $s \in I$, $s \to P(t,s)$ is a horizontal curve for each $t \in I$, and $P(t,0) = v(t)$ for $t \in I$ and $P(0,s) = h(s)$ for $s \in I$, is called the *vertical-horizontal* homotopy. If for each pair of vertical and horizontal curves $v, h : I \to M$ with $h(0) = v(0)$, there exists the corresponding vertical-horizontal homotopy P, then we say that the distribution H is an Ehresmann connection for the foliation F (see [3]).

Let $x_0 \in M$ and $\gamma(t, x_0)$ be the trajectory of the following system with the initial condition $x(0) = x_0$:

$$\dot x = \operatorname{grad} f(x). \tag{2}$$

Since the function f has no critical points, for each point $x_0 \in M$, the trajectory $\gamma(t, x_0)$ is a smooth curve different from a point. Due to the fact that the function f is metric, each trajectory of the system 2 is a geodesic line of the Riemannian manifold M (see [11]). This means that

$$\nabla_Z Z = 0, \quad Z = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}.$$

Now we show that every leaf of the foliation F is a totally geodesic submanifold of M. Let L_0 be a leaf of the foliation F and $v : [0, l_0] \to L_0$ be the shortest path in L_0 parametrized by the arclength. Here L_0 is considered as a Riemannian manifold with the Riemannian metric induced from M. We extend the gradient line $\gamma(t,s)$ of the function f parametrized by the arclength from each point $v(t)$ (in this case, $\gamma(t,s)$ satisfies the system of differential equations (2) with the initial condition $\gamma(t,0) = v(t)$).

Due to the fact that the function f has no critical points and M is complete, $\gamma(t,s)$ is defined for all $s \in \mathbb{R}^1$ (see [2]). The flow of the unit vector field $Z = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$ turns level surfaces into level surfaces (see [13]). Therefore, if $\gamma(t,s)$ is a gradient line parametrized by the arclength emanating from $v(t)$ at $s = 0$, then the curve $t \to \gamma(t,s)$ lies on the same level surface for each t.

Consider the following two-dimensional surface:

$$\Phi = \Big\{\gamma(t,s) : \ t \in [0, l_0], \ s \in (-\infty, +\infty)\Big\}.$$

We set

$$l_1 = \big\{\gamma(0,s), \ s \in \mathbb{R}^1\big\}, \quad l_2 = \big\{\gamma(l_0,s), \ s \in \mathbb{R}^1\big\}.$$

By the equality $\nabla_Z Z = 0$, the sets l_1 and l_2 are one-dimensional, totally geodesic submanifolds of M. By the uniqueness of solutions to the Cauchy problem for the system (2), the gradient lines l_1 and l_2 do not intersect; moreover, they are closed subsets of M. We show that the gradient lines $\gamma_t : s \to \ gamma(t,s)$ are straight lines in Φ. We consider the restriction of the Riemannian metric g to Φ. If we take (t,s) as curvilinear coordinates on Φ, then the restriction of the Riemannian metric g to Φ has the form

$$E(t,s)dt^2 + ds^2,$$

where $E(t,s) = |X(t,s)|^2$, and $|X(t,s)|$ is the length of the tangent vector $X(t,s)$ of the curve $t \to \gamma(t,s)$ at the point $p = \gamma(t,s)$. If $A = \gamma(t, s_1)$ and $B = \gamma(t, s_2)$, $s_2 > s_1$, then the length of the segment AB is $s_2 - s_1$.

Let $\tilde{\gamma}$ be another curve lying in Φ and joining points A and B. Then the length of the curve $\tilde{\gamma}$ is equal to the integral

$$l = \int_{\tau_1}^{\tau_2} \sqrt{Edt^2 + ds^2}$$

taken over the curve $\tilde{\gamma}$ parametrized by τ.

Since $E(t, s) = |X(t, s)|^2 \geq 0$, we have

$$\sqrt{Edt^2 + ds^2} \geq |ds|, \quad l = \int_{\tau_1}^{\tau_2} \sqrt{Edt^2 + ds^2} \geq \int_{\tau_1}^{\tau_2} |ds| \geq |\int ds| = s_2 - s_1.$$

Thus, the segment AB of the geodesic line $\gamma_t : s \to \gamma(t, s)$ realizes the shortest distance between the points A and B.

Let π be a two-dimensional plane in the tangent space T_pM generated by the vectors $Z(t, s)$ and $X(t, s)$, where $p = \gamma(t, s)$. Then the surface Φ lies in the two-dimensional section $\exp_p(\pi)$, where $\exp_p$ is the exponential mapping at the point p. The Gaussian curvature Φ at the point p is equal to the curvature of the two-dimensional section of the manifold M constructed by the vectors $Z(t, s)$ and $X(t, s)$. By the conditions of the theorem, the curvature of the surface Φ is constant and non-negative. On the other hand, the surface Φ contains a straight line, so its curvature should be zero (see [8]). Therefore, it lies in a surface, which is isometric to the Euclidean plane. Under an isometry, all straight lines (gradient lines of the function f) pass to parallel lines of the Euclidean plane. Therefore, if the shortest path is orthogonal to one gradient line in Φ, then it is orthogonal to all gradient lines, since they are parallel (see [8]).

Since l_2 is a closed subset and M is complete, for each point $q_1 \in l_1$, there exists a point $q_2 \in l_2$ such that the shortest line q_1q_2 realizes the distance from q_1 to l_2. The shortest line q_1q_2 is orthogonal to l_2 since gradient lines are parallel. This means that the shortest line q_1q_2 lies in the leaf of foliation passing through the point $q_1 \in l_1$.

Therefore, if the shortest line $v : [0, l_1] \to L_0$ of the leaf L_0 realizes the shortest distance from the point $q_1 = v(0)$ to l_2 in L_0, then it is a geodesic in M and realizes the shortest distance on the fibers between l_1 and l_2. Then the variation δl of the length of the curve

$$t \to \gamma(s, v(t)), \quad t \in [0, l_0], \tag{3}$$

vanishes for $s = 0$. Let us calculate this variation. The length of the curve (3) is

$$l(s) = \int_0^{l_0} |X(s, t)dt.$$

Clearly, $l(0) = l_0$ and $|X(0, t)| = 1$ for all $t \in [0, l_0]$ and

$$\delta l = \int_0^{l_0} \frac{\nabla_Z |X|^2}{2|X|} dt.$$

It is easy to verify that $[X, Z] = 0$, where $[X, Z]$ is the Lie bracket of the vector fields X and Z; moreover,

$$Z(t, s) = \frac{\operatorname{grad} f\big(\gamma(t, s)\big)}{|\operatorname{grad} f\big(\gamma(t, s)\big)|}.$$

Since the Levi-Civita connection ∇ is torsion-free, we obtain the following equality:

$$\nabla_Z X - \nabla_X Z = [Z, X]. \tag{4}$$

In addition, since ∇ is a metric connection, the following equality holds:

$$Wg(Y_1, Y_2) = g(\nabla_W Y_1, Y_2) \tag{5}$$

for any vector fields $Y_1, Y_2, W \in V(M)$ (see [4]). Since $|Z| = 1$, from Eq. (5) we obtain

$$0 = Xg(Z, Z) = 2g(\nabla_X Z, Z), \quad \text{i.e.,} \quad g(\nabla_X Z, Z) = 0. \tag{6}$$

From the relation $|X| = g(X, X)^{\frac{1}{2}}$ we get (see [5–7])

$$\nabla_Z |X|^2 = Zg(X, X) = 2g(\nabla_Z X, Z) = 2g(\nabla_X Z, X). \tag{7}$$

From Eq. (7) we obtain

$$\delta l = \int_0^{l_0} \frac{g(\nabla_X Z, X)}{|X|} dt.$$

Since the vector fields X and Z are mutually orthogonal,

$$0 = Xg(X, Z) = g(\nabla_X X, Z) + g(X, \nabla_X Z). \tag{8}$$

Equality (8) implies

$$g(\nabla_X Z, X) = -g(\nabla_X X, Z);$$

therefore, the variation δl has the form

$$\delta l = -\int_0^{l_0} \frac{g(\nabla_X X, Z)}{|X|} dt.$$

Since the geodesic $t \to v(t)$ of the leaf L_0 is also a geodesic in M, the equality $\nabla_X X = 0$ holds for $s = 0$. From (5) we see that $\nabla_X Z$ is tangent to the foliation F. Therefore, decreasing l_0 if necessary, we can assume that $g(\nabla_X Z, X) \geq 0$ for $s = 0$ at all points $v(t)$ for $t \in [0, l_0]$. Since $|X(t, 0)| = 1$ for $t \in [0, l_0]$, the equality

$$\int_0^{l_0} g(\nabla_X Z, X)|_{s=0} dt = 0 \tag{9}$$

implies that $g(\nabla_X Z, X) = 0$ for $s = 0$. The vector $\nabla_X Z$ at all points $v(t)$ lies in the plane of the vectors $X(v(t))$ and $Z(v(t))$; therefore, Eq. (9) implies $\nabla_X Z = 0$ for $s = 0$. This means that the vector field $Z(t, 0)$ is parallel along the curve $v(t)$.

If we denote by γ_X the shortest line from the point $\gamma(0, s)$ of the gradient line $s \to \gamma(0, s)$ to l_2, then it is also a vertical curve and its length is equal to the length of $v : [0, l_0] \to L_0$. Since the manifold M is complete and the foliation F is Riemannian, the distribution H is an Ehresmann connection for F (see [3]). Due to the uniqueness of the vertically horizontal homotopy, we conclude that the vertical curve $\gamma_X : [0, l_0] \to M$ coincides with the curve $t \to \gamma(t, s)$. This means that $\nabla_X X = 0$ at all points of the surface, and the curves $t \to \gamma(t, s)$ for each s are geodesic lines of length l_0.

Let $v : (a, b) :\to M$ be an arbitrary geodesic in M orthogonal to the gradient line of f at a certain point. Then in a sufficiently small neighborhood of each of its points, it is a shortest arc. Therefore, repeating the above reasoning, we find that it lies on the level surface passing through the point at which it is orthogonal to the gradient line of the function, and under the flow of the unit gradient vector field it turns to a geodesic line of the same length.

Thus, every geodesic line of M that touches a leaf of the foliation F, does not leave this leaf. In addition, the flow of a unit gradient field Z turns it into a geodesic line of the same length of the corresponding leaf. Hence it follows that all level surfaces are totally geodesic submanifolds; moreover, they are mutually isometric. $\square$

Theorem 2 allows one to prove the following assertion.

Theorem 3. *Let M be a smooth, complete, simply connected Riemannian manifold of constant non-negative sectional curvature, and $f : M \to \mathbb{R}^1$ be a metric function without critical points. Then the Riemannian manifold M is isometric to the direct product $L \times S$, where L is a level surface and S is a gradient line of the function f.*

Proof. By Theorem 2, level surface of the metric function f form a totally geodesic foliation F of codimension 1. The above results imply that F is a Riemannian foliation. Thus, the foliation F is Riemannian and totally geodesic simultaneously.

Gradient lines of the function f form a one-dimensional foliation $F^{\perp}$. Hence, $F^{\perp}$ is a geodesic foliation (see [11]. Since F is a totally geodesic foliation, $F^{\perp}$ is also a Riemannian and totally geodesic foliation simultaneously.

Let $\mu : I \to M$ be a smooth curve, $\mu(0) = p_0$ and $\mu(1) = p$, where $p_0, p \in M$. Since the manifold M is complete and the foliation F is Riemannian, for each piecewise smooth curve $\mu : I \to M$ there exists a unique vertical-horizontal homotopy $P_\mu : I \to M$ such that $\mu_t = P_\mu(t, t)$ for $t \in I$.

The curve $t \to v(t) = P(t, s)$ lies in the leaf $L(p_0)$ of the foliation F and $v(0) = p_0$. The curve $s \to \gamma(s) = P(t, s)$ is orthogonal to level surfaces and $\gamma(0) = p_0$. The curve $t \to v(t) = P(t, s)$, $t \in [0, 1]$, is called the *projection* of the curve $\mu : I \to M$ on $L(p_0)$, whereas the curve $s \to \gamma(s) = P(t, s)$, $s \in [0, 1]$, is called the *projection* of the curve $\mu : I \to M$ on $S(p_0)$, where $S(p_0)$ is the gradient line of the function f. The points $p_1 = v(1)$ and $p_2 = \gamma(1)$ are called the *projections* of p in $S(p_0)$, respectively.

Since H is completely integrable, the projection p depends only on the homotopy class of the curve γ (see [7]). Therefore, if M is simply connected, the mapping $G : p \to (p_1, p_2)$ is well defined. Since the foliation F is Riemannian and totally geodesic simultaneously, the mapping $G : p \to (p_1, p_2)$ is an isometric immersion (see [7]). Since $\dim M = \dim\{L(p_0) \times S(p_0)\}$, the mapping f is a cover; therefore, it is an isometry (see [7]). □

Corollary 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be a metric function without critical points. Then each level surface of the function f is isometric to $\mathbb{R}^{n-1}$.*

Indeed, by Theorem 3, $\mathbb{R}^n$ is isometric to the direct product $L \times S$, where L is a level surface and S is a gradient line of the function f.

It was proved in [6] that the curvature of each gradient line S of a metric function is zero. Therefore, since the function has no critical points, S is a straight line. Consequently, $\mathbb{R}^n$ is isometric to the direct product $L \times \mathbb{R}^1$. This implies that each level surface is isometric to the hyperplane $\mathbb{R}^{n-1}$.

REFERENCES

1. I. Ya. Bakelman, A. A. Verner, and B. E. Kantor, *Introduction to Differential Geometry "in the Whole"* [in Russian], Nauka, Moscow (1973).
2. Yu. N. Bibikov, *Ordinary Differential Equations* [in Russian], Lenigrad (1981).
3. R. Blumenthal and J. Hebda, "Complementary distributions which preserve the leaf geometry and applications to totally geodesic foliations," *Quart. J. Math.*, **35**, 383–392 (1984).
4. D. Gromoll, W. Klingenberg, and W. Meyer, *Riemannsche Geometrie im Großen*, Springer-Verlag, Berlin–Heidelberg–New York (1968).
5. G. Kaipnazarova and A. Ya. Narmanov, "Topology of foliations generated by level surfaces," *Uzbek. Mat. Zh.*, **2**, 53–60 (2008).
6. G. Kh. Kaipnazarova, *Geometry of foliations generated by level surfaces* [in Russian], Ph.D. thesis, Tashkent (2009).

7. Sh. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Interscience, New York–London (1963).
8. Sh. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. II, Interscience, New York–London (1969).
9. P. Molino, "Riemannian foliations," *Prog. Math.*, **73**, Birkhäuser, Boston (1988).
10. A. Narmanov and S. Sharapov, "on level surfaces of submersions," *Uzbek. Mat. Zh.*, **2**, 62–66 (2004).
11. A. Narmanov and G. Kaipnazarova, "Metric functions on Riemannian manifolds," *Uzbek. Mat. Zh.*, **1**, 11–20 (2010).
12. J. Palis and W. De Melo, *Geometric Theory of Dynamical Systems*, Springer-Verlag, New York–Heidelberg–Berlin (1982).
13. P. Tondeur, *Foliations on Riemannian Manifolds*, Springer-Verlag (1988).

A. M. Bayturaev
National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan
E-mail: abayturaev@mail.ru