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ON THE LAW OF THE ITERATED LOGARITHM WITHOUT ASSUMPTIONS ABOUT THE EXISTENCE OF MOMENTS

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Sufficient conditions are found for the applicability of the generalized law of the iterated logarithm for sums of random variables without conditions of independence and existence of moments. Bibliography: 2 titles.

Let $\{X_n; n = 1, 2, \dots\}$ be a sequence of random variables on a probability space and let $\{a_n; n = 1, 2, \dots\}$ be a nondecreasing sequence of positive numbers such that $a_n \to \infty$ as $n \to \infty$. Set

$$S_n = \sum_{k=1}^{n} X_k \quad \text{and} \quad \overline{S_n} = \max_{1\le k\le n} S_k. \tag{1}$$

Let us introduce the following condition (Condition D): For any positive ε and $\varepsilon_0 < \varepsilon$ there exists a number $\gamma > 0$ such that

$$\mathbf{P}(\overline{S_n} > (1+\varepsilon)a_n) \le \gamma \mathbf{P}(S_n > (1+\varepsilon_0)a_n) \tag{2}$$

for all n large enough.

Take a number $c > 1$. There exists a sequence of positive integers $\{n_k; k = 1, 2 \dots\}$ such that

$$a_{n_k - 1} \le c^k < a_{n_k} \tag{3}$$

for all k large enough. Clearly, this sequence grows unboundedly.

Theorem. *Let Condition D be satisfied and let*

$$\sum_{k=1}^{\infty} \mathbf{P}(S_{n_k} > (1+\varepsilon)a_{n_k}) < \infty \tag{4}$$

for the introduced sequence $\{n_k\}$ and for every $\varepsilon > 0$ and $c > 1$. Then

$$\limsup S_n/a_n \le 1 \quad a.s. \tag{5}$$

Proof. Let ε be an arbitrary positive number. Since the sequence $\{a_n\}$ is nondecreasing,

$$\begin{aligned}\mathbf{P}(S_n > (1+\varepsilon)a_n\, i.o.) &\le \mathbf{P}\left(\max_{n_{k-1}<n\le n_k} S_n > (1+\varepsilon)a_{n_{k-1}} \text{ i.o.}\right) \\ &\le \mathbf{P}(\overline{S_{n_k}} > (1+\varepsilon)a_{n_{k-1}} \text{ i.o.}).\end{aligned} \tag{6}$$

It follows from (3) that $c^{k-1} < a_{n_{k-1}} \le c^k$ and $a_{n_k} < c^2 a_{n_{k-1}}$ for all k large enough. Hence,

$$\mathbf{P}(\overline{S_{n_k}} > (1+\varepsilon)a_{n_{k-1}} \text{ i.o.}) \le \mathbf{P}(\overline{S_{n_k}} > (1+\varepsilon)a_{n_k}/c^2 \text{ i.o.}) \le \mathbf{P}(\overline{S_{n_k}} > (1+\varepsilon/2)a_{n_k} \text{ i.o.}) \tag{7}$$

if we take c close enough to 1.

Due to relations (6) and (7) and the Borel–Cantelli lemma, to prove (5), it is enough to show that

$$\sum_{k=1}^{\infty} \mathbf{P}(\overline{S_{n_k}} > (1+\varepsilon)a_{n_k}) < \infty \tag{8}$$

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for every $\varepsilon > 0$ and $c > 1$. The last statement is a corollary of relation (4) and Condition D. The theorem is proved. □

The following condition may have a wider field of applications compared to Condition D: For every $\delta > 0$ there exists a number $\gamma > 0$ such that

$$\mathbf{P}(\overline{S_n} \geq x) \leq \gamma \mathbf{P}(S_n \geq x - \delta a_n) \tag{9}$$

for every x and all n large enough.

The source of both inequalities (2) and (9) is the Kolmogorov inequality $\mathbf{P}(\overline{S_n} \geq x) \leq 2\mathbf{P}(S_n \geq x - \sqrt{2B_n})$ for every x and for the sum of independent random variables X_n with zero expectations and finite variances, where $B_n = \sum_{k=1}^{n} \mathbf{E}X_k^2$. Precisely this inequality is the key ingredient in the Kolmogorov proof of the historically first general form of the law of the iterated logarithm for sequences of independent random variables (see also [2, Theorem 7.1]).

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REFERENCES

1. A. Kolmogoroff, "Über das Gesetz des iterierten Logarithmus," *Math. Ann.*, **101**, 126–135 (1929).
2. V. V. Petrov, *Limit Theorems of Probability Theory*, Oxford Univ. Press, New York (1995).