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A SOLUTION TO THE CAUCHY PROBLEM FOR PARABOLIC EQUATION WITH SINGULAR COEFFICIENTS

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The Cauchy problem for the second order parabolic equation with singular coefficients with respect to t at the first order spatial derivatives, is considered. A solution to this problem is constructed in explicit form. To this purpose a weighted Hölder space with positive power of t as the weight, is defined. The existence, uniqueness, and estimates of the solution are proved. Bibliography: 9 *titles.*

Dedicated to the 85th jubilee of V. A. Solonnikov

The paper is devoted to study the Cauchy problem for a parabolic equation with coefficients singular in t at the first derivatives with respect to spatial variables. To such a problem, the problems for parabolic equations in domains with moving or free (unknown) boundaries are reduced when the required smoothness of the solutions is higher than the smoothness of the boundary of the domain.

Consider, for example, a one-dimensional problem in the domain $\Omega(t) := \{x : bt^\gamma < x < \infty\}$, $0 < \gamma < 1,\ b > 0$,

$$\partial_t u - a\partial_x^2 u = f(x,t) \quad \text{in } \Omega(t),\ 0 < t < T,$$
$$u\big|_{t=0} = u_0(x) \quad \text{in } \Omega(0), \quad u\big|_{x=bt^\gamma} = \varphi(t),\ 0 < t < T.$$

After change of the variable $y = x - bt^\gamma,\ t = t_1$, the problem reduces to a problem with unknown function $u(y + bt_1^\gamma, t_1) =: v(y, t_1)$ in the domain $(0,\infty)$,

$$\partial_{t_1} v - a\partial_y^2 v - b\gamma \frac{1}{t_1^{1-\gamma}}\partial_y v = f(y - bt_1^{1-\gamma}, t_1),\ y \in (0,\infty),\ t_1 \in (0,T),$$

$$v\big|_{t_1=0} = u_0(y), \quad y \in (0,\infty),\ v\big|_{y=0} = \varphi(t_1),\ t_1 \in (0,T),$$

where $1-\gamma \in (0,1)$.

We see that boundary-value problems in noncylindrical domains are reduced to problems for parabolic equations with coefficients singular in t at the first derivatives with respect to spatial variables. A study the problems for smooth functions in noncylindrical domains with moving boundaries whose smoothness is less than the smoothness of solutions was started by M. Gevrey [1]. L. I. Kamynin obtained results on the solvability of one-dimensional boundary-value problems in domains with boundary satisfying the Gevrey condition [2, 3]. In papers by E. A. Baderko [4–6], the studies of one-dimensional and multidimensional boundary-value problems were continued in domains with boundaries of less smoothness as compared with the smoothness of the solution in Hölder spaces. It should be noted that all the studies in the indicated papers [1–6] were carried out by methods of theory of heat potentials and by reducing the problems to the Volterra integral equations of the second kind.

In [7], V. P. Mikhailov proved that if the boundary of the domain is given by the equation $x = -\sqrt{t}\ln t,\ \ t \in (0,1)$, then the solution to a parabolic equation with such a boundary

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is not unique. In this case, after change of the variable $y = x + \sqrt{t}\ln t$, the heat equation $\partial_t u - a\partial_x^2 u = 0$, $x \in (-\sqrt{t}\ln t, \infty)$ takes the form

$$\partial_t v - a\partial_y^2 v + \Big(\frac{1}{2\sqrt{t}}\ln t + \frac{1}{\sqrt{t}}\Big)\partial_y v = 0, \quad y \in (0,\infty).$$

In the present paper, we prove that the Cauchy problem (1), (2) has a unique solution in the weighted Hölder space, and also $\beta_i \in (0, 1/2]$, $i = 1, \dots, n$ in equation (1). For this, the solution of the problem is constructed in explicit form. Then Theorem 2 is established for the problem (1), (2) with $\beta_1 = \ldots = \beta_n = \beta$; this enables us to prove Theorem 1.

Set $D := \mathbb{R}^n$, $n \geq 1$, $D_T := D \times (0,T)$, $x = (x_1, \dots, x_n)$.

In what follows, $C_1, C_2, \dots$ are positive constants.

We consider the Cauchy problem for a parabolic equation the coefficients of which are singular with respect to t at the first derivatives with respect to spatial variables,

$$\partial_t u - a\Delta u - \sum_{i=1}^{n} \frac{b_i}{t^{\beta_i}}\partial_{x_i} u = f(x,t) \text{ in } D_T, \tag{1}$$

$$u|_{t=0} = u_0(x) \text{ in } D, \tag{2}$$

where $a, b_1, \dots, b_n$ are constant coefficients, $a > 0$, and $\beta_i \in (0, 1/2]$, $i = 1, \dots, n$.

In the one-dimensional case, hereinafter we have an equation in the form

$$\partial_t u - a\partial_x^2 u - \frac{b}{t^\beta}\partial_x u = f(x,t).$$

The problem (1), (2) is studied in the classical and weighted Hölder spaces $C_{x\ \ t}^{\,l,l/2}(\overline{\Omega}_T)$ and $C_{\beta}^{2+\alpha}(\Omega_T)$, where l is a nonintegral positive number, $\alpha \in (0,1)$, $\beta > 0$, $\Omega_T = \Omega \times (0,T)$, and $\Omega \subset \mathbb{R}^n$, $n \geq 1$.

The norms $|u|_{\Omega_T}^{(l)}$ and $|u|_{\beta,\Omega_T}^{(2+\alpha)}$ in these spaces are defined by

$$\begin{aligned} |u|_{\Omega_T}^{(l)} = \sum_{2m_0+|m|=0}^{[l]} |\partial_t^{m_0}\partial_x^m u|_{\Omega_T} &+ \sum_{2m_0+|m|=[l]} \Big([\partial_t^{m_0}\partial_x^m u]_{x,\Omega_T}^{(\alpha)} + [\partial_t^{m_0}\partial_x^m u]_{t,\Omega_T}^{(\alpha/2)}\Big) \\ &+ \begin{cases} \sum\limits_{2m_0+|m|=[l]-1} [\partial_t^{m_0}\partial_x^m u]_{t,\Omega_T}^{(\frac{1+\alpha}{2})}, & [l] \geq 1, \\ 0, & [l] = 0, \end{cases} \end{aligned} \tag{3}$$

where $\alpha = l - [l] \in (0,1)$, $m = (m_1, \dots, m_n)$, the m_i are nonnegative numbers, $i = 0, 1, \dots, n$, $|m| = m_1 + \ldots + m_n$,

$$|v|_{\Omega_T} = \max_{(x,t)\in\overline{\Omega}_T} |v|, \ [v]_{x,\Omega_T}^{(\alpha)} = \max_{(x,t),(z,t)\in\overline{\Omega}_T} \frac{\big|v(x,t) - v(z,t)\big|}{|x-z|^\alpha},$$

$$[v]_{t,\Omega_T}^{(\alpha)} = \max_{(x,t),(x,t_1)\in\overline{\Omega}_T} \frac{\big|v(x,t) - v(x,t_1)\big|}{|t-t_1|^\alpha},$$

and

$$\begin{aligned} |u|_{\beta,\Omega_T}^{(2+\alpha)} = \sum_{|m|=0}^{2} |D_x^m u|_{\Omega_T} &+ \sup_{t\leq T} t^\beta |\partial_t u|_{\Omega_t'} + \sum_{|m|=2} \Big([D_x^m u]_{x,\Omega_T}^{(\alpha)} + [D_x^m u]_{t,\Omega_T}^{(\alpha/2)}\Big) \\ &+ \sup_{t\leq T} t^\beta [D_t u]_{x,\Omega_t'}^{(\alpha)} + \sup_{t\leq T} t^{\beta+\alpha/2} [\partial_t u]_{t,\Omega_t'}^{(\alpha/2)} + \sum_{|m|=1} \sup_{t\leq T} t^\beta [D_x^m u]_{\Omega_t'}^{(\frac{1+\alpha}{2})}, \end{aligned} \tag{4}$$

where $\Omega_t' = \Omega \times (t/2, t)$.

Now we state the main result of the paper.

Theorem 1. *Let $\beta_i \in (0, 1/2]$, $i = 1, \dots, n$, $\beta = \max(\beta_1, \dots, \beta_n)$, and $\alpha \in (0, 1)$. Then for any functions*

$$u_0(x) \in C^{2+\alpha}(\overline{D}), \quad f(x,t) \in C^{\alpha,\,\alpha/2}_{x\quad t}(\overline{D}_T),$$

the problem (1), (2) *has a unique solution*

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

such that $u_1(x,t) \in C^{2+\alpha}_{\beta}(D_T)$, $u_2(x,t) \in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\overline{D}_T)$, and the following estimates hold:

$$|u_1|^{(2+\alpha)}_{\beta,D_T} \le C_1 |u_0|^{(2+\alpha)}_{D}, \tag{5}$$

$$|u_2|^{(2+\alpha)}_{D_T} \le C_2 |f|^{(\alpha)}_{D_T}, \tag{6}$$

where u_1 and u_2 are solutions to the problem (1), (2) *with $f(x,t) = 0$, and $u_0(x) = 0$, respectively.*

Remark 1. In Theorem 1, $u_2(x,t)$ belongs to the classical Hölder space $C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\overline{D}_T)$. This is because in contrast to $u_1(x,t)$, we have $u_2|_{t=0} = 0$, $\partial_{x_i} u_2|_{t=0} = 0$, $\partial^2_{x_i x_j} u_2|_{t=0} = 0$, $i, j = 1, \dots, n$.

We find a solution to the problem (1), (2) explicitly.

Lemma 1. *Let $\beta_i \in (0, 1)$, $i = 1, \dots, n$. The solution to the problem* (1), (2) *is of the form*

$$u(x,t) = u_1(x,t) + u_2(x,t), \tag{7}$$

$$u_1(x,t) = \int\limits_{\mathbb{R}^n} u_0(\xi)\Gamma(x - \xi + (c_1 t^{1-\beta_1}, \dots, c_n t^{1-\beta_n}), t)\, d\xi, \tag{8}$$

$$u_2(x,t) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} f(\xi,\tau)\Gamma(x-\xi+(c_1(t^{1-\beta_1}-\tau^{1-\beta_1}), \dots, c_n(t^{1-\beta_n}-\tau^{1-\beta_n})), t)\, d\xi, \tag{9}$$

where $c_i = \frac{b_i}{1-\beta_i}$, $i = 1, \dots, n$, and

$$\Gamma(x,t) = \frac{1}{(2\sqrt{a\pi t})^n}\, e^{-\frac{x^2}{4at}}$$

is a fundamental solution to the heat equation $\partial_t u - a\Delta u = 0$.

Proof. We apply the integral Fourier transform with respect to $x = (x_1, \dots, x_n)$ to the problem (1), (2), see [8]:

$$F[u] = \widetilde{u}(s,t) = \int\limits_{\mathbb{R}^n} u(x,t) e^{-ixs} dx, \quad s = (s_1, \dots, s_n).$$

Then we get the Cauchy problem for the ordinary differential equation

$$\widetilde{u}\,' + \Big(as^2 - i\sum_{i=1}^n b_i t^{1-\beta_i}\Big)\widetilde{u} = \widetilde{f}(s,t), \ t > 0, \quad \widetilde{u}|_{t=0} = \widetilde{u}_0(s). \tag{10}$$

A solution to the problem (10) has the form

$$\widetilde{u}(s,t) = \widetilde{u}_0(s) e^{-as^2 t + i\sum\limits_{k=1}^n \frac{b_k}{1-\beta_k} t^{1-\beta_k} s_k} + \int\limits_0^t f(s,\tau) e^{-as^2(t-\tau)} e^{i\sum\limits_{k=1}^n \frac{b_k}{1-\beta_k}(t^{1-\beta_k} - \tau^{1-\beta_k}) s_k}\, d\tau.$$

After applying the inverse Fourier transform formulas to $\widetilde{u}$,

$$F^{-1}[e^{-as^2t}] = \frac{1}{(2\sqrt{a\pi t})^n} e^{-\frac{x^2}{4at}} = \Gamma(x,t),$$
$$F^{-1}[\widetilde{f}(s)e^{ids}] = f(x+d), \quad f(x) = F^{-1}[\widetilde{f}(s)], \quad d = (d_1,\dots,d_n),$$

we obtain a solution to the problem (1), (2) in the form (7)–(9).

Substituting functions (8), (9) into equation (1) and condition (2), we see that formula (7) gives a solution to the problem (1), (2). ☐

To prove Theorem 1, we first consider the problem

$$\begin{gathered} \partial_t u - a\Delta u - \frac{c}{t^\beta}\nabla^T u = f(x,t) \quad \text{in } D_T, \\ u|_{t=0} = u_0(x) \quad \text{in } D, \end{gathered} \tag{11}$$

where $c = (c_1,\dots,c_n)$, and $\nabla^T =: (\partial_{x_1},\dots,\partial_{x_n})$ is a column vector.

Theorem 2. *Let $\beta \in (0,1/2]$ and $\alpha \in (0,1)$. Then for any functions*

$$u_0(x) \in C^{2+\alpha}(\overline{D}), \quad f(x,t) \in C^{\alpha,\alpha/2}_{x\ \ t}(\overline{D}_T),$$

problem (11) *has a unique solution*

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

such that

$$u_1(x,t) \in C^{2+\alpha}_{\beta}(D_T), \quad u_2(x,t) \in C^{2+\alpha,1+\alpha/2}_{x\ \ \ \ t}(\overline{D}_T),$$

and the following estimates hold:

$$|u_1|^{(2+\alpha)}_{\beta,D_T} \le C_3|u_0|^{(2+\alpha)}_D, \tag{12}$$

$$|u_2|^{(2+\alpha)}_{D_T} \le C_4|f|^{(\alpha)}_{D_T}. \tag{13}$$

Set $\beta_i = \beta,\ i = 1,\dots,n$, in formulas (8) and (9). Then we obtain a solution to problem (11),

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

$$u_1(x,t) = \int\limits_{\mathbb{R}_n} u_0(\xi)\Gamma(x-\xi+ct^{1-\beta},t)\,d\xi, \tag{14}$$

$$u_2(x,t) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}_n} f(\xi,\tau)\Gamma(x-\xi+c(t^{1-\beta}-\tau^{1-\beta}),t)\,d\xi. \tag{15}$$

In formulas (14) and (15), we apply the substitution

$$y = x + ct^{1-\beta}.$$

Then

$$u_1(y-ct^{1-\beta},t) = \int\limits_{\mathbb{R}_n} u_0(\xi)\Gamma(y-\xi,t)\,d\xi =: v_1(y,t), \tag{16}$$

$$u_2(y - ct^{1-\beta}, t) = \int_0^t d\tau \int_{\mathbb{R}_n} f(\xi, \tau)\Gamma(y - \xi - c\tau^{1-\beta}, t - \tau)\, d\xi$$
$$= \int_0^t d\tau \int_{\mathbb{R}_n} f(\eta - c\tau^{1-\beta}, \tau)\Gamma(y - \eta, t - \tau) d\eta =: v_2(y, t); \tag{17}$$

in integral (17), we have also changed the integration variable $\eta = \xi + c\tau^{1-\beta}$ and got the function $f(\eta - c\tau^{1-\beta}, \tau)$.

Lemma 2. *Let* $\beta \in (0, 1/2]$ *and* $f(x, t) \in C_{x\ t}^{\alpha,\alpha/2}(\overline{D}_T)$*. Then* $f(y - ct^{1-\beta}, t) =: f_1(y, t) \in C_{y\ t}^{\alpha,\alpha/2}(\overline{D}_T)$ *and*

$$|f_1|_{D_T}^{(\alpha)} \le C_5 |f(y, t)|_{D_T}^{(\alpha)}. \tag{18}$$

Proof. The norm of the function $f_1(y, t)$ in $C_{y\ t}^{\alpha,\alpha/2}(\overline{D}_T)$ is defined by formula (3),

$$|f_1|_{D_T}^{(\alpha)} := |f_1|_{D_T} + [f_1]_{y,D_T}^{(\alpha)} + [f_1]_{t,D_T}^{(\alpha/2)}. \tag{19}$$

Obviously,

$$|f_1|_{D_T} = |f(y - ct^{1-\beta}, t)|_{D_T} = |f(y, t)|_{D_T}, \tag{20}$$
$$|f_1(y, t) - f_1(\xi, t)| = |f(y - ct^{1-\beta}, t) - f(\xi - ct^{1-\beta}, t)| \le [f(y, t)]_{y,D_T}^{(\alpha)} |y - \xi|^\alpha,$$

and

$$[f_1]_{y,D_T}^{(\alpha)} \le [f(y, t)]_{y,D_T}^{(\alpha)}. \tag{21}$$

We estimate the difference

$$|\Delta| = \big|f_1(y, t) - f_1(y, t_1)\big| \le \big|f(y - ct^{1-\beta}, t) - f(y - ct_1^{1-\beta}, t)\big| + \big|f(y - ct_1^{1-\beta}, t) - f(y - ct_1^{1-\beta}, t_1)\big|.$$

Let $t_1 < t$. Then

$$|\Delta| \le C_6\big([f(y, t)]_{y,D_T}^{(\alpha)} (t^{1-\beta} - t_1^{1-\beta})^\alpha + [f(y, t)]_{t,D_T}^{(\alpha/2)} (t - t_1)^{\alpha/2}\big).$$

Since

$$(t^{1-\beta} - t_1^{1-\beta})^\alpha \le C_7 (t - t_1)^{\alpha(1-\beta)} \le C_7 (t - t_1)^{\alpha/2}\, t^{\alpha(1/2-\beta)},$$

we have

$$|\Delta| = |f_1(y, t) - f_1(y, t_1)| \le C_8\big([f(y, t)]_{y,D_T}^{(\alpha)} + [f(y, t)]_{t,D_T}^{(\alpha/2)}\big)(t - t_1)^{\alpha/2}$$

and

$$[f_1]_{t,D_T}^{(\alpha/2)} \le C_8\big([f(y, t)]_{y,D_T}^{(\alpha)} + [f(y, t)]_{t,D_T}^{(\alpha/2)}\big). \tag{22}$$

Applying relations (20)–(22) in formula (19), we get estimate (18) which proves Lemma 2. □

Proof of Theorem 2. By direct estimates of potentials (14) and (15), one can prove the theorem as in [9]. But we use another way.

We have presented a solution to problem (11) in form (16), (17).

Let us consider the function $v_1(y, t)$ which is the potential (16). It is easily seen that $v_1(y, t)$ is a solution to the Cauchy problem

$$\partial_t v_1 - a\Delta_y v_1 = 0 \text{ in } D_T, \quad v_1|_{t=0} = u_0(y) \text{ in } D.$$

But then $v_1(y, t) \in C_{y\ \ t}^{2+\alpha,1+\alpha/2}(\overline{D}_T)$ (see [9]) and

$$|v_1(y, t)|_{D_T}^{(2+\alpha)} \le C_9 |u_0|_D^{(2+\alpha)}. \tag{23}$$

After the substitution $y = x + ct^{1-\beta}$ in formula (14), we put

$$u_1(y - ct^{1-\beta}, t) = v_1(y, t). \tag{24}$$

Relation (24) can be written in the form

$$u_1(x, t) = v_1(x + ct^{1-\beta}, t).$$

It is easily seen that

$$\begin{aligned} |u_1(x,t)|_{D_T} &= |v_1(x + ct^{1-\beta}, t)|_{D_T} = |v_1(y,t)|_{D_T}, \\ |\partial_{x_i} u_1(x,t)|_{D_T} &= |\partial_{x_i} v_1(x + ct^{1-\beta}, t)|_{D_T} = |\partial_{y_i} v_1(y,t)|_{D_T}, \\ |\partial^2_{x_i x_j} u_1(x,t)|_{D_T} &= |\partial^2_{x_i x_j} v_1(x + ct^{1-\beta}, t)|_{D_T} = |\partial^2_{y_i y_j} v_1(y,t)|_{D_T}, \end{aligned} \tag{25}$$

and

$$[\partial^2_{x_i x_j} u_1(x,t)]^{(\alpha)}_{x,D_T} = [\partial^2_{x_i x_j} v_1(x + ct^{1-\beta}, t)]^{(\alpha)}_{x,D_T} = [\partial^2_{y_i y_j} v_1(y,t)]^{(\alpha)}_{y,D_T}, \quad i,j = 1,\dots,n. \tag{26}$$

In what follows, we need the following estimates for $t/2 \le t_1 < t$ and $\beta \in (0, 1/2]$:

$$(t^{1-\beta} - t_1^{1-\beta})^\alpha \le (t - t_1)^{\alpha/2 + \alpha(1/2 - \beta)}, \tag{27}$$

$$\frac{1}{t_1^\beta} - \frac{1}{t^\beta} = \beta \int_{t_1}^{t} \frac{d\tau}{\tau^{1+\beta}} = \beta \int_{t_1}^{t} \frac{d\tau}{\tau^{1-\alpha/2}\, \tau^{\alpha/2+\beta}} \le C_{10} \frac{1}{t^{\beta+\alpha/2}} (t - t_1)^{\alpha/2}, \tag{28}$$

$$t^{1-\beta} - t_1^{1-\beta} = (1-\beta) \int_{t_1}^{t} \frac{\tau^{\frac{1-\alpha}{2}}}{\tau^{\beta + \frac{1-\alpha}{2}}} d\tau \le C_{11} \frac{1}{t^\beta} (t - t_1)^{\frac{1+\alpha}{2}} t^{\frac{1-\alpha}{2}}, \tag{29}$$

$$t^{1-\beta} - t_1^{1-\beta} \le C_{11} (t - t_1)^{\alpha/2}\, t^{1/2 - \beta + \frac{1-\alpha}{2}}. \tag{30}$$

Let us consider the difference

$$\begin{aligned} \partial^2_{x_i x_j} u_1(x,t) - \partial^2_{x_i x_j} u_1(x,t_1) &= \big(\partial^2_{x_i x_j} v_1(x + ct^{1-\beta}, t) - \partial^2_{x_i x_j} v_1(x + ct_1^{1-\beta}, t)\big) \\ &\quad + \big(\partial^2_{x_i x_j} v_1(x + ct_1^{1-\beta}, t) - \partial^2_{x_i x_j} v_1(x + ct_1^{1-\beta}, t_1)\big). \end{aligned} \tag{31}$$

Taking into account that $v_1(y,t) \in C^{\,2+\alpha, 1+\alpha/2}_{\;y \qquad t}(\overline{D}_T)$ and using inequality (27), we obtain

$$\begin{aligned} &|\partial^2_{x_i x_j} u_1(x,t) - \partial^2_{x_i x_j} u_1(x,t_1)| \\ &\quad \le C_{12} [\partial^2_{y_i y_j} v_1(y,t)]^{\alpha}_{y,D_T} (t^{1-\beta} - t_1^{1-\beta})^\alpha + C_{13} [\partial^2_{y_i y_j} v_1(y,t)]^{\alpha/2}_{t,D_T} (t - t_1)^{\alpha/2} \\ &\quad \le C_{14} \Big([\partial^2_{y_i y_j} v_1(y,t)]^{(\alpha)}_{y,D_T} + [\partial^2_{y_i y_j} v_1(y,t)]^{(\alpha/2)}_{t,D_T} \Big) (t - t_1)^{\alpha/2}, \end{aligned} \tag{32}$$

$i, j = 1, \dots, n$, and

$$[\partial^2_{x_i x_j} u_1]^{(\alpha/2)}_{t,D_T} \le C_{15} \Big([\partial^2_{y_i y_j} v_1(y,t)]^{(\alpha)}_{y,D_T} + [\partial^2_{y_i y_j} v_1(y,t)]^{(\alpha/2)}_{t,D_T} \Big). \tag{33}$$

We estimate the derivative

$$\partial_t u_1(x,t) = \partial_t v_1(x + ct^{1-\beta}, t) = (1-\beta) \frac{1}{t^\beta} c \nabla^T_x v_1(x + ct^{1-\beta}, t) + \partial_{t^*} v_1(x + ct^{1-\beta}, t^*)|_{t^* = t} \tag{34}$$

and its Hölder constants.

From (34), we obtain the estimates

$$|\partial_t u_1(x,t)| \le C_{16} \Big(\frac{1}{t^\beta} \sum_{i=1}^{n} |\partial_{y_i} v_1(y,t)|_{D_t} + |\partial_t v_1(y,t)|_{D_t} \Big) \tag{35}$$

and

$$\sup_{t\le T} t^{\beta}|\partial_t u_1|_{D_t'} \le C_{17}\Big(\sum_{i=1}^{n}|\partial_{y_i}v_1(y,t)|_{D_T}+|\partial_t v_1(y,t)|_{D_T}\Big), \tag{36}$$

where $D_t' = D\times(t/2,t)$. Taking formula (34) into account, we consider the difference

$$\begin{aligned}\Delta_1 := \partial_t u_1(x,t)-\partial_t u_1(x,t_1) &= \big(\partial_{t^*}v_1(x+ct^{1-\beta},t^*)|_{t^*=t}-\partial_{t^*}v_1(x+ct_1^{1-\beta},t^*)|_{t^*=t}\big)\\ &+\big(\partial_{t^*}v_1(x+ct_1^{1-\beta},t^*))|_{t^*=t}-\partial_{t^*}v_1(x+ct_1^{1-\beta},t^*))|_{t^*=t_1}\big)\\ &+(1-\beta)\Big(\frac{1}{t^{\beta}}-\frac{1}{t_1^{\beta}}\Big)c\nabla_x^T v_1(x+ct^{1-\beta},t)\\ &+\frac{1}{t_1^{\beta}}\big(c\nabla_x^T v_1(x+ct^{1-\beta},t)-c\nabla_x^T v_1(x+ct_1^{1-\beta},t)\big)\\ &+\frac{1}{t_1^{\beta}}\big(c\nabla_x^T v_1(x+ct_1^{1-\beta},t)-c\nabla_x^T v_1(x+ct_1^{1-\beta},t_1)\big)=:\sum_{i=1}^{5}\Delta_{1i}.\end{aligned} \tag{37}$$

In view of inequalities (27), (28), (30), and taking into account that $\beta\in(0,1/2]$, and $v_1(y,t)\in C^{\,2+\alpha,1+\alpha/2}_{\;y\;\;\;\;\;t}(\overline{D}_T)$, we have

$$\begin{aligned}|\Delta_{11}| &\le C_{18}[\partial_t v_1(y,t)]^{(\alpha)}_{y,D_T}(t-t_1)^{\alpha/2}t^{\alpha(1/2-\beta)},\\ |\Delta_{12}| &\le C_{19}[\partial_t v_1(y,t)]^{(\alpha/2)}_{t,D_T}(t-t_1)^{\alpha/2},\\ |\Delta_{13}| &\le C_{20}\frac{1}{t^{\beta+\alpha/2}}\sum_{i=1}^{n}|\partial_{y_i}v_1(y,t)|_{D_t}(t-t_1)^{\alpha/2},\\ |\Delta_{14}| &\le C_{21}\sum_{i,j=1}^{n}\frac{1}{t^{\beta}}|\partial^2_{y_iy_j}v_1(y,t)|(t-t_1)^{\alpha/2}\,t^{\frac{1-\alpha}{2}+(1/2-\beta)}\\ |\Delta_{15}| &\le C_{22}\sum_{i=1}^{n}[\partial_{y_i}v_1(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}(t-t_1)^{\alpha/2}\,t^{1/2-\beta}.\end{aligned}$$

Applying the established estimates for relation (37),

$$\begin{aligned}|\partial_t u_1(x,t)-\partial_{t_1}u_1(x,t_1)| \le C_{23}\big([\partial_t v_1(y,t)]^{(\alpha)}_{y,D_T}&+[\partial_t v_1(y,t)]^{(\alpha/2)}_{t,D_T}\big)\\ +\sum_{i=1}^{n}\Big(&\frac{1}{t^{\beta+\alpha/2}}|\partial_{y_i}v_1(y,t)|_{D_t}+[\partial_{y_i}v_1(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big)\\ +\sum_{i,j=1}^{n}&\frac{1}{t^{\beta}}|\partial^2_{y_iy_j}v_1(y,t)|t^{\frac{1-\alpha}{2}+1/2-\beta}\Big)(t-t_1)^{\alpha/2},\end{aligned} \tag{38}$$

we obtain

$$\sup_{t\le T} t^{\beta+\alpha/2}[\partial_t u_1]^{(\alpha/2)}_{t,D_t'} \le C_{24}|v_1(y,t)|^{(2+\alpha)}_{D_T}. \tag{39}$$

Taking into account formula (34), we consider the difference

$$\begin{aligned}\Delta_2 &= \partial_t u_1(x,t)-\partial_t u_1(z,t) = \partial_t v_1(x+ct^{1-\beta},t)-\partial_t v_1(z+ct^{1-\beta},t)\\ &=(1-\beta)\frac{1}{t^{\beta}}\big(c\nabla_x^T v_1(x+ct^{1-\beta},t)-c\nabla_z^T v_1(z+ct^{1-\beta},t)\big)\\ &\quad+\big(\partial_{t^*}v_1(x+ct^{1-\beta},t^*)\big|_{t^*=t}-\partial_{t^*}v_1(z+ct^{1-\beta},t^*)\big)|_{t^*=t_1},\end{aligned}$$

which can be estimated as follows:

$$|\Delta_2| \le \Big((1-\beta)\frac{1}{t^\beta}\frac{|c\nabla_x^T v_1(x+ct^{1-\beta},t) - c\nabla_z^T v_1(z+ct^{1-\beta},t))|^\alpha}{|x-z|^\alpha}$$
$$\times \left|c\nabla_x^T v_1(x+ct^{1-\beta},t) - c\nabla_z^T v_1(z+ct^{1-\beta},t)\right|^{1-\alpha} + [\partial_t v_1(y,t)]^{(\alpha)}_{y,D_T}\Big)|x-z|^\alpha$$
$$\le C_{25}\Big(\frac{1}{t^\beta}\big(\sum_{i,j=1}^n |\partial^2_{y_iy_j} v_1(y,t)|_{D_t}\big)^\alpha\big(\sum_{i=1}^n |\partial_{y_i} v_1(y,t)|_{D_t}\big)^{1-\alpha} + [\partial_t v_1(y,t)]^{(\alpha)}_{y,D_T}\Big)|x-z|^\alpha.$$

We apply Young's inequality,

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad 1/p + 1/q = 1, \quad p > 1, \tag{40}$$

with $1/p = \alpha$ and $1/q = 1-\alpha$. Then

$$|\Delta_2| \le C_{26}\Big(\frac{1}{t^\beta}\Big(\sum_{i,j=1}^n |\partial^2_{y_iy_j} v_1(y,t)|_{D_t} + \sum_{i=1}^n |\partial_{y_i} v_1(y,t)|_{D_t}\Big) + [\partial_t v_1(y,t)]^{(\alpha)}_{y,D_T}\Big)|x-z|^\alpha. \tag{41}$$

It follows that

$$\sup_{t\le T} t^\beta [\partial_t u_1]^{(\alpha)}_{x,D'_t} \le C_{27}\Big(\sum_{i,j=1}^n |\partial^2_{y_iy_j} v_1(y,t)|_{D_T} + \sum_{i=1}^n |\partial_{y_i} v_1(y,t)|_{D_T} + [\partial_t v_1(y,t)]^{(\alpha)}_{y,D_T}\Big). \tag{42}$$

Finally, we estimate the difference

$$\Delta_3 = \partial_{x_i} u_1(x,t) - \partial_{x_i} u_1(x,t_1)$$
$$= \big(\partial_{x_i} v_1(x+ct^{1-\beta},t) - \partial_{x_i} v_1(x+ct_1^{1-\beta},t)\big) + \big(\partial_{x_i} v_1(x+ct_1^{1-\beta},t) - \partial_{x_i} v_1(x+ct_1^{1-\beta},t_1)\big).$$

By inequality (29),

$$|\Delta_3| \le C_{28}\Big(\sum_{i,j=1}^n \frac{1}{t^\beta}|\partial^2_{y_iy_j} v_1(y,t)|_{D_t}\, t^{\frac{1-\alpha}{2}} + [\partial_{y_i} v_1(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_t}\Big)(t-t_1)^{\frac{1+\alpha}{2}}. \tag{43}$$

It follows that

$$\sum_{i=1}^n \sup_{t\le T} t^\beta [\partial_{x_i} u_1]^{(\frac{1+\alpha}{2})}_{D_T} \le C_{29}\Big(\sum_{i,j=1}^n |\partial^2_{y_iy_j} v_1(y,t)|_{D_T} + \sum_{i=1}^n [\partial_{y_i} v_1(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big). \tag{44}$$

Applying estimates (25), (26), (33), (36), (39), (42), (44), and also (23) for the function $v_1(y,t)$, we obtain the following estimate for the norm (4) of the function $u_1(x,t)$:

$$|u_1|^{(2+\alpha)}_{\beta,D_T} \le C_{30}|v_1(y,t)|^{(2+\alpha)}_{D_T} \le C_{31}|u_0|^{(2+\alpha)}_{D_T},$$

i.e., estimate (12) in Theorem 2.

Let us prove estimate (13) for the function

$$u_2(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n} f(\xi,\tau)\Gamma(x-\xi+c(t^{1-\beta}-\tau^{1-\beta}), t-\tau)\, d\xi.$$

After the substitution $y = x + ct^{1-\beta}$, it has been written in the form (17),

$$u_2(y-ct^{1-\beta},t) =: v_2(y,t) = \int_0^t d\tau \int_{\mathbb{R}^n} f(\eta - c\tau^{1-\beta},\tau)\Gamma(y-\eta,t-\tau)d\eta, \tag{45}$$

where

$$f(y-ct^{1-\beta},t)\in C_{y\ t}^{\alpha,\alpha/2}(\overline{D}_T)$$

and

$$|f(y-ct^{1-\beta},t)|_{D_T}^{(\alpha)}\le C_3|f(x,t)|_{D_T}^{(\alpha)},$$

see Lemma 2. But then (see [9]), $v_2(y,t)\in C_{y\ t}^{2+\alpha,1+\alpha/2}(\overline{D}_T)$ and

$$|v_2(y,t)|_{D_T}^{(2+\alpha)}\le C_{32}|f(y,t)|_{D_T}^{(\alpha)}.$$

First, we estimate the function $v_2(y,t)$ defined by formula (45),

$$|v_2(y,t)|\le|f(y,t)|_{D_T}t. \tag{46}$$

Then

$$\partial_{y_i}v(y,t)=\int_0^t d\tau\int_{\mathbb{R}^n}\big(f(\eta-c\tau^{1-\beta},\tau)-f(y-c\tau^{1-\beta},\tau)\big)\Gamma_{y_i}(y-\eta,t-\tau)d\eta,$$

$$\partial_{y_iy_j}v(y,t)=\int_0^t d\tau\int_{\mathbb{R}^n}\big(f(\eta-c\tau^{1-\beta},\tau)-f(y-c\tau^{1-\beta},\tau)\big)\Gamma_{y_iy_j}(y-\eta,t-\tau)d\eta,\quad i,j=1,\dots,n.$$

Since $f(y-ct^{1-\beta},t)\in C_{y\ t}^{\alpha,\alpha/2}$, we can use the estimate of the kernel $\Gamma(x,t)$,

$$|\partial_t^{m_0}\partial_x^m\Gamma(x,t)|\le C_{33}\frac{1}{t^{\frac{n+2m_0+|m|}{2}}}e^{-\frac{x^2}{8at}}$$

to obtain

$$|\partial_{y_i}v_2(y,t)|\le C_{34}[f(y,t)]_{y,D_T}^{(\alpha)}t^{\frac{1+\alpha}{2}}, \tag{47}$$

$$|\partial^2_{y_iy_j}v_2(y,t)|\le C_{35}[f(y,t)]_{y,D_T}^{(\alpha)}t^{\alpha/2},\ i,j=1,\dots,n. \tag{48}$$

Now we set $x=y-ct^{1-\beta}$ in the formula $u_2(y-ct^{1-\beta},t)=v_2(y,t)$. Then $u_2(x,t)=v_2(x+ct^{1-\beta},y)$.

For the function $u_2(x,t)$, we obtain the same estimates as for the function $u_1(x,t)=v_1(x+ct^{1-\beta},y)$. We make use of them with taking into account estimates (46)–(48). Then from formula (35), written for $u_2(x,t)$, and in view of (47), we have

$$\begin{aligned}|\partial_t u_2(x,t)|&\le C_{36}\Big(\frac{1}{t^\beta}\sum_{i=1}^n|\partial_{y_i}v_2(y,t)|+|\partial_t v_2(y,t)|\Big)\\&\le C_{37}\Big(\sum_{i=1}^n|\partial_{y_i}v_2(y,t)|t^{\frac{1+\alpha}{2}-\beta}+|\partial_t v_2(y,t)|\Big)\\&\le C_{38}\Big(\sum_{i=1}^n|\partial_{y_i}v_2(y,t)|_{D_T}+|\partial_t v_2(y,t)|_{D_T}\Big),\quad \beta\in(0,1/2].\end{aligned} \tag{49}$$

We estimate the Hölder constants of the function $u_2(x,t)$. To this end, we write inequality (38) for the function $u_2(x,t)$ with taking into account estimates (47) and (48) of the derivates $\partial_{y_i}v_2(y,t)$ and $\partial^2_{y_iy_j}v_2(y,t)$,

$$\begin{aligned}|\partial_t u_2(x,t)-\partial_{t_1}u_2(x,t_1)|&\le C_{39}\Big([\partial_t v_2(y,t)]_{y,D_T}^{(\alpha)}+[\partial_t v_2(y,t)]_{t,D_t}^{(\alpha/2)}\\&+\frac{1}{t^{\beta+\alpha/2}}[f(y,t)]_{y,D_T}^{(\alpha)}t^{\frac{1+\alpha}{2}}+[\partial_{y_i}v_2(y,t)]_{t,D_T}^{(\frac{1+\alpha}{2})}+\frac{1}{t^\beta}[f(y,t)]_{y,D_T}^{(\alpha)}t^{\frac{1-\alpha}{2}+1/2-\beta+\alpha/2}\Big)(t-t_1)^{\alpha/2},\end{aligned}$$

where in the third and fifth summands on the right-hand side of the inequality,

$$t^{\frac{1+\alpha}{2}-\beta-\alpha/2}=t^{1/2-\beta},\ \ t^{\frac{1-\alpha}{2}+1/2-\beta+\alpha/2-\beta}=t^{2(1/2-\beta)},$$

respectively, $\beta\in(0,1/2]$. This inequality gives the estimate

$$[\partial_t u_2]^{(\alpha/2)}_{t,D_T}\le C_{40}\Big(|v_2(y,t)|^{(2+\alpha)}_{D_T}+[f]^{(\alpha)}_{x,D_T}\Big). \tag{50}$$

We consider the difference obtained by formula (34),

$$\Delta_4=\partial_t u_2(x,t)-\partial_t u_2(z,t)=\partial_t v_2(x+ct^{1-\beta},t)-\partial_t v_2(z+ct^{1-\beta},t)=\Delta_{41}+\Delta_{42}, \tag{51}$$

$$\Delta_{41}=(1-\beta)\frac{1}{t^\beta}\Big(c\nabla^T_x v_2(x+ct^{1-\beta},t)-c\nabla^T_z v_2(z+ct^{1-\beta},t)\Big),$$
$$\Delta_{42}=\Big(\partial_{t^*}v_2(x+ct^{1-\beta},t^*)-\partial_{t^*}v_2(z+ct^{1-\beta},t^*)\Big)|_{t^*=t}.$$

Let us estimate Δ_{41} in a different way, not like Δ_2 for the function $\partial_t u_1(x,t)$ (see (41) and (42)). Here we take into account that

$$v_2|_{t=0}=0,\quad \partial_{x_i}v_2|_{t=0}=0,\quad \partial_{x_ix_j}v_2|_{t=0}=0.$$

The term Δ_{41} includes the derivatives $\partial_{x_i}v_2(x+ct^{1-\beta},t)$, $i=1,\dots,n$. Therefore for the sake of simplicity, we consider the difference of one derivative

$$\Delta_5=\partial_{x_i}v_2(x+ct^{1-\beta},t)-\partial_{x_i}v_2(x+ct^{1-\beta},0)-\partial_{z_i}v_2(z+ct^{1-\beta},t)+\partial_{z_i}v_2(z+ct^{1-\beta},0),\quad i=1,\dots,n.$$

We have

$$\begin{aligned}|\Delta_5|&=\frac{|\partial_{x_i}v_2(x+ct^{1-\beta},t)-\partial_{z_i}v_2(z+ct^{1-\beta},t)|^\alpha}{|x-z|^\alpha}\Big|\big(\partial_{x_i}v_2(x+ct^{1-\beta},t)-\partial_{x_i}v_2(x+ct^{1-\beta},0)\big)\\&\quad-\big(\partial_{z_i}v_2(z+ct^{1-\beta},t)-\partial_{z_i}v_2(z+ct^{1-\beta},0)\big)\Big|^{1-\alpha}|x-z|^\alpha\\&\le C_{41}\Big(\sum_{j=1}^n|\partial^2_{y_iy_j}v_2(y,t)|_{D_t}\Big)^\alpha\Big([\partial_{y_i}v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big)^{1-\alpha}t^{\frac{1+\alpha}{2}(1-\alpha)}|x-z|^\alpha.\end{aligned}$$

Applying estimate (48) to $|\partial^2_{y_iy_j}v_2|$, and then Young's inequality (40) with $1/p=\alpha$ and $1/q=1-\alpha$, we obtain

$$|\Delta_5|\le C_{42}t^{\alpha^2/2+1/2-\alpha^2/2}\Big([f(y,t)]^{(\alpha)}_{y,D_T}+[\partial_{y_i}v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big)|x-z|^\alpha.$$

For the difference Δ_{41} in (51), this gives

$$|\Delta_{41}|\le C_{43}\big([f(y,t)]^{(\alpha)}_{y,D_T}+\sum_{i=1}^n[\partial_{y_i}v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\big)t^{1/2-\beta}.$$

For the difference Δ_{42}, we have

$$|\Delta_{42}|\le C_{44}[\partial_t v_2(y,t)]^{(\alpha/2)}_{y,D_T}.$$

This together with (51) implies that

$$\begin{aligned}|\partial_t u_2(x,t)-\partial_t u_2(z,t)|\le C_{45}\Big(&\big([f(y,t)]^{(\alpha)}_{y,D_T}+\sum_{i=1}^n[\partial_{y_i}v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\big)t^{1/2-\beta}\\&+[\partial_t v_2(y,t)]^{(\alpha)}_{y,D_T}\Big)|x-z|^\alpha,\quad \beta\in(0,1/2],\end{aligned}$$

and

$$[\partial_t u_2]^{(\alpha)}_{x,D_T}\le C_{46}\Big(|v_2(y,t)|^{(2+\alpha)}_{D_T}+[f(y,t)]^{(\alpha)}_{y,D_T}\Big). \tag{52}$$

For the difference

$$\Delta_6 = \partial_{x_i} u_2(x,t) - \partial_{x_i} u_2(x,t_1),$$

estimate (43) holds with $v_1(y,t)$ replaced by $v_2(y,t)$. After taking into account inequality (48), $|\partial^2_{y_i y_j} v_2(y,t)| \le C_{35}[f]^{(\alpha)}_{y,D_T} t^{\alpha/2}$, we obtain

$$|\Delta_6| \le C_{47}\Big(\sum_{j=1}^n |\partial^2_{y_i y_j} v_2(y,t)|_{D_t} t^{\frac{1-\alpha}{2}-\beta} + [\partial_{y_i} v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big)(t-t_1)^{\frac{1+\alpha}{2}}$$
$$\le C_{48}\Big([f(y,t)]^{(\alpha)}_{y,D_T} t^{\alpha/2-\alpha/2+(1/2-\beta)} + [\partial_{y_i} v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big)(t-t_1)^{\frac{1+\alpha}{2}}.$$

It follows that

$$\sum_{i=1}^n [\partial_{x_i} u_2]^{(\frac{1+\alpha}{2})}_{t,D_T} \le C_{49}\Big([f(y,t)]^{(\alpha)}_{y,D_T} + \sum_{i=1}^n [\partial_{y_i} v_2(y,t)]^{(\frac{1+\alpha}{2})}_{t,D_T}\Big). \tag{53}$$

Collecting estimates (25), (26), (33) with $u_1(x,t)$ and $v_1(y,t)$ replaced by $u_2(x,t)$ and $v_2(y,t)$, and also (49), (50), (52), (53), we establish estimate (13): $|u_2|^{(2+\alpha)}_{\beta,D_T} \le C_3|f|^{(\alpha)}_{D_T}$. This completes the proof of Theorem 2. □

Proof of Theorem 1. The proof of the theorem is as that of Theorem 2. We consider the functions $u_1(x,t)$ and $u_2(x,t)$, defined by formulas (8) and (9), respectively. For the sake of convenience, set

$$d(t) = (c_1 t^{1-\beta_1}, \dots, c_n t^{1-\beta_n}), \quad c_i = \frac{b_i}{1-\beta_i}, \quad i = 1,\dots,n.$$

In formulas (8) and (9), we change the variable

$$x = y + (c_1 t^{1-\beta_1}, \dots, c_n t^{1-\beta_n}) \equiv x + d(t).$$

Changing the integration variable in (9), $\eta = \xi + d(\tau)$, we get

$$u_1(y-d(t),t) = \int_{\mathbb{R}_n} u_0(\xi)\Gamma(y-\xi,t)\,d\xi =: w_1(y,t), \tag{54}$$

$$u_2(y-d(t),t) = \int_0^t d\tau \int_{\mathbb{R}_n} f(\eta - d(\tau),\tau)\Gamma(y-\eta,t-\tau)d\eta =: w_2(y,t), \tag{55}$$

where according to Lemma 2 with $\beta = \max(\beta_1,\dots,\beta_n)$,

$$f(y-d(t),t) \in C^{\alpha,\alpha/2}_{y\ \ t}(\overline{D}_T)$$

and $|f_1|^{(\alpha)}_{D_T} \le C_5|f(y,t)|^{(\alpha)}_{D_T}$.

From formulas (54) and (55), it follows that

$$w_i(y,t) \in C^{2+\alpha,1+\alpha/2}_{y\ \ \ \ t}(\overline{D}_T), \quad i = 1,2,$$

see [9], and

$$|w_1|^{(2+\alpha)}_{D_T} \le C_{50}|u_0|^{(2+\alpha)}_{D}, \ |w_2|^{(2+\alpha)}_{D_T} \le C_{51}|f(y,t)|^{(\alpha)}_{D_T}. \tag{56}$$

We return to the variable x, and set $y = x + d(t)$ in formulas (54) and (55). Then

$$u_j(x,t) = w_j(x+d(t),t), \quad j = 1,2.$$

For the function $u_1(x,t)$, estimates (25) and (26) hold. We consider difference (31) with v_1 replaced by w_1, and apply estimates (32):

$$
\begin{aligned}
&|\partial^2_{x_ix_j}u_1(x,t)-\partial^2_{x_ix_j}u_1(x,t_1)|\\
&\quad=|\partial^2_{x_ix_j}w_1(x+d(t),t)-\partial^2_{x_ix_j}w_1(x+d(t_1),t_1)|\\
&\quad=\Big|\big(\partial^2_{x_ix_j}w_1(x+(c_1t^{1-\beta_1},\cdots,c_nt^{1-\beta_n}),t)-\partial^2_{x_ix_j}w_1(x+(c_1t_1^{1-\beta_1},\cdots,c_nt_1^{1-\beta_n}),t)\big)\\
&\qquad+\big(\partial^2_{x_ix_j}w_1(x+d(t_1),t)-\partial^2_{x_ix_j}w_1(x+d(t_1),t_1)\big)\Big|\\
&\quad\le C_{52}\sum_{k=1}^n[\partial^2_{x_ix_j}w_1(y,t)]^{(\alpha)}_{y,D_T}(t^{1-\beta_k}-t_1^{1-\beta_k})^\alpha+[\partial^2_{x_ix_j}w_1(y,t)]^{(\alpha/2)}_{t,D_T}(t-t_1)^{\alpha/2}.
\end{aligned}
$$

Taking into account estimate (27), $(t^{1-\beta}-t_1^{1-\beta})^\alpha\le(t-t_1)^{\alpha/2+\alpha(1/2-\beta)}$ for $t/2\le t_1<t$, $\beta\in(0,1/2]$, we obtain

$$(t^{1-\beta_k}-t_1^{1-\beta_k})^\alpha\le(t-t_1)^{\alpha/2+\alpha(1/2-\beta_k)},\quad \beta_k\in(0,1/2],\ k=1,\dots,n,$$

and

$$[\partial^2_{x_ix_j}u_1]^{(\alpha/2)}_{t,D_T}\le C_{53}\big([\partial^2_{x_ix_j}w_1(y,t)]^{(\alpha)}_{x,D_T}+[\partial^2_{x_ix_j}w_1(y,t)]^{(\alpha/2)}_{t,D_T}\big).$$

We estimate the derivative

$$
\begin{aligned}
\partial_tu_1(x,t)&=\partial_tw_1(x+(c_1t^{1-\beta_1},\cdots,c_nt^{1-\beta_n}),t)\\
&=\sum_{i=1}^n\frac{c_i(1-\beta_i)}{t^{\beta_i}}\partial_{x_i}w_1(x+d(t),t)+\partial_{t^*}w_1(x+d(t),t^*)|_{t^*=t}
\end{aligned}
$$

in the following way:

$$|\partial_tu_1(x,t)|\le C_{54}\sum_{i=1}^n\frac{1}{t^{\beta_i}}|\partial_{y_i}w_1(y,t)|_{D_T}+C_{55}|\partial_tw_1(y,t)|_{D_T}.$$

It follows that

$$\sup_{t\le T}t^\beta|\partial_tu_1|_{D'_t}\le C_{56}\Big(\sum_{i=1}^n|\partial_{y_i}w_1(y,t)|_{D_T}+|\partial_t|w_1(y,t)|_{D_T}\Big),$$

where $\beta=\max(\beta_1,\dots,\beta_n)$ and $D'_t=D\times(t/2,t)$.

All the other estimates for the functions

$$u_j(x,t)=w_j(x+(c_1t_1^{1-\beta_1},\dots,c_nt_1^{1-\beta_n}),t),\quad j=1,2,$$

are established in a similar way. As a result, we have

$$|u_1|^{(2+\alpha)}_{\beta,D_T}\le C_{57}|w_1(y,t)|^{(2+\alpha)}_{D_T},\quad |u_2|^{(2+\alpha)}_{D_T}\le C_{58}|w_2(y,t)|^{(2+\alpha)}_{D_T}.$$

By inequalities (56) for the functions $w_j(y,t)$, $j=1,2$, we obtain estimates (5) and (6):

$$|u_1|^{(2+\alpha)}_{\beta,D_T}\le C_1|u_0|^{(2+\alpha)}_{D},\ |u_2|^{(2+\alpha)}_{D_T}\le C_2|f|^{(\alpha)}_{D_T}.$$

Theorem 1 is proved. □

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