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ASYMPTOTIC PROPERTIES OF BAYESIAN-TYPE ESTIMATES IN THE COMPETING RISK MODEL UNDER RANDOM CENSORING

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Abstract. We prove the asymptotic normality of Bayesian-type estimates in the competing risk model with two-sided random censoring.

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1. Introduction. The likelihood ratio statistic (LRS) plays a fundamental role in the theory of decision-making, especially in the theory of verification of statistical hypotheses. Among various criteria, we mention criteria based on the LRS; they are actively used in the theory of hypothesis testing. According to the Neumann–Pearson lemma, criteria based on the LRS are optimal compared with other criteria constructed on the basis of other statistics. Interesting problems arise when alternatives H_1 depend on n and are "close" to H_0, i.e., $H_1 = H_1 n \to H_0$ for $n \to \infty$.

In such situations, asymptotic properties of the LRS appear; these properties are useful in the theory of estimating of unknown parameters and hypotheses testing. The most important property of statistical models is the property of local asymptotic normality (LAN) of LRS of a regular statistical experiment.

The essence of the LAN is that an LRS model admits approximation by functions of the form

$$\exp\Big\{u\omega_{n,\theta} - \frac{1}{2}u^2\Big\},$$

where $\omega_{n,\theta}$ are asymptotically (i.e., as $n \to \infty$) normal random variables with parameters $(0, 1)$. The properties of experiments satisfying the LAN condition in the case of independent and identically distributed observations were studied by A. Wald, L. Le Kam, and J. Haeck (see [5–10]. The results on approximation of the LRS by stochastic integrals in competing risk model (CRM) with random censoring of observations from the right and from both sides were established in [2, 3]; this version of the LAN generalizes the classical results. In the present paper, using the LAN property in the general statistical model, we examine asymptotic properties of estimates of Bayesian type for an unknown parameter, and prove its asymptotic efficiency.

2. Preliminaries. Let us consider a competing risk model (CRM) following [1]. Let X be a random variable defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$, which takes values in a measurable space $(\mathcal{X}, \mathcal{B})$. We consider the joint properties of random pairs $(X, A^{(i)})$, $i = \overline{1,k}$, where $A^{(1)}, \dots, A^{(k)}$ are pairwise incompatible events such that $P\Big(\bigcup\limits_{i=1}^{k} A^{(i)}\Big) = 1$.

This corresponds to the case where the object (a technical device or an individual) with the uptime X is exposed to k competing risks and breaks down under one of the events $A^{(i)}$, $i = \overline{1,k}$. Let $\delta^{(i)} =$

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$I\big(A^{(i)}\big)$ be the indicator of the event $A^{(i)}$, $i=\overline{1,k}$, and the joint distribution of the random vector $\big(X,\delta^{(1)},\dots,\delta^{(k)}\big)$ is given up to a parameter $\theta\in\Theta$:

$$Q_\theta\big(x,y^{(1)},\dots,y^{(k)}\big)=P\Big(X\le x,\ \delta^{(1)}=y^{(1)},\ \dots,\ \delta^{(k)}=y^{(k)}\Big),$$

where $x\in\mathbb{R}^1=(-\infty;+\infty)$, $y^{(i)}\in\{0,1\}$, $i=\overline{1,k}$, and Θ is an open set in $\mathbb{R}^1$. In particular, we define the marginal distributions $H(x;\theta)=P(X\le x)$ and assume that the following condition holds:

$$H^{(i)}(x;\theta)=P\Big(X\le x,\ \delta^{(i)}=1\Big),\quad i=\overline{1,k}.$$

It is easy to see that $\delta^{(1)}+\dots+\delta^{(k)}=1$ and for the subdistributions $H^{(i)}(x;\theta)$, $i=\overline{1,k}$, the equality

$$H^{(1)}(x;\theta)+\dots+H^{(k)}(x;\theta)=H(x;\theta)\tag{1}$$

holds for all $(x;\theta)\in\mathbb{R}^1\times\Theta$.

Assume that the distributions $H(x;\theta)$ and $H^{(i)}(x;\theta)$, $i=\overline{1,k}$, are absolutely continuous. We define the integral intensity functions

$$\Lambda(x;\theta)=\int_{-\infty}^{x}\frac{dH(u;\theta)}{1-H(u;\theta)}=-\log[1-H(x;\theta)],$$

$$\Lambda^{(i)}(x;\theta)=\int_{-\infty}^{x}\frac{dH^{(i)}(u;\theta)}{1-H(u;\theta)},\quad i=\overline{1,k}.$$

It is easy to see that $\Lambda^{(1)}(x;\theta)+\dots+\Lambda^{(k)}(x;\theta)=\Lambda(x;\theta)$; this implies

$$1-H(x;\theta)=\exp\Bigg\{-\sum_{i=1}^{k}\Lambda^{(i)}(x;\theta)\Bigg\}=\prod_{i=1}^{k}\Big[1-F^{(i)}(x;\theta)\Big],\tag{2}$$

where $F^{(i)}(x;\theta)=1-\exp\big\{\Lambda^{(i)}(x;\theta)\big\}$, $i=\overline{1,k}$.

It was established in [1] that the functions $F^{(i)}(x;\theta)$, $i=\overline{1,k}$, possess properties of subdistributions.

In the sequel, we consider a statistical scheme according to which in the model considered, the set $(X,A^{(1)},\dots,A^{(k)})$ is subjected to a random censoring from the right and from the left by random variables Y and L, respectively, with absolutely continuous unknown distribution functions $K(y)$, $y\in\mathbb{R}^1$, and $L(y)$, $y\in\mathbb{R}^1$.

Let

$$\begin{aligned}
&Z=\max(L,\min(X,Y))=L\vee(X\wedge Y),\\
&D^{(-1)}=\{\omega: X(\omega)\wedge Y(\omega)<L(\omega)\},\\
&D^{(0)}=\{\omega: L(\omega)\le Y(\omega)<X(\omega)\},\\
&D^{(i)}=A^{(i)}\cap\{\omega: L(\omega)\le X(\omega)\le Y(\omega)\},\ i=\overline{1,k}.
\end{aligned}$$

The set $\Big(Z;D^{(-1)},D^{(0)},D^{(1)},\dots,D^{(k)})$ is available for observation. Note that the events $\{D^{(-1)},D^{(0)},$ $D^{(1)},\dots,D^{(k)}\}$ also have the properties of the events $A^{(1)},\dots,A^{(k)}$. In this model, the random variables Y and L and the distribution functions K and L are considered to be disturbing.

Let $\big\{X_j,L_j,Y_j;D^{(-1)},D^{(0)},D^{(1)},\dots,D^{(k)}\big\}_{j=1}^{\infty}$ be a sequence of independent copies of the set $\big(X,L,$ $Y;D^{(-1)},D^{(0)},\ D^{(1)},\dots,D^{(k)}\big)$, and let at the nth step of the experiment, a sample of volume n be observed:

$$\tilde Z^{(n)}=(\tilde Z_1,\tilde Z_2,\dots,\tilde Z_n),\tag{3}$$

where

$$\tilde{Z}_j = \Big(Z_j; \Delta_j^{(-1)}, \Delta_j^{(0)}, \Delta_j^{(1)}, \ldots, \Delta_j^{(k)}\Big),$$
$$Z_j = L_j \vee (X_j \wedge Y_j),$$
$$\Delta_j^{(i)} = I(D_j^{(i)}), \quad i = \overline{-1, k}.$$

Note that in the sample (3), the pairs of interest $(X_j; A_j^{(i)})$ are observed only in the case where $\Delta_j^{(i)} = 1$, $i = \overline{1, k}$. It is easy to see that the observed random variables Z_j have the following distribution function:

$$N(x;\theta) = L(x)\Big(1 - \big(1 - K(x)\big)\big(1 - H(x;\theta)\big)\Big).$$

We introduce the subdistributions $T^{(i)}(x;\theta) = P\big(Z_j \le x; D_j^{(i)}\big)$, $i = \overline{-1, k}$, where the following identity holds:

$$T^{(-1)}(x;\theta) + T^{(0)}(x;\theta) + T^{(1)}(x;\theta) + \ldots + T^{(k)}(x;\theta) = N(x;\theta);$$

here

$$T^{(-1)}(x;\theta) = P(X_j \wedge Y_j < L_j; L_j \le x) = \int_{-\infty}^{x} \Big(1 - \big(1 - K(u)\big)\big(1 - H(u;\theta)\big)\Big) dL(u),$$

$$T^{(0)}(x;\theta) = P(L_j \le Y_j < X_j; Y_j \le x) = \int_{-\infty}^{x} L(u)(1 - H(u;\theta))dK(u) \tag{4}$$

$$T^{(i)}(x;\theta) = P(L_j \le X_j \le Y_j; X_j \le x; A_j^{(i)}) = \int_{-\infty}^{x} L(u)\big(1 - K(u)\big)dH(u;i).$$

Due to the censoring from the left, instead of $\Lambda^{(i)}(x;\theta)$ we must deal with the integral intensity functions truncated at some appropriate level τ:

$$\Lambda_\tau^{(i)}(x;\theta) = \Lambda^{(i)}(x;\theta) - \Lambda^{(i)}(\tau;\theta).$$

In this case, according to (4), it is easy to verify that

$$\Lambda_\tau^{(i)}(x;\theta) = \int_{\tau}^{x} \frac{dT^{(i)}(u;\theta)}{L(u)(1 - K(u))(1 - H(u;\theta))}, \quad i = \overline{1, k}. \tag{5}$$

Therefore, instead of $1 - F_\tau(i)(x;\theta)$ we can consider the following identity:

$$1 - F_\tau^{(i)}(x;\theta) = \big(1 - F^{(i)}(x;\theta)\big)\big(1 - F^{(i)}(\tau;\theta)\big)^{-1} = \exp\Big\{-\sum_{i=1}^{k} \Lambda_\tau^{(i)}(x;\theta)\Big\}, \quad i = \overline{1, k}.$$

Let $\Big(\mathcal{Y}^{(n)},\ \mathcal{U}^{(n)},\ \tilde{Q}_\theta^{(n)}\Big)$ be a sequence of statistical experiments generated by the observations (3). We denote by $\tilde{Z}$ the set of values of the random variable Z and obtain the relation

$$\mathcal{Y}^{(n)} = \Big\{\tilde{Z} \times \{0,1\}^{(k+2)}\Big\}^{(n)} = \overbrace{\Big\{\tilde{Z} \times \{0,1\}^{(k+2)}\Big\} \times \ldots \times \Big\{\tilde{Z} \times \{0,1\}^{(k+2)}\Big\}}^{n},$$

where $\{0,1\}^{(k+2)} = \overbrace{\{0,1\} \times \ldots \times \{0,1\}}^{k+2}$, $\mathcal{U}^{(n)}$ is the σ-algebra of Borel sets in $\mathcal{Y}^{(n)}$, and $\tilde{Q}_\theta^{(n)}$ is the distribution on $\big(\mathcal{Y}^{(n)}, \mathcal{U}^{(n)}\big)$, which is the n-fold direct product of the following "one-dimensional" distributions:

$$\tilde{Q}_\theta(x, y^{(-1)}, y^{(0)}, y^{(1)}, \ldots, y^{(k)}) \\ = P\Big(Z_j \le x,\ \Delta_j^{(-1)} = y^{(-1)},\ \Delta_j^{(0)} = y^{(0)},\ \Delta_j^{(1)} = y^{(1)},\ \ldots,\ \Delta_j^{(k)} = y^{(k)}\Big).$$

Let $f^{(i)}(x;\theta)$ be the density of the subdistribution $F^{(i)}(x;\theta)$, $i = \overline{1,k}$. Then the distribution $\tilde{Q}_\theta^{(n)}$ is absolutely continuous with respect to the measure $\nu^{(n)}$ and its density for any $\theta \in \Theta$ is defined on the sample space $\mathcal{Y}^{(n)}$ by the following formula:

$$\frac{d\tilde{Q}_\theta^{(n)}(\tilde{z}^{(n)})}{d\nu^{(n)}(\tilde{z}^{(n)})} = p_n(\tilde{z}^{(n)}\theta) = \prod_{m=1}^{n}\prod_{i=1}^{k}\Big\{l(z_m)\Big(1-(1-(K(z_m))(1-H(z_m;\theta))\Big)\Big\}^{y_m^{(-1)}} \\ \times \Big\{L(z_m)(1-K(z_m))f^{(i)}(z_m;\theta)\cdot\prod_{\substack{j=1\\ j\neq i}}^{k}[1-F^{(j)}(z_m;\theta)]\Big\}^{y_m^{(i)}} \\ \times \Big\{L(z_m)k(z_m)(1-H(z_m;\theta))\Big\}^{y_m^{(0)}},\ \tilde{z}^{(n)} \in \mathcal{Y}^{(n)}, \quad (6)$$

where $k(x) = K'(x)$, $l(x) = L'(x)$, $d\nu^{(n)}(\tilde{z}^{(n)}) = d\nu(\tilde{z}_1)\times\ldots\times d\nu(\tilde{z}_n)$, $d\nu(\tilde{z}_m) = \varepsilon_{y_m^{(i)}} \times dx_m$, $i = \overline{-1,k}$, $m = \overline{1,n}$, and $\varepsilon_{y_m^{(i)}}$ is the counting measure concentrated at the point $y_m^{(i)} \in \{0,1\}$.

Assume that the following condition is fulfilled:

$$h^{(i)}(x;\theta) = f^{(i)}(x;\theta)\prod_{j\neq i}\Big(1-F^{(j)}(x;\theta)\Big), \quad i = \overline{1,k},$$

where θ_0 is the true value of the parameter θ and $\gamma(x;\theta) = 1-\big(1-K(x)\big)\big(1-H(x;\theta)\big)$.

For $u \in \mathbb{R}^1$, the following identity holds:

$$\theta_0 + \frac{u}{\sqrt{n}} = \theta_n \in \Theta.$$

According to (6), we specify the LRS

$$\frac{d\tilde{Q}_{\theta_n}^{(n)}(\tilde{Z}^{(n)})}{d\tilde{Q}_{\theta_0}^{(n)}(\tilde{Z}^{(n)})} = \frac{p_n(\tilde{Z}^{(n)};\theta_n)}{p_n(\tilde{Z}^{(n)};\theta_0)} = \prod_{m=1}^{n}\Big\{\prod_{i=1}^{k}\Big[\frac{h^{(i)}(z_m;\theta_n)}{h^{(i)}(z_m;\theta_0)}\Big]\Big\}^{y_m^{(i)}}\Big\{\frac{\gamma(z_m;\theta_n)}{\gamma(z_m;\theta_0)}\Big\}^{y_m^{(-1)}}\Big\{\frac{1-H(z_m;\theta_n)}{1-H(z_m;\theta_0)}\Big\}^{y_m^{(0)}}$$

We take the logarithm of the LRS:

$$L_n(u) = \log\Big\{\frac{d\tilde{Q}_{\theta_n}^{(n)}(\tilde{Z}^{(n)})}{d\tilde{Q}_{\theta_0}^{(n)}(\tilde{Z}^{(n)})}\Big\} = n\sum_{i=1}^{k}\int_{-\infty}^{+\infty}\log\Big[\frac{h^{(i)}(x;\theta_n)}{h^{(i)}(x;\theta_0)}\Big]dT_n^{(i)}(x) \\ + \int_{-\infty}^{+\infty}\log\Big[\frac{\gamma(x;\theta_n)}{\gamma(x;\theta_0)}\Big]dT_n^{(-1)}(x) + \int_{-\infty}^{+\infty}\log\Big[\frac{1-H(x;\theta_n)}{1-H(x;\theta_0)}\Big]dT_n^{(0)}(x) \quad (7)$$

For $u \in \mathbb{R}^1$, we define a "close alternative" of $\theta_0 + \frac{u}{\sqrt{n}} = \theta_n \in \Theta$, where θ_0 is the true value of the parameter θ. Now we formulate the regularity conditions under which the LAN of the family of distributions $\{\tilde{Q}_\theta^{(n)}, \theta \in \Theta\}$ holds at the point $\theta = \theta_0$.

Condition 1. The supports $N^{(i)} = \{x : f^{(i)}(x;\theta) > 0\}$, $i = \overline{1,k}$, are independent of θ and the set $\bigcap\limits_{i=1}^{k} N^{(i)}$ is nonempty.

Condition 2. For any two points $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, and $x \in N^{(i)}$, we the inequalities $f^{(i)}(x;\theta_1) \neq f^{(i)}(x;\theta_2)$ hold, $i = 1, \dots, k$.

Condition 3. For all x, there exist finite derivatives $\partial^l f^{(i)}(x;\theta)/\partial\theta^l$, $l = 1, 2$, $i = 1, \dots, k$; moreover,

$$\int\limits_{-\infty}^{\infty} \left| \frac{\partial^l f^{(i)}(x;\theta)}{\partial\theta^l} \right| dx < \infty, \quad l = 1, 2, \quad i = 1, \dots, k.$$

Condition 4. The derivatives $\dfrac{\partial \log f^{(i)}(x;\theta_0)}{\partial\theta}$ and $\dfrac{\partial \log h^{(i)}(x;\theta_0)}{\partial\theta}$, $i = \overline{1,k}$, are functions of bounded variation.

Condition 5. The Fisher information functions are finite and positive at the point $\theta = \theta_0$:

$$J^{(i)}(\theta) = \int\limits_{-\infty}^{\infty} \left(\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta} \right)^2 dT^{(i)}(x;\theta) + \int\limits_{-\infty}^{\infty} \left(\frac{\partial \log(1 - H(x;\theta))}{\partial\theta} \right)^2 dT^{(-1)}(x;\theta)$$
$$+ \int\limits_{-\infty}^{\infty} \left(\frac{\partial \log(1 - H(x;\theta))}{\partial\theta} \right)^2 dT^{(0)}(x;\theta), \quad i = \overline{1,k}.$$

We introduce the notation $J(\theta) = J^{(1)}(\theta) + \dots + J^{(k)}(\theta)$ and note that the Fisher information function of the sample (3) is equal to $nJ(\theta)$. The following theorem is valid.

Theorem 1 (see [3])**.** *Let the conditions* 1–5 *be valid. Then for each* $u \in \mathbb{R}^1$, *the following representation of the LRS holds*:

$$\frac{d\widetilde{Q}_{\theta_n}^{(n)}(\widetilde{Z}^{(n)})}{d\widetilde{Q}_{\theta_0}^{(n)}(\widetilde{Z}^{(n)})} = \exp\{uW_n - \frac{u^2}{2} J(\theta_0) + R_n(u)\},$$

where

$$W_n = \sum_{i=1}^{k} \int\limits_{-\infty}^{\infty} \frac{\partial \log h^{(i)}(x;\theta_0)}{\partial\theta} dn^{-1/2} \widetilde{W}_i \Big(T^{(i)}(x); n \Big)$$
$$+ \int\limits_{-\infty}^{\infty} \frac{\partial \log(1 - H(x;\theta_0))}{\partial\theta} dn^{-1/2} \widetilde{W}_i \Big(T^{(-1)}(x); n \Big)$$
$$+ \int\limits_{-\infty}^{\infty} \frac{\partial \log(1 - H(x;\theta_0))}{\partial\theta} dn^{-1/2} \widetilde{W}_i \Big(T^{(0)}(x); n \Big),$$

$R_n(u) \to 0$ *as* $n \to \infty$ *by* $\widetilde{Q}_{\theta_0}^{(n)}$*-probability. Here* $\widetilde{W}_i(y;n)$ *are two-parameter Wiener processes on* $[0,1] \times (0,\infty)$ *and the components of the vector* $(\widetilde{W}_1, \dots, \widetilde{W}_k)$ *are independent.*

Remark 1. Due to the properties of the processes $\widetilde{W}_i$, the random variable W_n is the sum of independent stochastic Ito integrals, each of which coincides by distribution with the corresponding normally distributed random variable $N(0, J^{(i)}(\theta_0))$, $i = \overline{1,k}$. Therefore,

$$W_n \stackrel{D}{=} N(0, J(\theta_0)). \tag{8}$$

Taking into account the relation (8), the statement of Theorem 1 can be written in the following form:

$$L_n(u) = uJ^{1/2}(\theta_0)\zeta - \frac{u^2}{2}J(\theta_0) + R_n(u) \tag{9}$$

for each $u \in \mathbb{R}^1$. Here ζ is the standard normal random variable and the equality is understood in the sense of the distribution $\widetilde{Q}_{\theta_0}^{(n)}$. The property (9) is called the LAN for the LRS.

3. Main result. Let $\{\pi(u), u \in \Theta\}$ be a nonnegative measurable function and $l(d;\theta) = (d-\theta)^2$ be the loss function on the set $D \times \Theta$, where D is the set of possible estimates for θ.

We consider the estimates $\theta_n \in D$ defined by the following equation:

$$\hat{\theta}_n = \arg\min_{d \in D} \frac{\int\limits_{\Theta} l(d;\theta) p_n(\tilde{Z}^{(n)};\theta)\pi(\theta)d\theta}{\int\limits_{\Theta} p_n(\tilde{Z}^{(n)};\theta)\pi(\theta)d\theta}. \tag{10}$$

Note that if θ is a random quantity with a priori density π, then θ_n is the Bayesian estimate for θ. We prove the asymptotic normality of the estimates θ_n whose limit distributions are independent on the functions π.

Theorem 2. *Assume that the regularity conditions* 1–5 *are fulfilled and the function $\pi(\theta)$ is continuous in a neighborhood of a point θ_0, $\pi(\theta_0) \neq 0$. Then*

$$\sqrt{n}(\theta_n - \theta_0) \Rightarrow N(0, (J(\theta_0))^{-1}) \quad \textit{as } n \to \infty.$$

Proof. Under the conditions 1–5, the LAN of (9) follows from Theorem 1. According to (10), the estimate $\hat{\theta}_n$ has the following representation:

$$\hat{\theta}_n = \frac{\int\limits_{\Theta} \theta p_n(\tilde{Z}^{(n)};\theta)\pi(\theta)d\theta}{\int\limits_{\Theta} p_n(\tilde{Z}^{(n)};\theta)\pi(\theta)d\theta}. \tag{11}$$

In the integrals (11), we replace the variable θ by its close alternative $\theta_0 + \frac{u}{\sqrt{n}} = \theta_n \in \Theta$, $u \in \mathbb{R}^1$. Then, using $L_n(u)$, we obtain

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \frac{\int\limits_{-\infty}^{+\infty} u \exp(L_n(u))\pi\Big(\theta_0 + \frac{u}{\sqrt{n}}\Big)du}{\int\limits_{-\infty}^{+\infty} \exp(L_n(u))\pi\Big(\theta_0 + \frac{u}{\sqrt{n}}\Big)du}. \tag{12}$$

Let

$$L(u) = uJ^{1/2}(\theta_0)\zeta - \frac{u^2}{2}J(\theta_0).$$

Then, according to (9), for every $u \in \mathbb{R}^1$ we conclude that $\exp(L_n(u)) \Rightarrow \exp(L(u))$ as $n \to \infty$. This implies that the finite-dimensional distributions of the process $L_n(u)$ converge to finite-dimensional distributions of the process $L(u)$. In (12) we formally pass to the limit under the integral sign and obtain

$$\zeta J^{1/2}(\theta_0) = \frac{\int\limits_{-\infty}^{+\infty} u \exp(L(u))du}{\int\limits_{-\infty}^{+\infty} \exp(L(u))du}. \quad (13)$$

To justify the passage to the limit, we choose a fixed number $C > 0$ and prove the continuity of the process $L_n(u)$ for $u \in [-C, C]$. Let numbers u_1 and u_2 be such that $\theta_0 + u_j \in [-C, C]$, $j = 1, 2$. We show that for sufficiently large n, the following inequality holds:

$$M_{\theta_0}\Big(L_n(u_1) - L_n(u_2)\Big)^2 \le \alpha(u_1 - u_2)^2, \quad \alpha > 0. \quad (14)$$

Since

$$\sum_{i=1}^{k} \Delta_m^{(i)} + \Delta_m^{(-1)} + \Delta_m^{(0)} = 1, \quad m = \overline{1, n},$$

the following calculations are valid:

$$\begin{aligned} M_{\theta_0}\Big(L_n(u_1) - L_n(u_2)\Big)^2 &= M_{\theta_0}\left\{\sum_{m=1}^{n}\left\{\Delta_m^{(-1)}\left[\log\gamma\left(Z_m; \theta_0 + \frac{u_1}{\sqrt{n}}\right) - \log\gamma\left(Z_m; \theta_0 + \frac{u_2}{\sqrt{n}}\right)\right]\right.\right. \\ &+ \sum_{i=1}^{k}\Delta_m^{(i)}\left[\log h^{(i)}\left(Z_m; \theta_0 + \frac{u_1}{\sqrt{n}}\right) - \log h^{(i)}\left(Z_m; \theta_0 + \frac{u_2}{\sqrt{n}}\right)\right] \\ &\left.\left. + \Delta_m^{(0)}\left[\log\left(1 - H\left(Z_m; \theta_0 + \frac{u_1}{\sqrt{n}}\right)\right) - \log\left(1 - H\left(Z_m; \theta_0 + \frac{u_2}{\sqrt{n}}\right)\right)\right]\right\}\right\}^2 \\ &\le n\left\{\int\limits_{-\infty}^{+\infty}\left[\log\gamma\left(x; \theta_0 + \frac{u_1}{\sqrt{n}}\right) - \log\gamma\left(x; \theta_0 + \frac{u_2}{\sqrt{n}}\right)\right]^2 dT^{(-1)}(x; \theta_0)\right. \\ &+ \sum_{i=1}^{k}\int\limits_{-\infty}^{+\infty}\left[\log h^{(i)}\left(x; \theta_0 + \frac{u_1}{\sqrt{n}}\right) - \log h^{(i)}\left(x; \theta_0 + \frac{u_2}{\sqrt{n}}\right)\right]^2 dT^{(i)}(x; \theta_0) \\ &\left. + \int\limits_{-\infty}^{+\infty}\left[\log\left(1 - H\left(x; \theta_0 + \frac{u_1}{\sqrt{n}}\right)\right) - \log\left(1 - H\left(x; \theta_0 + \frac{u_2}{\sqrt{n}}\right)\right)\right]^2 dT^{(0)}(x; \theta_0)\right\} \\ &= J(\theta_0)(u_1 - u_2)^2, \quad (15) \end{aligned}$$

which proves (14). Thus, according to (15), the process $\{L_n(u),\ u \in [-C, C]\}$ is an element of the space $\mathbb{C}[-C; C]$.

On the other hand, for any t_1 and t_2, the following functional is continuous in ψ:

$$\Phi(\psi) = t_1 \int\limits_{-C}^{C} u\,\psi(u)du + t_2 \int\limits_{-C}^{C} u\,\psi(u)du$$

According to the Cramer–Wald theorem, due to the continuity of $L_n(u)$ and the condition (7), from [4] we conclude that the distributions of the random vectors

$$\left(\int_{-C}^{C} u\exp(L_n(u))\pi\left(\theta_0+\frac{u}{\sqrt{n}}\right)du,\ \int_{-C}^{C}\exp(L_n(u))\pi\left(\theta_0+\frac{u}{\sqrt{n}}\right)du\right)$$

converge to the distribution of the vector

$$\left(\pi(\theta_0)\int_{-C}^{C} u\exp(L(u))du,\ \pi(\theta_0)\int_{-C}^{C}\exp(L(u))du\right).$$

On the other hand, for any $\varepsilon>0$, there exists $\delta>0$ such that the following relations are valid:

$$P\left(\pi(\theta_0)\int_{|u|>C} u\exp(L(u))du>\delta\right)<\varepsilon, \tag{16}$$

$$P\left(\pi(\theta_0)\int_{|u|>C}\exp(L(u))du>\delta\right)<\varepsilon. \tag{17}$$

For sufficiently large n, the inequalities of the type (16) and (17) are also valid for $L_n(u)$. Moreover, for sufficiently large C and n, the following inequality holds:

$$P\left(t_1\int_{|u|>C} u\exp(L_n(u))\pi\left(\theta_0+\frac{u}{\sqrt{n}}\right)du+t_2\int_{|u|>C}\exp(L_n(u))\pi\left(\theta_0+\frac{u}{\sqrt{n}}\right)du>\frac{1}{C^N}\right)$$
$$\le\sum_{|l|>C}P\left(\int_{l}^{l+1}(|u|+1)\exp(L_n(u))\pi\left(\theta_0+\frac{u}{\sqrt{n}}\right)du>\frac{1}{l^N(t_1\vee t_2)}\right)$$
$$\le\sum_{|l|>C}P\left(\max_{u\in[l,l+1]}\left\{\exp(L_n(u))\right\}>\frac{l^{-(N+M+2)}}{(t_1\vee t_2)}\right)\le\frac{\lambda_N}{C^N}. \tag{18}$$

From (17), for large n we have

$$\sqrt{n}(\hat{\theta}_n-\theta_0)=\frac{\int_{-C}^{C} u\exp(L_n(u))du}{\int_{-C}^{C}\exp(L_n(u))du}+r_n(C),$$

where $P\left(|r_n(C)|>\delta\right)<\varepsilon$. Thus, Eq. (13) is valid. The theorem is proved. □

Remark 2. Due to Theorem 2, according to Fisher's definition (see [6]), the estimate $\hat{\theta}_n$ can be considered asymptotically effective.

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