DOI 10.1007/s10958-020-04672-w
Journal of Mathematical Sciences, Vol. 244, No. 6, February, 2020

ON TYPE I BLOW UP FOR THE NAVIER–STOKES EQUATIONS NEAR THE BOUNDARY

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For suitable weak solutions to the Navier–Stokes equations, a new sufficient condition for the uniform boundedness of the scale invariant energy functionals near a boundary point is established. Bibliography: 23 titles.

1. Introduction and Main Results

Let $\mathcal{C} := \{\ x \in \mathbb{R}^3 :\ x_1^2 + x_2^2 < 1,\ |x_3| < 1\ \}$ and $\mathcal{Q} := \mathcal{C} \times (-1, 0)$. We consider the Navier–Stokes equations in $\mathcal{Q}$,

$$\left\{\begin{array}{c} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{array}\right. \quad \text{in } \ \mathcal{Q}, \tag{1.1}$$

where $u : \mathcal{Q} \to \mathbb{R}^3$ and $p : \mathcal{Q} \to \mathbb{R}$ are the velocity field and pressure, respectively. Together with system (1.1), we consider the Navier–Stokes equations near the boundary,

$$\left\{\begin{array}{c} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{x_3=0} = 0 \end{array}\right. \quad \text{in } \ \mathcal{Q}^+, \tag{1.2}$$

where $\mathcal{Q}^+ := \mathcal{C}^+ \times (-1, 0)$ with $\mathcal{C}^+ := \mathcal{C} \cap \{x_3 > 0\}$.

In the present paper, we are interested in the local regularity for weak solutions to system (1.2), satisfying the estimate

$$|u(x,t)| \le \frac{C}{\sqrt{x_1^2 + x_2^2}} \tag{1.3}$$

for a.e. $(x,t) \in \mathcal{Q}$ and a positive constant C.

Our interest is partly motivated by studying possible behavior of axially symmetric solutions to the Navier–Stokes equations near the boundary (that is why we use cylinders $\mathcal{C}$, $\mathcal{C}^+$, etc., rather than standard balls). We remind that that a solution u, p to equations (1.1) or (1.2) is said to be axially symmetric if

$$u(x,t) = u_r(r,z,t)\mathbf{e}_r + u_\varphi(r,z,t)\mathbf{e}_\varphi + u_r(r,z,t)\mathbf{e}_z, \quad p(x,t) = p(r,z,t),$$

where $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ is the cylindrical basis in $\mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, and $z = x_3$. We say that the solution is axi-symmetric without swirl if

$$u(x,t) = u_r(r,z,t)\mathbf{e}_r + u_r(r,z,t)\mathbf{e}_z, \qquad p(x,t) = p(r,z,t).$$

For axially symmetric solutions, condition (1.3) is one of the scale invariant conditions which characterize so called Type I blow up at the axis of symmetry, see terminology in [15] or [20].

It is a well-known fact that the internal case axi-symmetric solutions without swirl are locally regular, see [7,9] and [6]. In contrast, in the boundary case the corresponding result is unknown and an axi-symmetric solution without swirl can potentially have a singularity near origin (i.e., at the point of intersection of the axis of symmetry with the domain boundary, see, for example, [5]).

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Published in *Zapiski Nauchnykh Seminarov POMI*, Vol. 477, 2018, pp. 136–149. Original article submitted September 27, 2018.

On the other hand, it was proved in [6] and [15] that in the internal case, axi-symmetric weak solutions with swirl satisfying (1.3) are regular. The analogous result near the boundary is unknown.

In our approach we replace condition (1.3) by a more general condition

$$\sup_{r<1} A_w(u,r) \le C_0, \tag{1.4}$$

where

$$A_w(u,r) := \frac{1}{\sqrt{r}} \operatorname*{ess\,sup}_{t\in(-r^2,0)} \|u(\,\cdot\,,t)\|_{L_{2,w}(\mathcal{C}^+(r))}.$$

Set $\mathcal{C}^+(r) := \{\ x\in\mathbb{R}^3:\ \sqrt{x_1^2+x_2^2}<r,\ 0<x_3<r\ \}$, and for any domain $\Omega\subset\mathbb{R}^3$ denote by $L_{2,w}(\Omega)$ a weak Lebesgue space equipped with the quasinorm

$$\|f\|_{L_{2,w}(\Omega)} := \sup_{\lambda>0} \lambda |\{x\in\Omega : |f(x)|>\lambda\}|^{1/2}.$$

Note that every measurable function u satisfying (1.3) meets condition (1.4) as well.

To formulate our main results we recall the notion of boundary suitable weak solutions. The notion of suitable weak solutions to the Navier–Stokes system was introduced in celebrated paper [2]. For the boundary case, we use the following definition, see, for example, [18] (the notation for functional spaces are explained at the end of this section).

Definition 1.1. *We say that a pair of functions u and p is a boundary suitable weak solution to the Navier–Stokes system in $\mathcal{Q}^+$ if*

- $u\in L_{2,\infty}(\mathcal{Q}^+)\cap W_2^{1,0}(\mathcal{Q}^+)$, $p\in L_{\frac32}(\mathcal{Q}^+)$,
- $u|_{x_3=0}=0$ *in the sense of traces,*
- u *and* p *satisfy the Navier–Stokes system in* $\mathcal{Q}^+$ *in the sense of distributions,*
- *for a.a.* $t\in(-1,0)$, *the pair* u *and* p *satisfies the local energy inequality in* Q^+,

$$\begin{aligned}\int\limits_{\mathcal{C}^+} \zeta(x,t)|u(x,t)|^2\,dx + 2\int\limits_{-1}^{t}\int\limits_{\mathcal{C}^+} \zeta|\nabla u|^2\,dx\,dt \\ \le \int\limits_{-1}^{t}\int\limits_{\mathcal{C}^+} |u|^2\ (\partial_t\zeta+\Delta\zeta)\ dx\,dt + \int\limits_{-1}^{t}\int\limits_{\mathcal{C}^+} u\cdot\nabla\zeta\ \big(|u|^2+2p\big)\ dx\,dt,\end{aligned} \tag{1.5}$$

for any nonnegative test function $\zeta\in C^\infty(\mathbb{R}^3\times\mathbb{R})$ vanishing near the parabolic boundary $\partial'\mathcal{Q}=(\partial\mathcal{C}\times[-1,0])\cup(\bar{\mathcal{C}}\times\{t=-1\})$.

To formulate our results we also introduce the following scale invariant functionals:

$$\begin{aligned}
A(u,r) &= \operatorname*{ess\,sup}_{t\in(-r^2,0)} \Big(\ \frac1r \int\limits_{\mathcal{C}^+(r)} |u(x,t)|^2\ dx\ \Big)^{1/2},\\
C(u,r) &= \Big(\ \frac{1}{r^2} \int\limits_{\mathcal{Q}^+(r)} |u(x,t)|^3\ dxdt\ \Big)^{1/3},\\
E(u,r) &= \Big(\ \frac1r \int\limits_{\mathcal{Q}^+(r)} |\nabla u(x,t)|^2\ dxdt\ \Big)^{1/2},\\
D(p,r) &= \Big(\ \frac{1}{r^2} \int\limits_{\mathcal{Q}^+(r)} |p(x,t)-[p]_{\mathcal{C}^+(r)}(t)|^{3/2}\ dx\,dt\Big)^{2/3}.
\end{aligned} \tag{1.6}$$

The main result of the present paper is the following theorem.

Theorem 1.1. *Assume that the pair u and p is a boundary suitable weak solution to system* (1.2)*. Assume that there exists $C_0 > 0$ such that condition* (1.4) *is satisfied. Then*

$$\sup_{r<1}\Big(A(u,r)+C(u,r)+E(u,r)+D(p,r)\Big)<+\infty. \tag{1.7}$$

Theorem 1.1 implies that suitable weak solutions with $A_w(u,r)$-norm, uniformly bounded in r, can have Type I singularities at the origin only (see terminology in [20]).

It is interesting to compare Theorem 1.1 with other known relevant results. The first important result was obtained in [14] in the internal case. Namely, it was shown there that if

$$\min\Big\{\sup_{r<1}A(u,r),\ \sup_{r<1}C(u,r),\ \sup_{r<1}E(u,r)\Big\}<+\infty,$$

then (1.7) holds. In [10], the same statement was proved near the boundary. In [13] (see also [21]), an analogous result was established in the internal case under the condition

$$\sup_{r<1}C_{s,l}(u,r)<+\infty,\quad \max\Big\{2-\frac{1}{l},\frac{3}{2}+\frac{1}{2l}\Big\}<\frac{3}{s}+\frac{2}{l}<2,$$

where $s\in(3,+\infty)$, $l\in(2,+\infty)$, and

$$C_{s,l}(u,r):=r^{1-\frac{3}{s}-\frac{2}{l}}\Big(\int\limits_{-r^2}^{0}\Big(\int\limits_{B(r)}|u|^s\,dx\Big)^{l/s}dt\Big)^{1/l}.$$

Under assumption (1.3), statement (1.7) in the internal case was proved in [15].

Condition (1.4) can also be interpreted as the inequality

$$\operatorname*{ess\,sup}_{t\in(-1,0)}\|u(\,\cdot\,,t)\|_X<+\infty, \tag{1.8}$$

where X is a Morrey-type class with the scale-invariant quasi-norm

$$\|w\|_X=\sup_{r<1}\frac{1}{\sqrt{r}}\|w\|_{L_{2,w}(\mathcal{C}^+(r))}.$$

Statement (1.7) is known in the internal case if (1.8) is satisfied and X is one of the following spaces: $X=L_3(\mathcal{C})$, $L_{3,w}(\mathcal{C})$, or $BMO^{-1}(\mathcal{C})$ (the notation is explained at the end of this section). Namely, in the case $X=L_3$ condition (1.8) implies the Hölder continuity of u near the origin both in the internal and boundary cases, see [3,12]. In the case of $X=L_{3,w}$, the regularity of u is unknown and estimate (1.7) is available only (this result easily follows from Theorem 1.1). In the case of $X=BMO^{-1}$, estimate (1.7) was obtained in the internal case in [8,16]. Moreover, a similar result was proved by G. Seregin and D. Zhou [22] in the internal case if X is the (globally defined) Besov space $\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

A simple consequence of our approach is the following ε-regularity condition. Similar conditions in the boundary case can be found in [17].

Theorem 1.2. *There exists an absolute constant $\varepsilon>0$ such that if a boundary suitable suitable weak solution u, p to* (1.2) *satisfies condition* (1.3) *with $C_0<\varepsilon$, then u is Hölder continuous in some neighborhood of the origin.*

The paper is organized as follows. In Sec. 2, we recall some known facts from the theory of functions. In Sec. 3, we prove Theorems 1.1 and 1.2.

We use the following notation:

- $\mathbb{R}^3_+:=\{x\in\mathbb{R}^3:\ x_3>0\}$,

- $\mathcal{C}(r) := \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < r, \ |x_3| < r\}, \ \mathcal{C} := \mathcal{C}(1)$,
- $\mathcal{C}^+(r) := \mathcal{C}^+(r) \cap \mathbb{R}^3_+, \quad \mathcal{C}^+ := \mathcal{C} \cap \mathbb{R}^3_+$,
- $L_q(\Omega)$, $W^k_q(\Omega)$, $\overset{\circ}{W}{}^k_q(\Omega)$ are the standard Lebesgue and Sobolev spaces,
- for any measurable $f : \Omega \to \mathbb{R}$, we set $d_f(\lambda) := |\{ x \in \Omega : \ |f(x)| \geq \lambda \}|$,
- for $q \in [1, +\infty)$, $L_{q,w}(\Omega)$ is a weak Lebesgue space equipped with the quasi-norm

$$\|f\|_{L_{q,w}(\Omega)} := \sup_{\lambda > 0} \ \lambda \, d_f(\lambda)^{1/q};$$

  here, for $q = \infty$, we set $L_{\infty,w}(\Omega) := L_\infty(\Omega)$,
- for $q \in (0, +\infty)$ and $s \in (0, +\infty)$, we denote by $L^{q,s}(\Omega)$ the Lorentz space equipped with the quasi-norm

$$\|f\|_{L^{q,s}(\Omega)} := q^{\frac{1}{s}} \left(\int\limits_0^{+\infty} \lambda^{s-1} d_f(\lambda)^{\frac{s}{q}} \ d\lambda \right)^{\frac{1}{s}}; \tag{1.9}$$

  if $s = \infty$, then we put $L^{q,\infty}(\Omega) := L_{q,w}(\Omega)$,
- $BMO(\Omega)$ is the space of functions with bounded mean oscillation in Ω, equipped with the norm

$$\|f\|_{BMO(\Omega)} := \sup_{B(x_0,R) \subset \Omega} \frac{1}{|B(R)|} \int\limits_{B(x_0,R)} |f - [f]_{B(x_0,R)}| \ dx,$$

$$[f]_{B(x_0,R)} := \frac{1}{|B(R)|} \int\limits_{B(x_0,R)} f dx,$$

  and $BMO^{-1}(\Omega) := \{\operatorname{div} F \in \mathcal{D}'(\Omega) \ : \ F \in BMO(\Omega)\}$,
- $\mathcal{Q}(r) := \mathcal{C}(r) \times (-r^2, 0), \ \mathcal{Q} := \mathcal{Q}(1)$,
- $[p]_{\mathcal{C}}$ and $(p)_{\mathcal{Q}}$ denote the spatial and total averages of the function $p(x,t)$, respectively,

$$[p]_{\mathcal{C}}(t) := \frac{1}{|\mathcal{C}|} \int\limits_{\mathcal{C}} p(x,t) \ dx, \quad (p)_{\mathcal{Q}} := \frac{1}{|\mathcal{Q}|} \int\limits_{\mathcal{Q}} p(x,t) \ dx\, dt,$$

- $\mathcal{Q}^+(r) := \mathcal{C}^+(r) \times (-r^2, 0), \ \mathcal{Q}^+ := \mathcal{Q}^+(1)$,
- $L_{q,l}(\mathcal{Q}(r))$ is an anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{q,l}(\mathcal{Q}(r))} := \Big(\int\limits_{-r^2}^{0} \|f(\cdot,t)\|^l_{L_q(\mathcal{C}(r))} dt \Big)^{1/l};$$

  in the case $l = \infty$, we set $\quad L_{q,\infty}(\mathcal{Q}(r)) := L_\infty(-r^2, 0; L_q(\mathcal{C}(r)))$,

$$\|f\|_{L_{q,\infty}(\mathcal{Q}(r))} := \operatorname*{ess\,sup}_{t \in (-r^2,0)} \|f(\,\cdot\,,t)\|_{L_q(\mathcal{C}(r))},$$

- $W^{1,0}_{q,l}(\mathcal{Q}(r)) := \{u \in L_{q,l}(\mathcal{Q}(r)) : \ \nabla u \in L_{q,l}(\mathcal{Q}(r))\}$,

$$\|u\|_{W^{1,0}_{q,l}(\mathcal{Q}(r))} := \|u\|_{L_{q,l}(\mathcal{Q}(r))} + \|\nabla u\|_{L_{q,l}(\mathcal{Q}(r))},$$

- $W^{2,1}_{q,l}(\mathcal{Q}(r)) := \{u \in W^{1,0}_{q,l}(\mathcal{Q}(r)) : \ \nabla^2 u \in L_{q,l}(\mathcal{Q}(r)), \ \partial_t u \in L_{q,l}(\mathcal{Q}(r))\}$,

$$\|u\|_{W^{2,1}_{q,l}(\mathcal{Q}(r))} := \|u\|_{W^{1,0}_{q,l}(\mathcal{Q}(r))} + \|\nabla^2 u\|_{L_{q,l}(\mathcal{Q}(r))} + \|\partial_t u\|_{L_{q,l}(\mathcal{Q}(r))};$$

  in the case of $q = l$, we set $W^{1,0}_q(\mathcal{Q}) := W^{1,0}_{q,q}(\mathcal{Q})$ etc.,
- $L_{q,w;\infty}(\mathcal{Q}(r)) := L_\infty(-r^2, 0; L_{q,w}(\mathcal{C}(r)))$.

2. Some results from the function theory

First we recall an interpolation result concerning Lorentz spaces, see [1, Theorem 5.3.1].

Lemma 2.1. *Assume that* $1 \le q_1 < q < q_2 \le \infty$ *and* $\theta \in (0,1)$ *are such that*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then for any $0 < s \le \infty$*, there is a constant* $c = c(q_1, q_2, q, s) > 0$ *such that if* $\Omega \subset \mathbb{R}^n$ *is any domain and* $u \in L_{q_1,w}(\Omega) \cap L_{q_2,w}(\Omega)$*, then* $u \in L^{q,s}(\Omega)$ *and the estimate*

$$\|u\|_{L^{q,s}(\Omega)} \;\le\; c\, \|u\|^{1-\theta}_{L_{q_1,w}(\Omega)} \|u\|^{\theta}_{L_{q_2,w}(\Omega)} \tag{2.1}$$

holds.

The next result is a trivial combination of Lemma 2.1 and Sobolev's embedding theorem.

Lemma 2.2. *Assume that* $1 \le q \le p \le 6$ *and* $\theta \in [0,1]$ *are such that*

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{6}.$$

Then any $f \in L_{q,w}(\mathcal{C}^+(r)) \cap W^1_2(\mathcal{C}^+(r))$ *belongs to* $L_p(\mathcal{C}^+(r))$*, and there exists a positive constant* $c = c(p,q)$ *(independent of* $r > 0$*) such that if, in addition,* $f|_{x_3=0} = 0$*, then*

$$\|f\|_{L_p(\mathcal{C}^+(r))} \le c\, \|f\|^{1-\theta}_{L_{q,w}(\mathcal{C}^+(r))} \|\nabla f\|^{\theta}_{L_2(\mathcal{C}^+(r))}. \tag{2.2}$$

Next we recall the well-known O'Neils inequality, see [4, Exercise 1.4.19].

Lemma 2.3. *If* $q_1,\ q_2,\ q \in (1,+\infty]$ *and* $s_1,\ s_2,\ s \in (0,+\infty]$ *are such that*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \quad \textit{and} \quad \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s},$$

then

$$\|fg\|_{L^{q,s}(\mathcal{C}^+(r))} \le c(q_1,q_2,s_1,s_2)\ \|f\|_{L^{q_1,s_1}(\mathcal{C}^+(r))} \|g\|_{L^{q_2,s_2}(\mathcal{C}^+(r))}.$$

We use the following modification of the O'Neils inequality for three functions.

Lemma 2.4. *If* $q_1,\ q_2,\ q_3,\ q \in (1,+\infty]$ *and* $s_1,\ s_2,\ s_3,\ s \in (0,+\infty]$ *are such that*

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q} \quad \textit{and} \quad \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = \frac{1}{s},$$

then

$$\|fgh\|_{L^{q,s}(\mathcal{C}^+(r))} \;\le\; c(q_i,s_i)\ \|f\|_{L^{q_1,s_1}(\mathcal{C}^+(r))} \|g\|_{L^{q_2,s_2}(\mathcal{C}^+(r))} \|h\|_{L^{q_3,s_3}(\mathcal{C}^+(r))} \tag{2.3}$$

3. Proof of the Main Results

We start with the following interpolation inequality. Below we set $A_w(r) := A_w(u,r)$, $C(r) := C(u,r)$, etc., see the definition of the functionals in (1.6).

Theorem 3.1. *Let* u *and* p *be a boundary suitable weak solution to the Navier–Stokes equations in* Q^+*. Then*

$$C(r) \le c\ A_w^{\frac{1}{2}}(r)\ E^{\frac{1}{2}}(r). \tag{3.1}$$

Proof. Follows from (2.2) with $p = 3,\ q = 2$, and $\theta = \frac{1}{2}$. □

Our next result gives an estimate for the pressure. This estimate is the crucial point of our approach, different from the analogous estimate in the internal case, because it involves stronger (energy-type) norms on the right-hand side. This leads to additional technical difficulties which do not arise in the internal case. To obtain the result we adopt a technique developed in [11] for studying boundary regularity to the Navier–Stokes equations.

Theorem 3.2. *For any $\delta \in (0,1)$, there exist positive constants c_1, c_2 such that for any boundary suitable weak solution u, p to the Navier–Stokes equation in $\mathcal{Q}^+$,*

$$D(\theta r) \leq c_1 \theta^{\frac{4}{3}} \big(C(r) + D(r) \big) + c_2 \theta^{-\frac{4}{3}} E^{1+\delta}(r) A_w^{1-\delta}(r) \tag{3.2}$$

for all $r \in (0,1)$ and $\theta \in (0, \frac{1}{2})$.

Proof. First, we prove (3.2) for $r = 1$. We decompose $p = p_1 + p_2$ and $u = u_1 + u_2$, where u_1 and p_1 is a solution to the initial-boundary value problem for the linear system

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla p_1 = (u \cdot \nabla) u \\ \operatorname{div} u_1 = 0 \qquad\qquad\qquad \text{in} \quad \mathcal{Q}^+. \\ u_1|_{\partial \mathcal{Q}^+} = 0 \end{cases} \tag{3.3}$$

Then $u_2 = u - u_1$ and $p_2 = p - p_1$ satisfy the following system:

$$\begin{cases} \partial_t u_2 - \Delta u_2 + \nabla p_2 = 0 \\ \operatorname{div} u_2 = 0 \qquad\qquad \text{in} \quad \mathcal{Q}^+. \\ u_2|_{x_3 = 0} = 0 \end{cases}$$

Moreover, we may assume that for a.e. $t \in (-1,0)$, $[p]_{\mathcal{C}^+} = [p_1]_{\mathcal{C}^+} = [p_2]_{\mathcal{C}^+} = 0$. The right-hand side $(u \cdot \nabla) u$ of system (3.3) belongs to $L_{\frac{9}{8}, \frac{3}{2}}(\mathcal{Q})$. Applying the coercive estimate of solutions to the Stokes problem in anisotropic Sobolev spaces (see [23]) for any $\varepsilon \in (0, \frac{1}{8}]$, we obtain

$$\|u_1\|_{W^{2,1}_{1+\varepsilon, \frac{3}{2}}(\mathcal{Q}^+)} + \|\nabla p_1\|_{L_{1+\varepsilon, \frac{3}{2}}(\mathcal{Q}^+)} \leq c \, \|(u \cdot \nabla) u\|_{L_{1+\varepsilon, \frac{3}{2}}(\mathcal{Q}^+)}.$$

To estimate the right-hand side of the last inequality, we split

$$|(u \cdot \nabla) u| \leq |u|^{\frac{1}{3}} |u|^{\frac{2}{3}} |\nabla u|$$

and apply (2.3) with exponents $q_1 = 2$, $q_2 = 3$, $q_3 = \frac{6(1+\varepsilon)}{1-5\varepsilon}$ and $r_1 = 2$, $r_2 = \infty$, $r_3 = \frac{2(1+\varepsilon)}{1-\varepsilon}$:

$$\frac{1}{1+\varepsilon} = \frac{1}{2} + \frac{1}{3} + \frac{1-5\varepsilon}{6(1+\varepsilon)}, \qquad \frac{1}{1+\varepsilon} = \frac{1}{2} + \frac{1}{\infty} + \frac{1-\varepsilon}{2(1+\varepsilon)}.$$

For a.e. $t \in (-1,0)$, we obtain

$$\|(u \cdot \nabla) u\|_{L_{1+\varepsilon}(\mathcal{C}^+)} \leq c \, \|\nabla u\|_{L_2(\mathcal{C}^+)} \||u|^{\frac{2}{3}}\|_{L_{3,w}(\mathcal{C}^+)} \||u|^{\frac{1}{3}}\|_{L^{\frac{6(1+\varepsilon)}{1-5\varepsilon}, \frac{2(1+\varepsilon)}{1-\varepsilon}}(\mathcal{C}^+)}.$$

Taking into account the property of the Lorentz norm $\big\||u|^\theta\big\|_{L^{q,s}(\mathcal{C}^+)} = \|u\|^\theta_{L^{\theta q, \theta s}(\mathcal{C}^+)}$, where $q, \theta \in (0, +\infty)$ and $s \in (0, +\infty]$, we get

$$\|(u \cdot \nabla) u\|_{L_{1+\varepsilon}(\mathcal{C}^+)} \leq c \, \|\nabla u\|_{L_2(\mathcal{C}^+)} \|u\|^{\frac{2}{3}}_{L_{2,w}(\mathcal{C}^+)} \|u\|^{\frac{1}{3}}_{L^{\frac{2(1+\varepsilon)}{1-5\varepsilon}, \frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)}.$$

Applying the Hölder inequality with exponents $l_1 = 2$, $l_2 = \infty$, $l_3 = 6$,

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{\infty} + \frac{1}{6},$$

we arrive at

$$\|(u\cdot\nabla)u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} \le c\,\|\nabla u\|_{L_2(\mathcal{Q}^+)}\,\|u\|^{\frac23}_{L_{2,w;\infty}(\mathcal{Q}^+)}\,\|u\|^{\frac13}_{L_2\left(-r^2,0;\;L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)\right)}.$$

Using (2.1), we get

$$\|u\|_{L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)} \le c\,\|u\|^{1-\delta'}_{L_{2,w}(\mathcal{C}^+(r))}\|u\|^{\delta'}_{L_6(\mathcal{C}^+)}.$$

Here, $\frac{1-5\varepsilon}{2(1+\varepsilon)} = \frac{1-\delta'}{2} + \frac{\delta'}{6}$ and $\delta' = \frac{3\varepsilon}{1-5\varepsilon}$. Now using the Sobolev embedding theorem, we obtain

$$\|u\|_{L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)} \le c\,\|u\|^{1-\delta'}_{L_{2,w}(\mathcal{C}^+)}\|\nabla u\|^{\delta'}_{L_2(\mathcal{C}^+)}.$$

Therefore,

$$\|u\cdot\nabla u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} \le c\,\|\nabla u\|^{1+\frac{\delta'}{3}}_{L_2(\mathcal{Q}^+)}\|u\|^{\frac23+\frac{1-\delta'}{3}}_{L_{2,w;\infty}(\mathcal{Q}^+)} = c\,\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)},$$

where $\delta := \frac{\delta'}{3} = \frac{\varepsilon}{1-5\varepsilon}$. Thus,

$$\|u_1\|_{W^{2,1}_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|\nabla p_1\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} \le c\,\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)}.$$

Now we turn to the derivation of the estimate for p_2. From the local regularity theory for the linear Stokes system near the boundary (see, for example, [18, Theorem 2.3]) for any $m\in(1,+\infty)$, it follows that $p_2 \in W^{1,0}_{m,\frac32}(\mathcal{Q}^+(\frac12))$ and for any $\rho < \frac12$ the following estimate holds:

$$\begin{aligned}\|\nabla p_2\|_{L_{m,\frac32}(\mathcal{Q}^+(\frac12))} &\le c\left(\|u_2\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|p_2\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}\right)\\ &\le c\left(\|u_1\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|p\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|p_1\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}\right)\\ &\le c\left(\|u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|p\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)}\right).\end{aligned} \tag{3.4}$$

Taking any $\theta < \frac12$ and using Poincare inequality, we obtain

$$\|p_2 - [p_2]_{\mathcal{C}^+(\theta)}\|_{L_{\frac32}(\mathcal{Q}^+(\theta))} \le c\,\theta^\beta\|\nabla p_2\|_{L_{m,\frac32}(\mathcal{Q}^+(\frac12))},$$

where $\beta > 0$ depends on $m\in(1,+\infty)$. Choosing $m = 9$, we get $\beta = \frac83$. Finally, we can estimate $p = p_1 + p_2$ as follows:

$$\begin{aligned}\|p - [p]_{\mathcal{C}^+(\theta)}\|_{L_{\frac32}(\mathcal{Q}^+(\theta))} &\le 2\|p_1\|_{L_{\frac32}(\mathcal{Q}^+(\theta))} + \|p_2 - [p_2]_{C^+(\theta)}\|_{L_{\frac32}(\mathcal{Q}^+(\theta))}\\ &\le c\Big(\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)} + \theta^{\frac83}\|\nabla p_2\|_{L_{9,\frac32}(\mathcal{Q}^+(\frac12))}\Big).\end{aligned}$$

Taking into account (3.4) with $m = 9$, we obtain

$$\|p - [p]_{\mathcal{C}^+(\theta)}\|_{L_{\frac32}(\mathcal{Q}^+(\theta))} \le c\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)} + c\theta^{\frac83}\left(\|u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)} + \|p\|_{L_{\frac32}(\mathcal{Q}^+)}\right). \tag{3.5}$$

Using the definition of the functionals $D(r) := D(p,r)$, etc., we arrive at the estimate

$$D(\theta) \le c_1\,\theta^{\frac43}(C(1) + D(1)) + c_2\theta^{-\frac43}E^{1+\delta}(1)A_w^{1-\delta}(1).$$

To complete the proof, we use standard scaling arguments to get (3.2) for all $r\in(0,1)$ and $\theta\in(0,\frac12)$. □

Now we can prove Theorem 1.1.

Proof. As u, p is a boundary suitable weak solution and (3.1) holds, we have

$$\sup_{r<1} A_w(r) \le C_0, \qquad A\Big(\frac{3}{4}\Big) + E\Big(\frac{3}{4}\Big) \le C_1 < \infty.$$

From (3.1), it follows that

$$C(r) \le c(C_0)\ E^{\frac{1}{2}}(r). \tag{3.6}$$

Set $\mathcal{E}(r) = E(r) + A(r) + D(r)$. Using the local energy inequality (1.5) for any $\theta \in (0, \frac{1}{16})$, we get

$$\mathcal{E}(\theta r) \le C(2\theta r) + C^{\frac{3}{2}}(2\theta r) + C^{\frac{1}{2}}(2\theta r) D^{\frac{1}{2}}(2\theta r) + D(\theta r).$$

By Young's inequality this implies that

$$\mathcal{E}(\theta r) \le c\,(C(2\theta r) + C^{\frac{3}{2}}(2\theta r) + D(2\theta r)). \tag{3.7}$$

Taking $\delta = \frac{1}{7}$ in (3.2) and using (3.6), we obtain

$$D(\theta r) + D(2\theta r) \le c\,\theta^{\frac{4}{3}}\Big[C\Big(\frac{r}{4}\Big) + D\Big(\frac{r}{4}\Big)\Big] \ +\ c(C_0)\theta^{-\frac{4}{3}} E^{\frac{8}{7}}\Big(\frac{r}{4}\Big). \tag{3.8}$$

Combining (3.6), (3.7), and (3.8), we get

$$\begin{aligned} \mathcal{E}(\theta r) &\le c(C_0)\Big[E^{\frac{1}{2}}(2\theta r) + E^{\frac{3}{4}}(2\theta r) + \theta^{-\frac{4}{3}} E^{\frac{8}{7}}\Big(\frac{r}{4}\Big)\Big] + c\theta^{\frac{4}{3}}\Big(C\Big(\frac{r}{4}\Big) + D\Big(\frac{r}{4}\Big)\Big) \\ &\le c(C_0)\Big[\theta^{-\frac{1}{4}}\mathcal{E}^{\frac{1}{2}}(r) + \theta^{-\frac{3}{8}}\mathcal{E}^{\frac{3}{4}}(r) + \theta^{-\frac{4}{3}} E^{\frac{8}{7}}\Big(\frac{r}{4}\Big)\Big] + c\theta^{\frac{4}{3}}\mathcal{E}\Big(\frac{r}{4}\Big). \end{aligned} \tag{3.9}$$

One of the terms on the right-hand side of (3.9) has exponent $\frac{8}{7} > 1$. Therefore, to estimate it, we use (3.6) and (3.7) again:

$$E^{\frac{8}{7}}(\frac{r}{4}) \le \Big(C\Big(\frac{r}{2}\Big) + C^{\frac{3}{2}}\Big(\frac{r}{2}\Big) + D\Big(\frac{r}{2}\Big)\Big)^{\frac{8}{7}} \le c(C_0)\Big(\mathcal{E}^{\frac{1}{2}}(r) + \mathcal{E}^{\frac{3}{4}}(r)\Big)^{\frac{8}{7}} \le c(C_0)\Big(\mathcal{E}^{\frac{4}{7}}(r) + \mathcal{E}^{\frac{6}{7}}(r)\Big).$$

Combining the last estimate with (3.9), we arrive at

$$\mathcal{E}(\theta r) \le c(C_0)\Big[\theta^{-\frac{1}{4}}\mathcal{E}^{\frac{1}{2}}(r) + \theta^{-\frac{3}{8}}\mathcal{E}^{\frac{3}{4}}(r) + \theta^{-\frac{4}{3}}\Big(\mathcal{E}^{\frac{4}{7}}(r) + \mathcal{E}^{\frac{6}{7}}(r)\Big)\Big] + c\theta^{\frac{4}{3}}\mathcal{E}(r).$$

Taking $\varepsilon > 0$ and using Young's inequality $\theta^{\beta}\mathcal{E}^{\alpha}(r) \le \varepsilon\mathcal{E}(r) + c(\varepsilon, \theta, \alpha, \beta)$ for any $\alpha < 1$, $\beta \in \mathbb{R}$, we proceed to

$$\mathcal{E}(\theta r) \le \mathcal{E}(r)(\varepsilon + c\theta^{\frac{4}{3}}) + F(\varepsilon, C_0, \theta)$$

where $F(\varepsilon, C, \theta)$ is a continuous function which is nondecreasing with respect to C and has the following property:

$$\text{for any fixed } \ \varepsilon, \theta \in (0,1), \quad F(\varepsilon, C, \theta) \to 0 \quad \text{as} \quad C \to +0.$$

Let us fix $\theta \in (0, \frac{1}{16})$ and then fix $\varepsilon \in (0,1)$ so that $\varepsilon + c\theta^{\frac{4}{3}} \le \frac{1}{2}$. Then

$$\mathcal{E}(\theta r) \le \frac{1}{2}\,\mathcal{E}(r) + F(C_0) \quad \text{for all} \quad r \in (0,1).$$

Using standard iteration technique, we conclude that

$$\sup_{r<1} \mathcal{E}(r) \le c\,F(C_0) < +\infty. \tag{3.10}$$

Theorem 1.1 is proved. □

We complete the paper with the proof of Theorem 1.2.

Proof. Assume that $C_0 \leq \varepsilon$. As the function $F(C)$ in (3.10) is continuous, nondecreasing, and tends to zero as $C \to +0$, we can fix $\varepsilon > 0$ so that

$$\sup_{r<1} \mathcal{E}(r) \leq \varepsilon_*,$$

where $\varepsilon_* > 0$ is the absolute constant from the boundary analog of the Caffarelli–Kohn–Nirenberg theorem, see [11]. Then Theorem 1.2 follows from results of [11], see also [18,19]. □

The author thanks Timofey Shilkin for the statement of the problem and Alexander Mikhaylov for valuable discussions.

REFERENCES

1. J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer (1976).
2. L. Caffarelli, R. V. Kohn, and L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier–Stokes equations," *Comm. Pure Appl. Math.*, **35**, 771–831 (1982).
3. L. Escauriaza, G. Seregin, and V. Sverak, "$L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness," *Russian Math. Surveys*, **58**, No. 2, 211–250, (2003).
4. L. Grafakos, *Classical Fourier Analysis*, Springer, New York (2009).
5. K. Kang, "Regularity of axially symmetric flows in a half-space in three dimensions," *SIAM J. Math. Anal.*, **35**, No. 6, 1636–1643 (2004).
6. G. Koch, N. Nadirashvili, G. Seregin, and V. Sverak, "Liouville theorems for the Navier–Stokes equations and applications," *Acta Math.*, **203**, No. 1, 83–105 (2009).
7. O. A. Ladyzhenskaya, "On the unique solvability in large of a three-dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry," *Zap. Nauchn. Semin. LOMI* **7**, 155–177 (1968).
8. Z. Lei and Q. Zhang, "A Liouville theorem for the axi-symmetric Navier–Stokes equations," *J. Funct. Anal.*, **261**, No. 8, 2323–2345 (2011).
9. S. Leonardi, J. Malek, J. Necas, and M. Pokorny, "On axially symmetric flows in $\mathbb{R}^3$," *J. Anal. Appl.*, **18**, No. 3, 639–649 (1999).
10. A. Mikhaylov, "Local regularity for suitable weak solutions of the Navier–Stokes equations near the boundary," *Zap. Nauchn. Semin. LOMI* **370**, 73–93 (2009).
11. G. A. Seregin, "Local regularity of suitable weak solutions to the Navier–Stokes equations near the boundary," *J. Math. Fluid Mech.*, **4**, No. 1, 1–29 (2002).
12. G. A. Seregin, "On smoothness of $L_{3,\infty}$-solutions to the Navier–Stokes equations up to boundary," *Math. Ann.*, **332**, 219–238 (2005).
13. G. Seregin and W. Zajaczkowski, "A sufficient condition of local regularity for the Navier–Stokes equations," *Zap. Nauchn. Semin. POMI*, **336**, 46–54 (2006).
14. G. A. Seregin, "Local regularity for suitable weak solutions of the Navier–Stokes equations," *Russian Math. Surveys*, **62**, No. 3, 595–614 (2007).
15. G. Seregin and V. Sverak, "On type I singularities of the local axi-symmetric solutions of the Navier–Stokes equations," *Comm. PDE's*, **34**, No. 1–3, 171–201 (2009).
16. G. A. Seregin, "Note on bounded scale-invariant quantities for the Navier–Stokes equations," *Zap. Nauchn. Semin. POMI*, **397**, 150–156 (2011).
17. G. Seregin and V. Sverak, "Rescalings at possible singularities of Navier–Stokes equations in half-space," *Algebra Anal.*, **25**, No. 5, 146–172 (2013).
18. G. A. Seregin and T. N. Shilkin, "The local regularity theory for the Navier–Stokes equations near the boundary," *Proc. St. Petersburg Math. Soc.*, **15**, 219–244 (2014).
19. G. A. Seregin, *Lecture Notes on Regularity Theory for the Navier–Stokes Equations*, World Scientific Publishing Co., Hackensack, New Jork (2015).

20. G. Seregin and T. Shilkin, "Liouville-type theorems for the Navier–Stokes equations," *Russian Math. Surveys*, **73**, No. 4, 103–170 (2018).
21. G. Seregin and V. Sverak, "Regularity criteria for Navier–Stokes solutions," in: *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, Springer (2018), pp. 829–867.
22. G. Seregin and D. Zhou, "Regularity of solutions to the Navier–Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$," https://arxiv.org/abs/1802.03600 (2018).
23. V. A. Solonnikov, "On estimates of solutions of the non-stationary Stokes problem in anisotropic Sobolev spaces and on estimates for the resolvent of the Stokes operator," *Russian Math. Surveys*, **58**, No. 2, 331–365 (2003).