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ASYMPTOTIC BEHAVIOR OF A SOLUTION TO THE RADIATIVE TRANSFER EQUATION IN A MULTILAYERED MEDIUM WITH DIFFUSE REFLECTION AND REFRACTION CONDITIONS

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We consider the radiative transfer problem in a multilayered semitransparent medium composed of m plane vertical layers of optic thickness $\varepsilon = \tau_*/m$. It is assumed that the interfaces separating the layers as well as the left and right boundaries of the multilayered system are diffuse reflecting and diffuse refracting. We obtain asymptotics of the radiation intensity I_ε and its density U_ε as $\varepsilon \to 0$. Bibliography: 2 titles.

1 Introduction

We consider the radiative transfer problem in a multilayered semitransparent medium composed of m planar vertical layers of optic thickness $\varepsilon = \tau_*/m$, where τ_* is the total optic thickness of the layer system which is assumed to be fixed. The jth layer is associated with the interval (τ_{j-1}, τ_j), $1 \leqslant j \leqslant m$, where $\tau_j = \varepsilon j$. We assume that the interface separating layers and the left and right boundaries are diffuse reflecting and refracting. All layers possess the same optic properties. The scattering inside the layers and the density of volume radiation sources are isotropic.

The unknown is the radiation intensity $I_\varepsilon(\mu, \tau)$ defined on the set

$$D_\varepsilon = \bigcup_{j=1}^{m} D_{j,\varepsilon},$$

where $D_{\varepsilon,j} = D^-_{\varepsilon,j} \cup D^+_{\varepsilon,j}$, $D^+_{\varepsilon,j} = (0,1] \times (\tau_{j-1}, \tau_j)$, $D^-_{\varepsilon,j} = [-1,0) \times (\tau_{j-1}, \tau_j)$, $1 \leqslant j \leqslant m$. The variable μ is interpreted as the cosine of the angle between the direction of radiation propagation and the perpendicular to the system of layers. The value of the intensity of radiation propagating inside the jth layer is denoted by $I_{\varepsilon,j}(\mu, \tau)$. We denote by $I^+_{\varepsilon,j}(\mu, \tau)$ and $I^-_{\varepsilon,j}(\mu, \tau)$ the values of the radiation intensity corresponding to $\mu > 0$ and $\mu < 0$, i.e., the values of the intensity of

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radiation propagating inside the jth layer "to the right" and "to the left." The functions $I_{\varepsilon,j}$, $I^+_{\varepsilon,j}$, and $I^-_{\varepsilon,j}$ are the restrictions of I_ε on $D_{\varepsilon,j}$, $D^+_{\varepsilon,j}$, and $D^-_{\varepsilon,j}$ respectively. We set

$$I^{\pm}_{\varepsilon,j,\ell}(\mu) = I^{\pm}_{\varepsilon,j}(\mu, \tau_{j-1} + 0), \quad I^{\pm}_{\varepsilon,j,r}(\mu) = I^{\pm}_{\varepsilon,j}(\mu, \tau_j - 0).$$

The sought function I_ε is a solution to the problem

$$\mu \frac{d}{d\tau} I_{\varepsilon,j} + I_{\varepsilon,j} = \varpi \mathscr{S}(I_{\varepsilon,j}) + (1-\varpi)F, \quad (\mu,\tau) \in D_{\varepsilon,j}, \quad 1 \leqslant j \leqslant m, \tag{1.1}$$

$$I^+_{\varepsilon,j+1,\ell} = \mathscr{R}^+(I^-_{\varepsilon,j+1,\ell}) + \mathscr{P}^+(I^+_{\varepsilon,j,r}), \quad \mu \in (0,1], \quad 1 \leqslant j < m, \tag{1.2}$$

$$I^-_{\varepsilon,j,r} = \mathscr{R}^-(I^+_{\varepsilon,j,r}) + \mathscr{P}^-(I^-_{\varepsilon,j+1,\ell}), \quad \mu \in [-1,0), \quad 1 \leqslant j < m, \tag{1.3}$$

$$I^+_{\varepsilon,1,\ell} = \mathscr{R}^+_\ell(I^-_{\varepsilon,1,\ell}) + (1-\theta_\ell)J_\ell, \quad \mu \in (0,1], \tag{1.4}$$

$$I^-_{\varepsilon,m,r} = \mathscr{R}^-_r(I^+_{\varepsilon,m,r}) + (1-\theta_r)J_r, \quad \mu \in [-1,0). \tag{1.5}$$

The function $F \in C^2[0,\tau_*]$ characterizing the density of isotropic volume radiation sources and constants J_ℓ, J_r characterizing the radiation falling on the layer system at the left and at the right are given.

Equation (1.1) describes radiative transfer inside the jth layer, where $\mathscr{S}$ is the scattering operator

$$\mathscr{S}(I_{\varepsilon,j})(\tau) = \frac{1}{2}\int_{-1}^{1} I_{\varepsilon,j}(\mu,\tau)\,d\mu,$$

$\varpi \in [0,1)$ is the *albedo coefficient* of the medium, i.e., the quotient of dividing the scattering coefficient by the sum of the absorption and scattering coefficients.

The conditions (1.2) and (1.3) describe diffuse reflection and diffuse refraction of radiation on the interface separating the jth and $(j+1)$th layers, where $\mathscr{R}^\pm$ and $\mathscr{P}^\pm$ are the diffuse reflection and diffuse refraction operators

$$\mathscr{R}^+(I^-_{\varepsilon,j+1,\ell}) = 2\theta \int_{-1}^{0} I^-_{\varepsilon,j+1,\ell}(\mu)\,|\mu|\,d\mu, \quad \mathscr{P}^+(I^+_{\varepsilon,j,r}) = 2(1-\theta)\int_0^1 I^+_{\varepsilon,j,r}(\mu)\,\mu\,d\mu,$$

$$\mathscr{R}^-(I^+_{\varepsilon,j,r}) = 2\theta \int_0^1 I^+_{\varepsilon,j,r}(\mu)\,\mu\,d\mu, \quad \mathscr{P}^-(I^-_{\varepsilon,j+1,\ell}) = 2(1-\theta)\int_{-1}^{0} I^-_{\varepsilon,j+1,\ell}(\mu)\,|\mu|\,d\mu.$$

Here, $0 < \theta < 1$ is the coefficient characterizing the reflection properties of the interface separating layers.

The conditions (1.4) and (1.5) describe diffuse reflection and diffuse refraction of radiation on the left and right boundaries of the layer system, where

$$\mathscr{R}^+_\ell(I^-_{\varepsilon,1,\ell}) = 2\theta_\ell \int_{-1}^{0} I^-_{\varepsilon,1,\ell}(\mu)\,|\mu|\,d\mu, \quad \mathscr{R}^-_r(I^+_{\varepsilon,m,r}) = 2\theta_r \int_0^1 I^+_{\varepsilon,m,r}(\mu)\,\mu\,d\mu,$$

$0 < \theta_\ell < 1$, $0 < \theta_r < 1$ are the coefficients characterizing reflection properties of the left and right boundaries of the layer system.

By (1.2)–(1.5), the values $I^+_{\varepsilon,j,\ell}$, $I^-_{\varepsilon,j,r}$ for all j are independent of μ.

The existence and uniqueness of a solution to the problem (1.1)–(1.5) follow from [1]. The corresponding result is contained in Section 2.

The goal of this paper is to study the asymptotic properties of the solution to the problem (1.1)–(1.5) as $\varepsilon \to 0$. We also indicate the important characteristic of radiation: the density

$$U_\varepsilon(\tau) = \frac{2\pi}{c} \int_{-1}^{1} I_\varepsilon(\mu,\tau)\,d\mu = \frac{4\pi}{c}\mathscr{S}(I_\varepsilon)(\tau),$$

where c is the radiation propagation velocity. It is convenient to represent the solution I_ε as the linear combination

$$I_\varepsilon = J_\ell I^I_\varepsilon + J_r I^{II}_\varepsilon + I^{III}_\varepsilon \tag{1.6}$$

of solutions to the following three problems:

Problem P^I with $J_\ell = 1$, $J_r = 0$, $F = 0$,

Problem P^{II} with $J_\ell = 0$, $J_r = 1$ and $F = 0$,

Problem P^{III} with $J_\ell = 0$ and $J_r = 0$.

Each of these problems is of an independent interest. Problems P^I and P^{II} govern the propagation of the radiative flux falling on the layer system at the left and at the right. Problem P^{III} describes the radiation generated by isotropic volume sources.

The paper is organized as follows. The main results of the paper are formulated in Section 2 and are proved in the remaining sections 3–8.

2 The Main Results

2.1. The unique solvability of the problem (1.1)–(1.5). We formulate the result about the solvability of the problem following from [1]. Let $1 \leqslant j \leqslant m$. We set

$$\overline{D}^-_{\varepsilon,j} = [-1,0) \times [\tau_{j-1},\tau_j], \quad \overline{D}^+_{\varepsilon,j} = (0,1] \times [\tau_{j-1},\tau_j], \quad \overline{D}_{\varepsilon,j} = \overline{D}^-_{\varepsilon,j} \cup \overline{D}^+_{\varepsilon,j}.$$

We introduce the space $\mathscr{W}^\infty(D_{\varepsilon,j})$ of functions $f(\mu,\tau)$ in $L^\infty(D_{\varepsilon,j})$ that can be extended to $\overline{D}_{\varepsilon,j}$ in such a way that $f(\mu,\cdot) \in W^{1,\infty}(\tau_{j-1},\tau_j) \cap C[\tau_{j-1},\tau_j]$ for almost all $\mu \in [-1,0) \cup (0,1]$ and $\mu \dfrac{d}{d\tau} f \in L^\infty(D_{\varepsilon,j})$.

For a function I_ε defined on D_ε we denote by $I_{\varepsilon,j}$ its restriction on $D_{\varepsilon,j}$, $1 \leqslant j \leqslant m$. We introduce the space $\mathscr{W}^\infty(D_\varepsilon)$ of functions $I_\varepsilon \in L^\infty(D_\varepsilon)$ such that $I_{\varepsilon,j} \in \mathscr{W}^\infty(D_{\varepsilon,j})$ for all $1 \leqslant j \leqslant m$.

By a *solution* to the problem (1.1)–(1.5) we mean a function $I_\varepsilon \in \mathscr{W}^\infty(D_\varepsilon)$ satisfying Equation (1.1) for almost all $(\mu,\tau) \in D_{\varepsilon,j}$, $1 \leqslant j \leqslant m$, the conditions (1.2), (1.4) for almost all $\mu \in (0,1]$, and the conditions (1.3), (1.5) for almost all $\mu \in [-1,0)$.

Theorem 2.1. *Let $F \in L^\infty(0,\tau_*)$. Then the problem* (1.1)–(1.5) *has a solution $I_\varepsilon \in \mathscr{W}^\infty(D_\varepsilon)$. The solution is unique and satisfies the estimates*

$$\|I_\varepsilon\|_{L^\infty(D_\varepsilon)} \leqslant \max\{\|F\|_{L^\infty(0,\tau_*)}, |J_\ell|, |J_r|\}, \tag{2.1}$$

$$\max_{1\leqslant j\leqslant m}\Big\|\mu\frac{d}{d\tau}I_{\varepsilon,j}\Big\|_{L^\infty(D_{\varepsilon,j})}\leqslant 2\max\{\|F\|_{L^\infty(0,\tau_*)},|J_\ell|,|J_r|\}. \tag{2.2}$$

2.2. Notation. We use the exponential integral function

$$E_k(\tau)=\int\limits_0^1 e^{-\tau/\mu}\mu^{k-2}\,d\mu,\quad k=1,2,3.$$

We recall the formulas [2]

$$\frac{d}{d\tau}E_2(\tau)=-E_1(\tau),\quad \frac{d}{d\tau}E_3(\tau)=-E_2(\tau),\quad \tau>0, \tag{2.3}$$

$$E_1(\tau)=-\gamma-\ln\tau+\tau+O(\tau^2),\quad \tau\to 0^+, \tag{2.4}$$

$$E_2(\tau)=1+\tau(\ln\tau+\gamma-1)+O(\tau^2),\quad \tau\to 0^+, \tag{2.5}$$

$$E_3(\tau)=\frac{1}{2}-\tau-\frac{1}{2}\tau^2(\ln\tau+\gamma-3/2)+O(\tau^3),\quad \tau\to 0^+, \tag{2.6}$$

where γ is the Euler constant: $\gamma=0,577215\ldots$. Formulas (2.3)–(2.6) will be often used below with any special references.

Assume that $\varphi=\varphi(\varepsilon,\mu,\tau,\varpi,\theta,\theta_\ell,\theta_r,\tau_*)$, $\varphi_j=\varphi_j(\varepsilon,\mu,\tau,\varpi,\theta,\theta_\ell,\theta_r,\tau_*)$, and $\psi=\psi(\varepsilon)$. We write $\varphi=O(\psi(\varepsilon))$, $\varphi_j=O(\psi(\varepsilon))$ if for all sufficiently small $\varepsilon>0$

$$|\varphi|\leqslant C|\psi(\varepsilon)|,\quad |\varphi_j|\leqslant C|\psi(\varepsilon)|,$$

where the constant $C>0$ can depend on ϖ, θ, θ_ℓ, θ_r, τ_*, but is independent of ε, μ, τ, j.

We also use the notation

$$\varepsilon_\varpi=(1-\varpi)\varepsilon,\quad \varepsilon_{\theta,\varpi}=\frac{1-\theta}{\theta}(1-\varpi)\varepsilon,$$

$$\lambda_0=\frac{1-\theta}{\theta(1-\varpi)},\quad \lambda(\varepsilon)=\frac{1}{4}\frac{\lambda_0+\dfrac{\varepsilon}{3}}{1+\dfrac{1}{2}\varepsilon_\varpi(\ln\varepsilon+2\lambda_0+\gamma+1/2)}.$$

2.3. Asymptotics of the solution to Problem P^I. Let I_ε be a solution to Problem P^I.

Theorem 2.2. *For all* $1\leqslant j\leqslant m$

$$I^+_{\varepsilon,j}(\mu,\tau)=[Z^+_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu}+\varpi Z_{\varepsilon,j}(1-e^{-(\tau-\tau_{j-1})/\mu})](1+O(\varepsilon\ln^2\varepsilon)),\quad (\mu,\tau)\in\overline{D}^+_{\varepsilon,j}, \tag{2.7}$$

$$I^-_{\varepsilon,j}(\mu,\tau)=[Z^-_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|}+\varpi Z_{\varepsilon,j}(1-e^{-(\tau_j-\tau)/|\mu|})](1+O(\varepsilon\ln^2\varepsilon)),\quad (\mu,\tau)\in\overline{D}^-_{\varepsilon,j}, \tag{2.8}$$

$$U_\varepsilon(\tau)=\frac{4\pi}{c}Z_{\varepsilon,j}(1+O(\varepsilon\ln^2\varepsilon)),\quad \tau\in(\tau_{j-1},\tau_j). \tag{2.9}$$

where

$$Z^+_{\varepsilon,j}=a^+_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}[1-b^+_r(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}],$$

$$Z^-_{\varepsilon,j}=a^-_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}[1-b^-_r(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}],$$

$$Z_{\varepsilon,j}=a^0_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}[1-b^0_r(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}],$$

$$a_\ell^+(\varepsilon) = \frac{1}{1 + \dfrac{2\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}}, \quad b_r^+(\varepsilon) = \frac{1}{1 + \dfrac{4}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}} + \dfrac{8}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}},$$

$$a_\ell^-(\varepsilon) = \frac{1}{1 + \dfrac{2}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}}, \quad b_r^-(\varepsilon) = \frac{1}{1 + \dfrac{4\theta_r}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}} + \dfrac{8\theta_r^2}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}},$$

$$a_\ell^0(\varepsilon) = \frac{1}{1 + \dfrac{1+\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}}, \quad b_r^0(\varepsilon) = \frac{1}{1 + \dfrac{2(1+\theta_r)}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}} + \dfrac{2(1+\theta_r)^2}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}}.$$

Remark 2.1. Important characteristic of optic systems are the reflection coefficient R and the transmission coefficient T which are equal to the quotients of dividing the radiation fluxes reflected or transmitted by the medium by the radiation flux falling on the body.

In the case under consideration, radiative flux falling on the multilayered medium at the left is equal to 1, the reflected radiative flux is equal to

$$\theta_\ell + 2(1-\theta_\ell)\int_{-1}^{0} I^-_{\varepsilon,1,\ell}(\mu)|\mu|\,d\mu,$$

and the transmitted radiative flux is equal to

$$2(1-\theta_r)\int_0^1 I^+_{\varepsilon,m,r}(\mu)\mu\,d\mu.$$

By Theorem 2.2, $R = \theta_\ell + (1-\theta_\ell)a_\ell^-(\varepsilon)(1 + O(\varepsilon\ln^2\varepsilon)) = 1 - 2\sqrt{\varepsilon_{\theta,\varpi}} + O(\varepsilon\ln^2\varepsilon) \to 1$ as $\varepsilon \to 0$ and $T = 2(1-\theta_r)O(e^{-\tau_*/\sqrt{\varepsilon\lambda(\varepsilon)}}) \to 0$ as $\varepsilon \to 0$. Thus, the behavior of the multilayered medium looks like an almost mat mirror.

Theorem 2.2 yields the asymptotics of the solution on $[0, \tau_*]$ with the relative accuracy $O(\varepsilon\ln^2\varepsilon)$. However, it is possible to specify the asymptotics if we ignore the behavior of the solution in a neighborhood of the point τ_*, where the radiation intensity is negligibly small.

We set $\delta(\varepsilon) = \sqrt{\varepsilon\lambda(\varepsilon)}\ln(1/\varepsilon)$.

Theorem 2.3. *If $\tau_j \leqslant \tau_* - \delta(\varepsilon)$, then*

$$I^+_{\varepsilon,j}(\mu,\tau) = \biggl[\widetilde{Z}^+_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}\widetilde{Z}_{\varepsilon,j}(\tau')\,d\tau'\biggr](1 + O(\varepsilon^{3/2}\ln^2\varepsilon)), \quad (\mu,\tau) \in \overline{D}^+_{\varepsilon,j}, \tag{2.10}$$

$$I^-_{\varepsilon,j}(\mu,\tau) = \biggl[\widetilde{Z}^-_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|} + \frac{\varpi}{|\mu|}\int_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}\widetilde{Z}_{\varepsilon,j}(\tau')\,d\tau'\biggr](1 + O(\varepsilon^{3/2}\ln^2\varepsilon)), \quad (\mu,\tau) \in \overline{D}^-_{\varepsilon,j}, \tag{2.11}$$

$$U_\varepsilon(\tau) = \frac{4\pi}{c}\widetilde{Z}_{\varepsilon,j}(\tau)(1+O(\varepsilon^{3/2}\ln^2\varepsilon)), \quad \tau\in(\tau_{j-1},\tau_j), \tag{2.12}$$

where

$$\widetilde{Z}^+_{\varepsilon,j} = \widetilde{a}^+_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon/\lambda(\varepsilon)}}, \quad \widetilde{a}^+_\ell(\varepsilon) = \frac{1}{1+\dfrac{2\theta_\ell}{1-\theta_\ell}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}+\varepsilon_\varpi)},$$

$$\widetilde{Z}^-_{\varepsilon,j} = \widetilde{a}^-_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon/\lambda(\varepsilon)}}, \quad \widetilde{a}^-_\ell(\varepsilon) = \frac{1}{1+\dfrac{2}{1-\theta_\ell}(\sqrt{\varepsilon_{\theta,\varpi}}+\varepsilon_{\theta,\varpi})+\dfrac{2\theta_\ell}{1-\theta_\ell}\varepsilon_\varpi},$$

$$\begin{aligned}\widetilde{Z}_{\varepsilon,j}(\tau) &= \frac{1}{2}\widetilde{Z}^+_{\varepsilon,j}\Big[\varpi+\Big(1-\frac{\varpi}{2}\Big)E_2(\tau-\tau_{j-1})-\frac{\varpi}{2}E_2(\tau_j-\tau)\Big]\\ &+\frac{1}{2}\widetilde{Z}^-_{\varepsilon,j}\Big[\varpi+\Big(1-\frac{\varpi}{2}\Big)E_2(\tau_j-\tau)-\frac{\varpi}{2}E_2(\tau-\tau_{j-1})\Big]\end{aligned}$$

2.4. Asymptotics of the solution to Problem P^{II}. Let I_ε be a solution to Problem P^{II}.

Theorem 2.4. *For all* $1\leqslant j\leqslant m$

$$I^+_{\varepsilon,j}(\mu,\tau) = [Y^+_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu}+\varpi Y_{\varepsilon,j}(1-e^{-(\tau-\tau_{j-1})/\mu})](1+O(\varepsilon\ln^2\varepsilon)), \quad (\mu,\tau)\in\overline{D}^+_{\varepsilon,j},$$

$$I^-_{\varepsilon,j}(\mu,\tau) = [Y^-_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|}+\varpi Y_{\varepsilon,j}(1-e^{-(\tau_j-\tau)/|\mu|})](1+O(\varepsilon\ln^2\varepsilon)), \quad (\mu,\tau)\in\overline{D}^-_{\varepsilon,j},$$

$$U_\varepsilon(\tau) = \frac{4\pi}{c}Y_{\varepsilon,j}(1+O(\varepsilon\ln^2\varepsilon)), \quad \tau\in(\tau_{j-1},\tau_j),$$

where

$$Y^+_{\varepsilon,j} = a^+_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}[1-b^+_\ell(\varepsilon)e^{-2\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}],$$

$$Y^-_{\varepsilon,j} = a^-_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}[1-b^-_\ell(\varepsilon)e^{-2\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}],$$

$$Y_{\varepsilon,j} = a^0_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}[1-b^0_\ell(\varepsilon)e^{-2\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}],$$

$$a^-_r(\varepsilon) = \frac{1}{1+\dfrac{2\theta_r}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}}, \quad b^-_\ell(\varepsilon) = \frac{1}{1+\dfrac{4}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{8}{(1-\theta_\ell)^2}\varepsilon_{\theta,\varpi}},$$

$$a^+_r(\varepsilon) = \frac{1}{1+\dfrac{2}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}}, \quad b^+_\ell(\varepsilon) = \frac{1}{1+\dfrac{4\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{8\theta_\ell^2}{(1-\theta_\ell)^2}\varepsilon_{\theta,\varpi}},$$

$$a^0_r(\varepsilon) = \frac{1}{1+\dfrac{1+\theta_r}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}}, \quad b^0_\ell(\varepsilon) = \frac{1}{1+\dfrac{2(1+\theta_\ell)}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{2(1+\theta_\ell^2)}{(1-\theta_\ell)^2}\varepsilon_{\theta,\varpi}}.$$

Theorem 2.5. *If* $\tau_{j-1}\geqslant\delta(\varepsilon)$, *then*

$$I^+_{\varepsilon,j}(\mu,\tau)=\Bigg[\widetilde{Y}^+_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu}+\frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau}e^{-(\tau-\tau')/\mu}\widetilde{Y}_{\varepsilon,j}(\tau')\,d\tau'\Bigg](1+O(\varepsilon^{3/2}\ln^2\varepsilon)), \ (\mu,\tau)\in\overline{D}^+_{\varepsilon,j},$$

$$I^-_{\varepsilon,j}(\mu,\tau)=\left[\widetilde{Y}^-_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|}+\frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j}e^{-(\tau'-\tau)/|\mu|}\widetilde{Y}_{\varepsilon,j}(\tau')\,d\tau'\right](1+O(\varepsilon^{3/2}\ln^2\varepsilon)),\ \ (\mu,\tau)\in\overline{D}^-_{\varepsilon,j},$$

$$U_\varepsilon(\tau)=\frac{4\pi}{c}\widetilde{Y}_{\varepsilon,j}(\tau)(1+O(\varepsilon^{3/2}\ln^2\varepsilon)),\quad \tau\in(\tau_{j-1},\tau_j),$$

where

$$\widetilde{Y}^-_{\varepsilon,j}=\widetilde{a}^-_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}},\quad \widetilde{a}^-_r(\varepsilon)=\frac{1}{1+\dfrac{2\theta_r}{1-\theta_r}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}+\varepsilon_\varpi)},$$

$$\widetilde{Y}^+_{\varepsilon,j}=\widetilde{a}^+_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}},\quad \widetilde{a}^+_r(\varepsilon)=\frac{1}{1+\dfrac{2}{1-\theta_r}(\sqrt{\varepsilon_{\theta,\varpi}}+\varepsilon_{\theta,\varpi})+\dfrac{2\theta_r}{1-\theta_r}\varepsilon_\varpi},$$

$$\widetilde{Y}_{\varepsilon,j}(\tau)=\frac{1}{2}\widetilde{Y}^+_{\varepsilon,j}\Big[\varpi+\Big(1-\frac{\varpi}{2}\Big)E_2(\tau-\tau_{j-1})-\frac{\varpi}{2}E_2(\tau_j-\tau)\Big]$$
$$+\frac{1}{2}\widetilde{Y}^-_{\varepsilon,j}\Big[\varpi+\Big(1-\frac{\varpi}{2}\Big)E_2(\tau_j-\tau)-\frac{\varpi}{2}E_2(\tau-\tau_{j-1})\Big].$$

2.5. Asymptotics of the solution to Problem P^{III}. Let I_ε be a solution to Problem P^{III}.

Theorem 2.6. *If $\tau_{j-1}<\delta(\varepsilon)$, then*

$$I^+_{\varepsilon,j}(\mu,\tau)=F(\tau)-F(0)\left[\widehat{Z}^+_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu}+\frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau}e^{-(\tau-\tau')/\mu}\widehat{Z}_{\varepsilon,j}(\tau')\,d\tau'\right]$$
$$+\|F\|_{C^2[0,\tau_*]}O(\varepsilon),\quad (\mu,\tau)\in\overline{D}^+_{\varepsilon,j},\tag{2.13}$$

$$I^-_{\varepsilon,j}(\mu,\tau)=F(\tau)-F(0)\left[\widehat{Z}^-_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|}+\frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j}e^{-(\tau'-\tau)/|\mu|}\widehat{Z}_{\varepsilon,j}(\tau')\,d\tau'\right]$$
$$+\|F\|_{C^2[0,\tau_*]}O(\varepsilon),\quad (\mu,\tau)\in\overline{D}^-_{\varepsilon,j},\tag{2.14}$$

$$U_\varepsilon(\tau)=\frac{4\pi}{c}[F(\tau)-F(0)\widehat{Z}_{\varepsilon,j}(\tau)+\|F\|_{C^2[0,\tau_*]}O(\varepsilon)],\quad \tau\in(\tau_{j-1},\tau_j),\tag{2.15}$$

where

$$\widehat{Z}^+_{\varepsilon,j}=a^+_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}},\quad \widehat{Z}^-_{\varepsilon,j}=a^-_\ell(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}},$$

$$\widehat{Z}_{\varepsilon,j}(\tau)=\frac{1}{2}\widehat{Z}^+_{\varepsilon,j}\Big[\varpi+\Big(1-\frac{\varpi}{2}\Big)E_2(\tau-\tau_{j-1})-\frac{\varpi}{2}E_2(\tau_j-\tau)\Big]$$
$$+\frac{1}{2}\widehat{Z}^-_{\varepsilon,j}\Big[\varpi+\Big(1-\frac{\varpi}{2}\Big)E_2(\tau_j-\tau)-\frac{\varpi}{2}E_2(\tau-\tau_{j-1})\Big].$$

If $\tau_j>\tau_*-\delta(\varepsilon)$, then

$$I^+_{\varepsilon,j}(\mu,\tau)=F(\tau)-F(\tau_*)\left[\widehat{Y}^+_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu}+\frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau}e^{-(\tau-\tau')/\mu}\widehat{Y}_{\varepsilon,j}(\tau')\,d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}^+_{\varepsilon,j}, \tag{2.16}$$

$$I^-_{\varepsilon,j}(\mu,\tau) = F(\tau) - F(\tau_*)\left[\widehat{Y}^-_{\varepsilon,j}e^{-(\tau_- \tau)/|\mu|} + \frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}\widehat{Y}_{\varepsilon,j}(\tau')\,d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}^-_{\varepsilon,j}, \tag{2.17}$$

$$U_\varepsilon(\tau) = \frac{4\pi}{c}[F(\tau) - F(\tau_*)\widehat{Y}_{\varepsilon,j}(\tau) + \|F\|_{C^2[0,\tau_*]}O(\varepsilon)], \quad \tau \in (\tau_{j-1},\tau_j), \tag{2.18}$$

where

$$\widehat{Y}^+_{\varepsilon,j} = a^+_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}, \quad \widehat{Y}^-_{\varepsilon,j} = a^-_r(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}},$$

$$\widehat{Y}_{\varepsilon,j}(\tau) = \frac{1}{2}\widehat{Y}^+_{\varepsilon,j}\Big[\varpi + \Big(1 - \frac{\varpi}{2}\Big)E_2(\tau - \tau_{j-1}) - \frac{\varpi}{2}E_2(\tau_j - \tau)\Big]$$

$$+ \frac{1}{2}\widehat{Y}^-_{\varepsilon,j}\Big[\varpi + \Big(1 - \frac{\varpi}{2}\Big)E_2(\tau_j - \tau) - \frac{\varpi}{2}E_2(\tau - \tau_{j-1})\Big].$$

If $\delta(\varepsilon) \leqslant \tau_{j-1} < \tau_j \leqslant \tau_* - \delta(\varepsilon)$, *then*

$$I_{\varepsilon,j}(\mu,\tau) = F(\tau) + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}_{\varepsilon,j}, \tag{2.19}$$

$$U_\varepsilon(\tau) = \frac{4\pi}{c}[F(\tau) + \|F\|_{C^2[0,\tau_*]}O(\varepsilon)], \quad \tau \in (\tau_{j-1},\tau_j). \tag{2.20}$$

2.6. Diffuse approximation. To find asymptotic approximations of the radiation density U_ε with accuracy of order $O(\varepsilon)$, one can use the singularly degenerated one-dimensional stationary diffusion equation

$$-\varepsilon\lambda(\varepsilon)\frac{d^2}{d\tau^2}u_\varepsilon + u_\varepsilon = F, \quad \tau \in (\varepsilon/2, \tau_* - \varepsilon/2), \tag{2.21}$$

with special boundary conditions

$$B_{\varepsilon,\ell}[u_\varepsilon](\varepsilon/2) = \frac{2a^0_\ell(\varepsilon)}{1 - b^0_\ell(\varepsilon)}J_\ell, \quad B_{\varepsilon,r}[u_\varepsilon](\tau_* - \varepsilon/2) = \frac{2a^0_r(\varepsilon)}{1 - b^0_r(\varepsilon)}J_r. \tag{2.22}$$

Here,

$$B_{\varepsilon,\ell}[u_\varepsilon] = -\sqrt{\varepsilon\lambda(\varepsilon)}\frac{d}{d\tau}u_\varepsilon + \frac{1 + b^0_\ell(\varepsilon)}{1 - b^0_\ell(\varepsilon)}u_\varepsilon, \quad B_{\varepsilon,r}[u_\varepsilon] = \sqrt{\varepsilon\lambda(\varepsilon)}\frac{d}{d\tau}u_\varepsilon + \frac{1 + b^0_r(\varepsilon)}{1 - b^0_r(\varepsilon)}u_\varepsilon.$$

More exactly, the following assertion holds.

Theorem 2.7. *Let* u_ε *be a solution to the problem* (2.21), (2.22). *Then*

$$U_\varepsilon(\tau_{j-1/2}) = \frac{4\pi}{c}[\varkappa(\varepsilon)u_\varepsilon(\tau_{j-1/2}) + (1 - \varkappa(\varepsilon))F(\tau_{j-1/2}) + MO(\varepsilon)], \quad 1 \leqslant j \leqslant m, \tag{2.23}$$

where $\tau_{j-1.2} = (\tau_{j-1} + \tau_j)/2$, $\varkappa(\varepsilon) = \varpi + (1 - \varpi)E_2(\varepsilon)$, $M = \max\{\|F\|_{C^2[0,\tau_*]}, |J_\ell|, |J_r|\}$. *Consequently,*

$$U_\varepsilon(\tau_{j-1/2}) = \frac{4\pi}{c}[u_\varepsilon(\tau_{j-1/2}) + MO(\varepsilon\ln\varepsilon)], \quad 1 \leqslant j \leqslant m. \tag{2.24}$$

Remark 2.2. A solution to the problem (2.21), (2.22) is defined on the segment $[\varepsilon/2, \tau_* - \varepsilon/2]$ but not on $[0,\tau_*]$.

3 The Radiative Transfer Integral Equation

An important role in our further consideration is played by the radiative transfer integral equations

$$\Psi(\tau) = \varpi\Lambda_\varepsilon(\Psi)(\tau) + f(\tau), \quad \tau \in [0,\varepsilon], \tag{3.1}$$

$$\Psi(\tau) = \varpi\Lambda_{\varepsilon,j}(\Psi)(\tau) + f(\tau), \quad \tau \in [\tau_{j-1},\tau_j], \tag{3.2}$$

where the operators $\Lambda_\varepsilon : C[0,\varepsilon] \to C[0,\varepsilon]$ and $\Lambda_{\varepsilon,j} : C[\tau_{j-1},\tau_j] \to C[\tau_{j-1},\tau_j]$ are given by

$$\Lambda_\varepsilon(\Psi)(\tau) = \frac{1}{2}\int\limits_0^\varepsilon \Psi(\tau')E_1(|\tau'-\tau|)\,d\tau', \quad \tau \in [0,\varepsilon],$$

$$\Lambda_{\varepsilon,j}(\Psi)(\tau) = \frac{1}{2}\int\limits_{\tau_{j-1}}^{\tau_j} \Psi(\tau')E_1(|\tau'-\tau|)\,d\tau', \quad \tau \in [\tau_{j-1},\tau_j].$$

We note that

$$\Lambda_\varepsilon(1)(\tau) = 1 - \frac{1}{2}E_2(\tau) - \frac{1}{2}E_2(\varepsilon-\tau). \tag{3.3}$$

Therefore,

$$\|\Lambda_\varepsilon(1)\|_{C[0,\varepsilon]} = 1 - E_2(\varepsilon/2),$$

$$\|\Lambda_\varepsilon(\Psi)\|_{C[0,\varepsilon]} \leqslant \|\Lambda_\varepsilon(1)\|_{C[0,\varepsilon]}\|\Psi\|_{C[0,\varepsilon]} = [1 - E_2(\varepsilon/2)]\|\Psi\|_{C[0,\varepsilon]} \quad \forall\,\Psi \in C[0,\varepsilon].$$

Consequently,

$$\|\Lambda_\varepsilon\| = \|\Lambda_\varepsilon\|_{C[0,\varepsilon]\to C[0,\varepsilon]} = 1 - E_2(\varepsilon/2). \tag{3.4}$$

Thus, $\|\varpi\Lambda_\varepsilon\| < \varpi < 1$ and for every $f \in C[0,\varepsilon]$ Equation (3.1) has a unique solution $\Psi^f \in C[0,\varepsilon]$ represented as the Neumann series

$$\Psi^f = \sum_{k=0}^\infty \varpi^k\Lambda_\varepsilon^k(f).$$

converging in $C[0,\varepsilon]$. We note that (3.4) and (2.5) imply

$$\|\Lambda_\varepsilon\| = O(\varepsilon\ln\varepsilon). \tag{3.5}$$

Therefore,

$$\Psi^f(\tau) = f(\tau) + \varpi\Lambda_\varepsilon(f)(\tau) + O(\varepsilon^2\ln^2\varepsilon)\|f\|_{C[0,\varepsilon]}, \quad \tau \in [0,\varepsilon]. \tag{3.6}$$

We consider the solutions $\Psi_\varepsilon^1(\tau)$, $\Psi_\varepsilon^{E_2}(\tau)$, $\Psi_\varepsilon^{E_2}(\varepsilon-\tau)$ of the equations

$$\Psi(\tau) = \varpi\Lambda_\varepsilon(\Psi)(\tau) + 1, \quad \tau \in [0,\varepsilon],$$

$$\Psi(\tau) = \varpi\Lambda_\varepsilon(\Psi)(\tau) + E_2(\tau), \quad \tau \in [0,\varepsilon],$$

$$\Psi(\tau) = \varpi\Lambda_\varepsilon(\Psi)(\tau) + E_2(\varepsilon-\tau) \quad \tau \in [0,\varepsilon]$$

respectively. By (3.3), (3.5), (2.5),

$$\Lambda_\varepsilon(E_2)(\tau) = \Lambda_\varepsilon(1)(\tau) + \Lambda_\varepsilon(E_2 - 1) = 1 - \frac{1}{2}E_2(\tau) - \frac{1}{2}E_2(\varepsilon - \tau) + O(\varepsilon^2 \ln^2 \varepsilon).$$

Therefore, (3.3) and (3.6) imply

$$\Psi^1_\varepsilon(\tau) = 1 + \varpi\Big(1 - \frac{1}{2}E_2(\tau) - \frac{1}{2}E_2(\varepsilon - \tau)\Big) + O(\varepsilon^2 \ln^2 \varepsilon),$$

$$\Psi^{E_2}_\varepsilon(\tau) = \varpi + \Big(1 - \frac{\varpi}{2}\Big)E_2(\tau) - \frac{\varpi}{2}E_2(\varepsilon - \tau) + O(\varepsilon^2 \ln^2 \varepsilon), \tag{3.7}$$

$$\Psi^{E_2}_\varepsilon(\varepsilon - \tau) = \varpi + \Big(1 - \frac{\varpi}{2}\Big)E_2(\varepsilon - \tau) - \frac{\varpi}{2}E_2(\tau) + O(\varepsilon^2 \ln^2 \varepsilon). \tag{3.8}$$

Roughening these formulas, we find

$$\begin{aligned} \Psi^1_\varepsilon(\tau) &= 1 + O(\varepsilon \ln \varepsilon), \\ \Psi^{E_2}_\varepsilon(\tau) &= 1 + O(\varepsilon \ln \varepsilon), \\ \Psi^{E_2}_\varepsilon(\varepsilon - \tau) &= 1 + O(\varepsilon \ln \varepsilon). \end{aligned} \tag{3.9}$$

It is clear that

$$\Lambda_{\varepsilon,j}(\Psi)(\tau) = \Lambda_\varepsilon(\Psi(\cdot - \tau_{j-1}))(\tau - \tau_{j-1}), \quad \tau \in [\tau_{j-1}, \tau_j].$$

Consequently,

$$\|\varpi\Lambda_{\varepsilon,j}\|_{C[\tau_{j-1},\tau_j]\to C[\tau_{j-1},\tau_j]} = \varpi\|\Lambda_\varepsilon\| < 1$$

and for every $f \in C[\tau_{j-1}, \tau_j]$ Equation (3.2) has a unique solution $\Psi^f_{\varepsilon,j} \in C[\tau_{j-1}, \tau_j]$; moreover,

$$\Psi^f_{\varepsilon,j}(\tau) = \Psi^{f_j}_\varepsilon(\tau - \tau_{j-1}), \quad \tau \in [\tau_{j-1}, \tau_j],$$

where $f_j(\cdot) = f(\tau_{j-1} + \cdot)$.

4 Auxiliary Problem of Finding $I^+_{\varepsilon,j,\ell}$ and $I^-_{\varepsilon,j,r}$, $1 \leqslant j \leqslant m$

Let $I_{\varepsilon,j}$ be the restriction of the solution to the problem (1.1)–(1.5) onto $D_{\varepsilon,j}$, $1 \leqslant j \leqslant m$. We set

$$S_{\varepsilon,j}(\tau) = \mathscr{S}(I_{\varepsilon,j})(\tau) = \frac{1}{2}\int_{-1}^{1} I_{\varepsilon,j}(\mu, \tau)\, d\mu, \quad \tau \in [\tau_{j-1}, \tau_j], \tag{4.1}$$

$$\Phi_{\varepsilon,j}(\tau) = \varpi S_{\varepsilon,j}(\tau) + (1 - \varpi)F(\tau), \quad \tau \in [\tau_{j-1}, \tau_j]. \tag{4.2}$$

If the function $\Phi_{\varepsilon,j}$ and constants $I^+_{\varepsilon,j,\ell}$, $I^-_{\varepsilon,j,r}$ are given, then $I_{\varepsilon,j}$ can be regarded as a solution to the problem

$$\mu\frac{d}{d\tau}I_{\varepsilon,j} + I_{\varepsilon,j} = \Phi_{\varepsilon,j}, \quad (\mu, \tau) \in D_{\varepsilon,j},$$

$$I^+_{\varepsilon,j}|_{\tau=\tau_{j-1}+0} = I^+_{\varepsilon,j,\ell}, \quad I^-_{\varepsilon,j}|_{\tau=\tau_j-0} = I^-_{\varepsilon,j,r}.$$

Solving this problem, we find

$$I^+_{\varepsilon,j}(\mu,\tau) = I^+_{\varepsilon,j,\ell} e^{-(\tau-\tau_{j-1})/\mu} + \frac{1}{\mu}\int_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}\Phi_{\varepsilon,j}(\tau')\,d\tau', \quad (\mu,\tau)\in \overline{D}^+_{\varepsilon,j}, \tag{4.3}$$

$$I^-_{\varepsilon,j}(\mu,\tau) = I^-_{\varepsilon,j,r} e^{-(\tau_j-\tau)/|\mu|} + \frac{1}{|\mu|}\int_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}\Phi_{\varepsilon,j}(\tau')\,d\tau', \quad (\mu,\tau)\in \overline{D}^-_{\varepsilon,j}. \tag{4.4}$$

Integrating (4.3) and (4.4) with respect to μ, we have

$$\begin{aligned} S_{\varepsilon,j}(\tau) &= \frac{1}{2}\int_0^1 \left[I^+_{\varepsilon,j,\ell} e^{-(\tau-\tau_{j-1})/\mu} + \frac{1}{\mu}\int_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}\Phi_{\varepsilon,j}(\tau')\,d\tau' \right] d\mu \\ &+ \frac{1}{2}\int_{-1}^0 \left[I^-_{\varepsilon,j,r} e^{-(\tau_j-\tau)/|\mu|} + \frac{1}{|\mu|}\int_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}\Phi_{\varepsilon,j}(\tau')\,d\tau' \right] d\mu \\ &= \frac{1}{2}\int_{\tau_{j-1}}^{\tau_j} \Phi_{\varepsilon,j}(\tau')E_1(|\tau'-\tau|)\,d\tau' + \frac{1}{2}I^+_{\varepsilon,j,\ell}E_2(\tau-\tau_{j-1}) + \frac{1}{2}I^-_{\varepsilon,j,r}E_2(\tau_j-\tau). \end{aligned}$$

Thus, the function $S_{\varepsilon,j}$ satisfies the equation

$$S_{\varepsilon,j}(\tau) = \varpi\Lambda_{\varepsilon,j}(S_{\varepsilon,j})(\tau) + \frac{1}{2}I^+_{\varepsilon,j,\ell}E_2(\tau-\tau_{j-1}) + \frac{1}{2}I^-_{\varepsilon,j,r}E_2(\tau_j-\tau) + (1-\varpi)\Lambda_{\varepsilon,j}(F)(\tau), \tag{4.5}$$

and the function $\Phi_{\varepsilon,j}$ satisfies the equation

$$\Phi_{\varepsilon,j}(\tau) = \varpi\Lambda_{\varepsilon,j}(\Phi_{\varepsilon,j})(\tau) + \frac{\varpi}{2}I^+_{\varepsilon,j,\ell}E_2(\tau-\tau_{j-1}) + \frac{\varpi}{2}I^-_{\varepsilon,j,r}E_2(\tau_j-\tau) + (1-\varpi)F(\tau),$$

which implies

$$\Phi_{\varepsilon,j}(\tau) = \frac{\varpi}{2}I^+_{\varepsilon,j,\ell}\Psi^{E_2}_{\varepsilon}(\tau-\tau_{j-1}) + \frac{\varpi}{2}I^-_{\varepsilon,j,r}\Psi^{E_2}_{\varepsilon}(\tau_j-\tau) + (1-\varpi)\Psi^F_{\varepsilon,j}(\tau). \tag{4.6}$$

We note that (4.3) implies

$$\begin{aligned} \int_0^1 I^+_{\varepsilon,j,r}(\mu)\mu\,d\mu &= \int_0^1 I^+_{\varepsilon,j,\ell}e^{-\varepsilon/\mu}\mu\,d\mu + \int_0^1 \left[\int_{\tau_{j-1}}^{\tau_j} e^{-(\tau_j-\tau')/\mu}\Phi_{\varepsilon,j}(\tau')\,d\tau' \right] d\mu \\ &= I^+_{\varepsilon,j,\ell}E_3(\varepsilon) + \int_{\tau_{j-1}}^{\tau_j} \Phi_{\varepsilon,j}(\tau')E_2(\tau_j-\tau')\,d\tau'. \end{aligned} \tag{4.7}$$

Similarly, from (4.4) it follows that

$$\int_{-1}^0 I^-_{\varepsilon,j+1,\ell}(\mu)|\mu|\,d\mu = I^-_{\varepsilon,j+1,r}E_3(\varepsilon) + \int_{\tau_j}^{\tau_{j+1}} \Phi_{\varepsilon,j+1}(\tau')E_2(\tau'-\tau_j)\,d\tau'. \tag{4.8}$$

Substituting (4.6) into (4.7) and (4.8), we find

$$\mathscr{R}^-(I^+_{\varepsilon,j,r}) = \theta(\alpha_\varepsilon I^+_{\varepsilon,j,\ell} + \beta_\varepsilon I^-_{\varepsilon,j,r} + f_{\varepsilon,j,r}), \quad 1 \leqslant j < m, \tag{4.9}$$

$$\mathscr{P}^+(I^+_{\varepsilon,j,r}) = (1-\theta)(\alpha_\varepsilon I^+_{\varepsilon,j,\ell} + \beta_\varepsilon I^-_{\varepsilon,j,r} + f_{\varepsilon,j,r}), \quad 1 \leqslant j < m, \tag{4.10}$$

$$\mathscr{R}^+(I^-_{\varepsilon,j+1,\ell}) = \theta(\alpha_\varepsilon I^-_{\varepsilon,j+1,r} + \beta_\varepsilon I^+_{\varepsilon,j+1,\ell} + f_{\varepsilon,j+1,\ell}), \quad 1 \leqslant j < m, \tag{4.11}$$

$$\mathscr{P}^-(I^-_{\varepsilon,j+1,\ell}) = (1-\theta)(\alpha_\varepsilon I^-_{\varepsilon,j+1,r} + \beta_\varepsilon I^+_{\varepsilon,j+1,\ell} + f_{\varepsilon,j+1,\ell}), \quad 1 \leqslant j < m, \tag{4.12}$$

$$\mathscr{R}^+_\ell(I^-_{\varepsilon,1,\ell}) = \theta_\ell(\alpha_\varepsilon I^-_{\varepsilon,1,r} + \beta_\varepsilon I^+_{\varepsilon,1,\ell} + f_{\varepsilon,1,\ell}), \tag{4.13}$$

$$\mathscr{R}^-_r(I^+_{\varepsilon,m,r}) = \theta_r(\alpha_\varepsilon I^+_{\varepsilon,m,\ell} + \beta_\varepsilon I^-_{\varepsilon,m,r} + f_{\varepsilon,m,r}), \tag{4.14}$$

where

$$\alpha_\varepsilon = 2E_3(\varepsilon) + \varpi \int\limits_0^\varepsilon \Psi^{E_2}_\varepsilon(\tau) E_2(\varepsilon-\tau)\,d\tau,$$

$$\beta_\varepsilon = \varpi \int\limits_0^\varepsilon \Psi^{E_2}_\varepsilon(\varepsilon-\tau) E_2(\varepsilon-\tau)\,d\tau = \varpi \int\limits_0^\varepsilon \Psi^{E_2}_\varepsilon(\tau) E_2(\tau)\,d\tau,$$

$$f_{\varepsilon,j,r} = 2(1-\varpi) \int\limits_{\tau_{j-1}}^{\tau_j} \Psi^F_{\varepsilon,j}(\tau) E_2(\tau_j - \tau)\,d\tau, \quad 1 \leqslant j \leqslant m,$$

$$f_{\varepsilon,j+1,\ell} = 2(1-\varpi) \int\limits_{\tau_j}^{\tau_{j+1}} \Psi^F_{\varepsilon,j+1}(\tau) E_2(\tau - \tau_j)\,d\tau, \qquad 0 \leqslant j < m.$$

It is clear that $0 < \alpha_\varepsilon$, $0 < \beta_\varepsilon$. We present another property of the coefficients.

Lemma 4.1. *The following relations hold*:

$$1 - \alpha_\varepsilon - \beta_\varepsilon = 2(1-\varpi) \int\limits_0^\varepsilon \Psi^1_\varepsilon(\tau) E_2(\varepsilon-\tau)\,d\tau, \tag{4.15}$$

$$1 - \alpha_\varepsilon - \beta_\varepsilon = 2(1-\varpi) \int\limits_0^\varepsilon \Psi^1_\varepsilon(\tau) E_2(\tau)\,d\tau. \tag{4.16}$$

Proof. From (3.3) it follows that

$$1 = \varpi \Lambda_\varepsilon(1)(\tau) + \frac{\varpi}{2} E_2(\tau) + \frac{\varpi}{2} E_2(\varepsilon-\tau) + 1 - \varpi.$$

Hence

$$1 = \frac{\varpi}{2} \Psi^{E_2}_\varepsilon(\tau) + \frac{\varpi}{2} \Psi^{E_2}_\varepsilon(\varepsilon-\tau) + (1-\varpi)\Psi^1_\varepsilon(\tau). \tag{4.17}$$

Multiplying (4.17) by $2E_2(\varepsilon-\tau)$ and integrating with respect to τ, we get the identity

$$1 - 2E_3(\varepsilon) = \varpi \int\limits_0^\varepsilon \Psi^{E_2}_\varepsilon(\tau) E_2(\varepsilon-\tau)\,d\tau + \varpi \int\limits_0^\varepsilon \Psi^{E_2}_\varepsilon(\tau) E_2(\tau)\,d\tau$$

$$+ 2(1-\varpi)\int\limits_0^{\varepsilon} \Psi^1_\varepsilon(\tau)E_2(\varepsilon-\tau)\,d\tau$$

which is equivalent to (4.15). Similarly, multiplying (4.17) by $2E_2(\tau)$ and integrating with respect to τ, we arrive at (4.16). □

Corollary 4.1. *The following relation holds*: $1-\alpha_\varepsilon-\beta_\varepsilon > 0$.

Substituting (4.9)–(4.14) into (1.2)–(1.5), we obtain the system

$$I^+_{\varepsilon,j+1,\ell} = \theta(\alpha_\varepsilon I^-_{\varepsilon,j+1,r} + \beta_\varepsilon I^+_{\varepsilon,j+1,\ell}) + (1-\theta)(\alpha_\varepsilon I^+_{\varepsilon,j,\ell} + \beta_\varepsilon I^-_{\varepsilon,j,r}) + f^+_{\varepsilon,j}, \quad 1 \leqslant j < m, \tag{4.18}$$

$$I^-_{\varepsilon,j,r} = \theta(\alpha_\varepsilon I^+_{\varepsilon,j,\ell} + \beta_\varepsilon I^-_{\varepsilon,j,r}) + (1-\theta)(\alpha_\varepsilon I^-_{\varepsilon,j+1,r} + \beta_\varepsilon I^+_{\varepsilon,j+1,\ell}) + f^-_{\varepsilon,j}, \quad 1 \leqslant j < m, \tag{4.19}$$

$$I^+_{\varepsilon,1,\ell} = \theta_\ell(\alpha_\varepsilon I^-_{\varepsilon,1,r} + \beta_\varepsilon I^+_{\varepsilon,1,\ell}) + (1-\theta_\ell)J_\ell + f^+_{\varepsilon,0}, \tag{4.20}$$

$$I^-_{\varepsilon,m,r} = \theta_r(\alpha_\varepsilon I^+_{\varepsilon,m,\ell} + \beta_\varepsilon I^-_{\varepsilon,m,r}) + (1-\theta_r)J_r + f^-_{\varepsilon,m}, \tag{4.21}$$

where $f^+_{\varepsilon,j} = (1-\theta)f_{\varepsilon,j,r} + \theta f_{\varepsilon,j+1,\ell}$, $f^-_{\varepsilon,j} = \theta f_{\varepsilon,j,r} + (1-\theta)f_{\varepsilon,j+1,\ell}$, $1 \leqslant j < m$, $f^+_{\varepsilon,0} = \theta_\ell f_{\varepsilon,1,\ell}$, $f^-_{\varepsilon,m} = \theta_r f_{\varepsilon,m,r}$. We set

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

we write $\mathbf{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix} > \mathbf{0}$ if $x^+ > 0$ and $x^- > 0$.

We set $\mathbf{I}_{\varepsilon,j} = \begin{pmatrix} I^+_{\varepsilon,j,\ell} \\ I^-_{\varepsilon,j,r} \end{pmatrix}$, $1 \leqslant j \leqslant m$, and write the system (4.18)–(4.21) in the matrix form

$$\mathbf{A}_\varepsilon \mathbf{I}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \mathbf{I}_{\varepsilon,j} = \mathbf{f}_{\varepsilon,j}, \quad 1 \leqslant j < m, \tag{4.22}$$

$$\mathbf{A}_{\varepsilon,\ell}\mathbf{I}_{\varepsilon,1} = (1-\theta_\ell)J_\ell + f^+_{\varepsilon,0}, \tag{4.23}$$

$$\mathbf{B}_{\varepsilon,r}\mathbf{I}_{\varepsilon,m} = (1-\theta_r)J_r + f^-_{\varepsilon,m}, \tag{4.24}$$

where

$$\mathbf{A}_\varepsilon = \begin{pmatrix} 1-\theta\beta_\varepsilon & -\theta\alpha_\varepsilon \\ -(1-\theta)\beta_\varepsilon & -(1-\theta)\alpha_\varepsilon \end{pmatrix}, \quad \mathbf{B}_\varepsilon = \begin{pmatrix} -(1-\theta)\alpha_\varepsilon & -(1-\theta)\beta_\varepsilon \\ -\theta\alpha_\varepsilon & 1-\theta\beta_\varepsilon \end{pmatrix},$$

$$\mathbf{A}_{\varepsilon,\ell} = \begin{pmatrix} 1-\theta_\ell\beta_\varepsilon & -\theta_\ell\alpha_\varepsilon \end{pmatrix}, \quad \mathbf{B}_{\varepsilon,r} = \begin{pmatrix} -\theta_r\alpha_\varepsilon & 1-\theta_r\beta_\varepsilon \end{pmatrix},$$

$$\mathbf{f}_{\varepsilon,j} = \begin{pmatrix} f^+_{\varepsilon,j} \\ f^-_{\varepsilon,j} \end{pmatrix}, \quad 1 \leqslant j < m.$$

Let us verify that the system (4.22)–(4.24) is uniquely solvable. For this purpose we consider the corresponding homogenous system

$$\mathbf{A}_\varepsilon \mathbf{x}_{j+1} + \mathbf{B}_\varepsilon \mathbf{x}_j = \mathbf{0}, \quad 1 \leqslant j < m, \tag{4.25}$$

$$\mathbf{A}_{\varepsilon,\ell}\mathbf{x}_1 = 0, \tag{4.26}$$

$$\mathbf{B}_{\varepsilon,r}\mathbf{x}_m = 0 \tag{4.27}$$

for unknown $\mathbf{x}_j = \begin{pmatrix} x_j^+ \\ x_j^- \end{pmatrix}$, $1 \leqslant j \leqslant m$. The matrix $\mathbf{A}_\varepsilon$ is nonsingular since $\det \mathbf{A}_\varepsilon = -(1-\theta)\alpha_\varepsilon \neq 0$. Therefore, the system (4.25) is equivalent to the system

$$\mathbf{x}_{j+1} = \mathbf{Q}_\varepsilon \mathbf{x}_j, \quad 1 \leqslant j < m,$$

where $\mathbf{Q}_\varepsilon = -\mathbf{A}_\varepsilon^{-1}\mathbf{B}_\varepsilon$. The eigenvalue q of the matrix $\mathbf{Q}_\varepsilon$ satisfies the equation

$$\det(\mathbf{B}_\varepsilon + q\mathbf{A}_\varepsilon) = 0,$$

i.e., $(1-\theta)\alpha_\varepsilon q^2 - [(1-\theta\beta_\varepsilon)^2 + (1-\theta)^2\alpha_\varepsilon^2 - (1-\theta)^2\beta_\varepsilon^2 - \theta^2\alpha_\varepsilon^2]q + (1-\theta)\alpha_\varepsilon = 0$ which can be transformed to the form

$$q^2 - 2(1+\rho_\varepsilon)q + 1 = 0,$$

where

$$\rho_\varepsilon = \frac{1}{2(1-\theta)\alpha_\varepsilon}[\theta(1-\beta_\varepsilon+\alpha_\varepsilon) + (1-\theta)(1-\alpha_\varepsilon+\beta_\varepsilon)](1-\alpha_\varepsilon-\beta_\varepsilon). \tag{4.28}$$

Since $\alpha_\varepsilon > 0$, $\beta_\varepsilon > 0$ and $1-\alpha_\varepsilon-\beta_\varepsilon > 0$, we have $\rho_\varepsilon > 0$. Thus, the matrix $\mathbf{Q}_\varepsilon$ has two eigenvalues $q_\varepsilon = 1+\rho_\varepsilon - \sqrt{2\rho_\varepsilon+\rho_\varepsilon^2}$ and $q_\varepsilon^{-1} = 1+\rho_\varepsilon+\sqrt{2\rho_\varepsilon+\rho_\varepsilon^2}$. It is clear that $0 < q_\varepsilon < 1$.

It is easy to see that the eigenvector $\mathbf{v}_{\varepsilon,1} = \begin{pmatrix} 1 \\ \sigma_\varepsilon \end{pmatrix}$ corresponds to the eigenvalue q_ε and the eigenvector $\mathbf{v}_{\varepsilon,2} = \begin{pmatrix} \sigma_\varepsilon \\ 1 \end{pmatrix}$ corresponds to the eigenvalue q_ε^{-1}, where

$$\sigma_\varepsilon = \frac{(1-\theta\beta_\varepsilon)q_\varepsilon - (1-\theta)\alpha_\varepsilon}{(1-\theta)\beta_\varepsilon + \theta\alpha_\varepsilon q_\varepsilon} = \frac{\theta\alpha_\varepsilon + (1-\theta)\beta_\varepsilon q_\varepsilon}{1-\theta\beta_\varepsilon - (1-\theta)\alpha_\varepsilon q_\varepsilon},$$

and the general solution to the system (4.25) has the form

$$\mathbf{x}_j = C_1 q_\varepsilon^{j-1}\mathbf{v}_{\varepsilon,1} + C_2 q_\varepsilon^{m-j}\mathbf{v}_{\varepsilon,2}, \quad 1 \leqslant j \leqslant m, \tag{4.29}$$

where C_1 and C_2 are arbitrary constants.

Lemma 4.2. *The following relation holds*:

$$0 < \sigma_\varepsilon < 1. \tag{4.30}$$

Proof. From the inequalities $0 < \alpha_\varepsilon$, $0 < \beta_\varepsilon$, $1-\alpha_\varepsilon-\beta_\varepsilon > 0$, $0 < q_\varepsilon < 1$ it follows that

$$1-\theta\beta_\varepsilon - (1-\theta)\alpha_\varepsilon q_\varepsilon > 1-\beta_\varepsilon-\alpha_\varepsilon > 0,$$

$$0 < \sigma_\varepsilon = \frac{\theta\alpha_\varepsilon + (1-\theta)\beta_\varepsilon q_\varepsilon}{1-\theta\beta_\varepsilon-(1-\theta)\alpha_\varepsilon q_\varepsilon} < \frac{\theta\alpha_\varepsilon + (1-\theta)\beta_\varepsilon}{1-\theta\beta_\varepsilon-(1-\theta)\alpha_\varepsilon} = 1 - \frac{1-\alpha_\varepsilon-\beta_\varepsilon}{1-\theta\beta_\varepsilon-(1-\theta)\alpha_\varepsilon} < 1.$$

The lemma is proved. □

We note that

$$\mathbf{A}_\varepsilon\mathbf{v}_{\varepsilon,1} = \begin{pmatrix} 1-\theta p_\varepsilon \\ -(1-\theta)p_\varepsilon \end{pmatrix}, \quad \mathbf{A}_\varepsilon\mathbf{v}_{\varepsilon,2} = \begin{pmatrix} \sigma_\varepsilon - \theta r_\varepsilon \\ -(1-\theta)r_\varepsilon \end{pmatrix},$$

$$\mathbf{B}_\varepsilon\mathbf{v}_{\varepsilon,1} = \begin{pmatrix} -(1-\theta)r_\varepsilon \\ \sigma_\varepsilon - \theta r_\varepsilon \end{pmatrix}, \quad \mathbf{B}_\varepsilon\mathbf{v}_{\varepsilon,2} = \begin{pmatrix} -(1-\theta)p_\varepsilon \\ 1-\theta p_\varepsilon \end{pmatrix},$$

where $p_\varepsilon = \alpha_\varepsilon \sigma_\varepsilon + \beta_\varepsilon$, $r_\varepsilon = \alpha_\varepsilon + \beta_\varepsilon \sigma_\varepsilon$. We note that $0 < p_\varepsilon < 1$, $0 < r_\varepsilon < 1$.

It is clear that

$$\mathbf{B}_\varepsilon \mathbf{v}_{\varepsilon,1} = -q_\varepsilon \mathbf{A}_\varepsilon \mathbf{v}_{\varepsilon,1}, \quad \mathbf{A}_\varepsilon \mathbf{v}_{\varepsilon,2} = -q_\varepsilon \mathbf{B}_\varepsilon \mathbf{v}_{\varepsilon,2}, \tag{4.31}$$

i.e.,

$$(1-\theta) r_\varepsilon = q_\varepsilon (1 - \theta p_\varepsilon), \quad \sigma_\varepsilon - \theta r_\varepsilon = q_\varepsilon (1-\theta) p_\varepsilon.$$

We set

$$\zeta_{\varepsilon,\ell} = \frac{\sigma_\varepsilon - \theta_\ell r_\varepsilon}{1 - \theta_\ell p_\varepsilon}, \quad \zeta_{\varepsilon,r} = \frac{\sigma_\varepsilon - \theta_r r_\varepsilon}{1 - \theta_r p_\varepsilon}$$

We note that (4.30) implies

$$0 < \zeta_{\varepsilon,\ell} < \sigma_\varepsilon, \quad 0 < \zeta_{\varepsilon,r} < \sigma_\varepsilon. \tag{4.32}$$

Substituting (4.29) with $j = 1$ into (4.26), we obtain the expression for the general solution to the system (4.25), (4.26)

$$\mathbf{x}_j = C q_\varepsilon^{m-j} (\mathbf{v}_{\varepsilon,2} - \zeta_{\varepsilon,\ell} q_\varepsilon^{2(j-1)} \mathbf{v}_{\varepsilon,1}), \tag{4.33}$$

where C is an arbitrary constant. If (4.27) holds, we obtain the identity

$$C(1 - \theta_r p_\varepsilon)(1 - \zeta_{\varepsilon,\ell} \zeta_{\varepsilon,r} q_\varepsilon^{2(m-1)}) = 0$$

which implies $C = 0$, Thus, the homogeneous system (4.25)–(4.27) has only trivial solution, and the system (4.22)–(4.24) is uniquely solvable.

We also note that the solution to the system (4.25), (4.27) has the form

$$\mathbf{x}_j = C q_\varepsilon^{j-1} (\mathbf{v}_{\varepsilon,1} - \zeta_{\varepsilon,r} q_\varepsilon^{2(m-j)} \mathbf{v}_{\varepsilon,2}), \tag{4.34}$$

where C is an arbitrary the constant.

For the further considerations, it is useful to represent the solution to the system (4.22)–(4.24) as

$$\mathbf{I}_{\varepsilon,j} = \Big[J_\ell + \frac{1}{1-\theta_\ell} f^+_{\varepsilon,0} \Big] \mathbf{z}_{\varepsilon,j} + \Big[J_r + \frac{1}{1-\theta_r} f^-_{\varepsilon,m} \Big] \mathbf{y}_{\varepsilon,j} + \sum_{i=1}^{m-1} f^+_{\varepsilon,i} \mathbf{g}^{i,1}_{\varepsilon,j} + \sum_{i=1}^{m-1} f^-_{\varepsilon,i} \mathbf{g}^{i,2}_{\varepsilon,j}, \quad 1 \leqslant j \leqslant m,$$

where $\mathbf{y}_{\varepsilon,j} = \begin{pmatrix} y^+_{\varepsilon,j} \\ y^-_{\varepsilon,j} \end{pmatrix}$ and $\mathbf{z}_{\varepsilon,j} = \begin{pmatrix} z^+_{\varepsilon,j} \\ z^-_{\varepsilon,j} \end{pmatrix}$, $1 \leqslant j \leqslant m$, are solutions to the systems

$$\mathbf{A}_\varepsilon \mathbf{y}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \mathbf{y}_{\varepsilon,j} = \mathbf{0}, \quad 1 \leqslant j < m, \tag{4.35}$$

$$\mathbf{A}_{\varepsilon,\ell} \mathbf{y}_{\varepsilon,1} = 0, \tag{4.36}$$

$$\mathbf{B}_{\varepsilon,r} \mathbf{y}_{\varepsilon,m} = 1 - \theta_r. \tag{4.37}$$

and

$$\mathbf{A}_\varepsilon \mathbf{z}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \mathbf{z}_{\varepsilon,j} = \mathbf{0}, \quad 1 \leqslant j < m, \tag{4.38}$$

$$\mathbf{A}_{\varepsilon,\ell} \mathbf{z}_{\varepsilon,1} = 1 - \theta_\ell, \tag{4.39}$$

$$\mathbf{B}_{\varepsilon,r} \mathbf{z}_{\varepsilon,m} = 0 \tag{4.40}$$

respectively, whereas $\mathbf{g}^{i,1}_{\varepsilon,j} = \begin{pmatrix} g^{i,1,+}_{\varepsilon,j} \\ g^{i,1,-}_{\varepsilon,j} \end{pmatrix}$ and $\mathbf{g}^{i,2}_{\varepsilon,j} = \begin{pmatrix} g^{i,2,+}_{\varepsilon,j} \\ g^{i,2,-}_{\varepsilon,j} \end{pmatrix}$, $1 \leqslant j \leqslant m$, are solutions to the systems

$$\mathbf{A}_\varepsilon \mathbf{g}^{i,1}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \mathbf{g}^{i,1}_{\varepsilon,j} = \delta_{ij}\mathbf{e}_1, \quad 1 \leqslant j < m, \tag{4.41}$$

$$\mathbf{A}_{\varepsilon,\ell}\mathbf{g}^{i,1}_{\varepsilon,1} = 0, \tag{4.42}$$

$$\mathbf{B}_{\varepsilon,r}\mathbf{g}^{i,1}_{\varepsilon,m} = 0, \tag{4.43}$$

and

$$\mathbf{A}_\varepsilon \mathbf{g}^{i,2}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \mathbf{g}^{i,2}_{\varepsilon,j} = \delta_{ij}\mathbf{e}_2, \quad 1 \leqslant j < m, \tag{4.44}$$

$$\mathbf{A}_{\varepsilon,\ell}\mathbf{g}^{i,2}_{\varepsilon,1} = 0, \tag{4.45}$$

$$\mathbf{B}_{\varepsilon,r}\mathbf{g}^{i,2}_{\varepsilon,m} = 0 \tag{4.46}$$

respectively. Here, $1 \leqslant i < m$ and δ_{ij} denotes the Kronecker symbol.

We obtain formulas for $\mathbf{y}_{\varepsilon,j}$, $\mathbf{z}_{\varepsilon,j}$, $\mathbf{g}^{i,1}_{\varepsilon,j}$, $\mathbf{g}^{i,2}_{\varepsilon,j}$, $1 \leqslant j \leqslant m$. Since $\mathbf{y}_{\varepsilon,j}$ and $\mathbf{z}_{\varepsilon,j}$ satisfy (4.35), (4.36) and (4.38), (4.40) respectively, we have

$$\mathbf{y}_{\varepsilon,j} = C_{\varepsilon,r}\, q_\varepsilon^{m-j}(\mathbf{v}_{\varepsilon,2} - \zeta_{\varepsilon,\ell} q_\varepsilon^{2(j-1)}\mathbf{v}_{\varepsilon,1}), \quad 1 \leqslant j \leqslant m, \tag{4.47}$$

$$\mathbf{z}_{\varepsilon,j} = C_{\varepsilon,\ell}\, q_\varepsilon^{j-1}(\mathbf{v}_{\varepsilon,1} - \zeta_{\varepsilon,r} q_\varepsilon^{2(m-j)}\mathbf{v}_{\varepsilon,2}), \quad 1 \leqslant j \leqslant m. \tag{4.48}$$

Substituting these formulas into (4.37) and (4.39), we find

$$C_{\varepsilon,r} = \frac{1-\theta_r}{(1-\theta_r p_\varepsilon)(1-\zeta_{\varepsilon,\ell}\zeta_{\varepsilon,r}q_\varepsilon^{2(m-1)})},$$

$$C_{\varepsilon,\ell} = \frac{1-\theta_\ell}{(1-\theta_\ell p_\varepsilon)(1-\zeta_{\varepsilon,r}\zeta_{\varepsilon,\ell}q_\varepsilon^{2(m-1)})}.$$

We note that from (4.30), (4.31), $0 < q_\varepsilon < 1$ it follows that

$$\mathbf{v}_{\varepsilon,2} - \zeta_{\varepsilon,\ell} q_\varepsilon^{2(j-1)}\mathbf{v}_{\varepsilon,1} > \mathbf{0}, \quad \mathbf{v}_{\varepsilon,1} - \zeta_{\varepsilon,r} q_\varepsilon^{2(m-j)}\mathbf{v}_{\varepsilon,2} > \mathbf{0}, \quad 1 \leqslant j \leqslant m. \tag{4.49}$$

Since $C_{\varepsilon,r} > 0$, $C_{\varepsilon,\ell} > 0$, from (4.47)–(4.49) it follows that

$$\mathbf{y}_{\varepsilon,j} > \mathbf{0}, \quad \mathbf{z}_{\varepsilon,j} > \mathbf{0}, \quad 1 \leqslant j \leqslant m. \tag{4.50}$$

We fix i, $1 \leqslant i < m$ and set

$$\mathbf{g}^{i,1}_{\varepsilon,j} = \begin{cases} C^{i,1}_{\varepsilon,1} q_\varepsilon^{i-j}(\mathbf{v}_{\varepsilon,2} - \zeta_{\varepsilon,\ell} q_\varepsilon^{2(j-1)}\mathbf{v}_{\varepsilon,1}), & 1 \leqslant j \leqslant i, \\ C^{i,1}_{\varepsilon,2} q_\varepsilon^{j-i-1}(\mathbf{v}_{\varepsilon,1} - \zeta_{\varepsilon,r} q_\varepsilon^{2(m-j)}\mathbf{v}_{\varepsilon,2}), & i < j \leqslant m. \end{cases}$$

We note that $\mathbf{g}^{i,1}_{\varepsilon,j}$ satisfies (4.41) for $j \neq i$ and also (4.42), (4.43). We choose constants $C^{i,1}_{\varepsilon,1}$ and $C^{i,1}_{\varepsilon,2}$ such that

$$\mathbf{A}_\varepsilon \mathbf{g}^{i,1}_{\varepsilon,i+1} + \mathbf{B}_\varepsilon \mathbf{g}^{i,1}_{\varepsilon,i} = \mathbf{e}_1,$$

i.e.,

$$C^{i,1}_{\varepsilon,2}(\mathbf{A}_\varepsilon\mathbf{v}_{\varepsilon,1}-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-i-1)}\mathbf{A}_\varepsilon\mathbf{v}_{\varepsilon,2})+C^{i,1}_{\varepsilon,1}(\mathbf{B}_\varepsilon\mathbf{v}_{\varepsilon,2}-\zeta_{\varepsilon,\ell}q_\varepsilon^{2(i-1)}\mathbf{B}_\varepsilon\mathbf{v}_{\varepsilon,1})=\mathbf{e}_1.$$

Taking into account the identities (4.31), we obtain the system

$$C^{i,1}_{\varepsilon,2}(\mathbf{A}_\varepsilon\mathbf{v}_{\varepsilon,1}+\zeta_{\varepsilon,r}q_\varepsilon^{2(m-i)-1}\mathbf{B}_\varepsilon\mathbf{v}_{\varepsilon,2})+C^{i,1}_{\varepsilon,1}(\mathbf{B}_\varepsilon\mathbf{v}_{\varepsilon,2}+\zeta_{\varepsilon,\ell}q_\varepsilon^{2i-1}\mathbf{A}_\varepsilon\mathbf{v}_{\varepsilon,1})=\mathbf{e}_1,$$

i.e.,

$$-[(1-\theta)p_\varepsilon-\zeta_{\varepsilon,\ell}q_\varepsilon^{2i-1}(1-\theta p_\varepsilon)]C^{i,1}_{\varepsilon,1}+[1-\theta p_\varepsilon-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-i)-1}(1-\theta)p_\varepsilon)]C^{i,1}_{\varepsilon,2}=1,$$

$$[1-\theta p_\varepsilon-\zeta_{\varepsilon,\ell}q_\varepsilon^{2i-1}(1-\theta)p_\varepsilon)]C^{i,1}_{\varepsilon,1}-[(1-\theta)p_\varepsilon-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-i)-1}(1-\theta p_\varepsilon)]C^{i,1}_{\varepsilon,2}=0.$$

Solving this system, we find

$$C^{i,1}_{\varepsilon,1}=\frac{1}{\Delta_\varepsilon}[(1-\theta)p_\varepsilon-\zeta_{\varepsilon,r}(1-\theta p_\varepsilon)q_\varepsilon^{2(m-i)-1}],$$

$$C^{i,1}_{\varepsilon,2}=\frac{1}{\Delta_\varepsilon}[1-\theta p_\varepsilon-\zeta_{\varepsilon,\ell}(1-\theta)p_\varepsilon q_\varepsilon^{2i-1}],$$

where

$$\Delta_\varepsilon=(1-\zeta_{\varepsilon,\ell}\zeta_{\varepsilon,r}q_\varepsilon^{2(m-1)})(1-p_\varepsilon)(1-\theta p_\varepsilon+(1-\theta)p_\varepsilon)>0. \tag{4.51}$$

Thus,

$$\mathbf{g}^{i,1}_{\varepsilon,j}=\begin{cases}\dfrac{1}{\Delta_\varepsilon}[(1-\theta)p_\varepsilon-\zeta_{\varepsilon,r}(1-\theta p_\varepsilon)q_\varepsilon^{2(m-i)-1}]q_\varepsilon^{i-j}(\mathbf{v}_{\varepsilon,2}-\zeta_{\varepsilon,\ell}q_\varepsilon^{2(j-1)}\mathbf{v}_{\varepsilon,1}),\\ \qquad 1\leqslant j\leqslant i<m,\\ \dfrac{1}{\Delta_\varepsilon}[1-\theta p_\varepsilon-\zeta_{\varepsilon,\ell}(1-\theta)p_\varepsilon q_\varepsilon^{2i-1}]q_\varepsilon^{j-i-1}(\mathbf{v}_{\varepsilon,1}-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)}\mathbf{v}_{\varepsilon,2}),\\ \qquad 1\leqslant i<j\leqslant m.\end{cases} \tag{4.52}$$

Similarly, we have

$$\mathbf{g}^{i,2}_{\varepsilon,j}=\begin{cases}\dfrac{1}{\Delta_\varepsilon}[1-\theta p_\varepsilon-\zeta_{\varepsilon,r}(1-\theta)p_\varepsilon q_\varepsilon^{2(m-i)-1}]q_\varepsilon^{i-j}(\mathbf{v}_{\varepsilon,2}-\zeta_{\varepsilon,\ell}q_\varepsilon^{2(j-1)}\mathbf{v}_{\varepsilon,1}),\\ \qquad 1\leqslant j\leqslant i<m,\\ \dfrac{1}{\Delta_\varepsilon}[(1-\theta)p_\varepsilon-\zeta_{\varepsilon,\ell}(1-\theta p_\varepsilon)q_\varepsilon^{2i-1}]q_\varepsilon^{j-i-1}(\mathbf{v}_{\varepsilon,1}-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)}\mathbf{v}_{\varepsilon,2})\\ \qquad 1\leqslant i<j\leqslant m.\end{cases} \tag{4.53}$$

We note that

$$(1-\theta)p_\varepsilon-\zeta_{\varepsilon,r}(1-\theta p_\varepsilon)q_\varepsilon^{2(m-i)-1}\geqslant(1-\theta)p_\varepsilon-\sigma_\varepsilon(1-\theta p_\varepsilon)q_\varepsilon$$

$$=(1-\theta)p_\varepsilon-\sigma_\varepsilon(1-\theta)r_\varepsilon=(1-\theta)(1-\sigma_\varepsilon^2)\beta_\varepsilon>0,$$

$$1-\theta p_\varepsilon-\zeta_{\varepsilon,\ell}(1-\theta)p_\varepsilon q_\varepsilon^{2i-1}\geqslant 1-\theta p_\varepsilon-(1-\theta)p_\varepsilon=1-p_\varepsilon>0.$$

Similarly,

$$(1-\theta)p_\varepsilon-\zeta_{\varepsilon,\ell}(1-\theta p_\varepsilon)q_\varepsilon^{2i-1}>0,\quad 1-\theta p_\varepsilon-\zeta_{\varepsilon,r}(1-\theta)p_\varepsilon q_\varepsilon^{2(m-i)-1}>0.$$

Therefore, from (4.49), (4.51) it follows that $\mathbf{g}^{i,1}_{\varepsilon,j}>\mathbf{0}$, $\mathbf{g}^{i,2}_{\varepsilon,j}>\mathbf{0}$, $1\leqslant j\leqslant m$.

5 Auxiliary Asymptotic Formulas

Lemma 5.1. *The following relations hold*:

$$\alpha_\varepsilon = 1 - (2-\varpi)\varepsilon - (1 - \varpi + \frac{\varpi^2}{2})\varepsilon^2(\ln\varepsilon + \gamma - 3/2) + O(\varepsilon^3\ln^2\varepsilon), \tag{5.1}$$

$$\beta_\varepsilon = \varpi\varepsilon + \varpi(1 - \frac{\varpi}{2})\varepsilon^2(\ln\varepsilon + \gamma - 3/2) + O(\varepsilon^3\ln^2\varepsilon), \tag{5.2}$$

$$1 - \alpha_\varepsilon - \beta_\varepsilon = 2\varepsilon_\varpi + \varepsilon_\varpi^2(\ln\varepsilon + \gamma - 3/2) + O(\varepsilon^3\ln^2\varepsilon). \tag{5.3}$$

Proof. Using (2.3) and (2.5), we find

$$\int\limits_0^\varepsilon E_2^2(\tau)\,d\tau = \int\limits_0^\varepsilon (2E_2(\tau) - 1)\,d\tau + \int\limits_0^\varepsilon (E_2(\tau) - 1)^2\,d\tau = 1 - 2E_3(\varepsilon) - \varepsilon + O(\varepsilon^3\ln^2\varepsilon), \tag{5.4}$$

$$\int\limits_0^\varepsilon E_2(\tau)E_2(\varepsilon-\tau)\,d\tau = \int\limits_0^\varepsilon (E_2(\tau) + E_2(\varepsilon-\tau) - 1)\,d\tau$$
$$+ \int\limits_0^\varepsilon (E_2(\tau) - 1)(E_2(\varepsilon-\tau) - 1)\,d\tau = 1 - 2E_3(\varepsilon) - \varepsilon + O\big(\varepsilon^3\ln^2\varepsilon\big). \tag{5.5}$$

Using (3.7), (5.4), (5.5), (2.6), we find

$$\alpha_\varepsilon = 2E_3(\varepsilon) + \varpi\int\limits_0^\varepsilon \Psi_\varepsilon^{E_2}(\tau)E_2(\varepsilon-\tau)\,d\tau$$

$$= 2E_3(\varepsilon) + \varpi\int\limits_0^\varepsilon \Big[\varpi + \Big(1 - \frac{\varpi}{2}\Big)E_2(\tau) - \frac{\varpi}{2}E_2(\varepsilon-\tau)\Big]E_2(\varepsilon-\tau)\,d\tau + O(\varepsilon^3\ln^2\varepsilon)$$

$$= 2E_3(\varepsilon) + \varpi\Bigg[\varpi\Big(\frac{1}{2} - E_3(\varepsilon)\Big) + \Big(1 - \frac{\varpi}{2}\Big)\int\limits_0^\varepsilon E_2(\tau)E_2(\varepsilon-\tau)\,d\tau$$

$$- \frac{\varpi}{2}\int\limits_0^\varepsilon E_2^2(\tau)\,d\tau\Bigg] + O(\varepsilon^3\ln^2\varepsilon)$$

$$= 2E_3(\varepsilon) + \varpi\Bigg[\varpi\Big(\frac{1}{2} - E_3(\varepsilon)\Big) + (1-\varpi)(1 - 2E_3(\varepsilon) - \varepsilon)\Bigg] + O(\varepsilon^3\ln^2\varepsilon)$$

$$= 1 - (2-\varpi)\varepsilon - (1 - \varpi + \frac{\varpi^2}{2})\varepsilon^2(\ln\varepsilon + \gamma - 3/2) + O(\varepsilon^3\ln^2\varepsilon),$$

$$\beta_\varepsilon = \varpi\int\limits_0^\varepsilon \Psi_\varepsilon^{E_2}(\tau)E_2(\tau)\,d\tau = \varpi\int\limits_0^\varepsilon \Big[\varpi + \Big(1 - \frac{\varpi}{2}\Big)E_2(\tau)$$

$$- \frac{\varpi}{2}E_2(\varepsilon-\tau)\Big]E_2(\tau)\,d\tau + O(\varepsilon^3\ln^2\varepsilon)$$

$$= \varpi\left[\varpi\Big(\frac{1}{2} - E_3(\varepsilon)\Big) + \Big(1 - \frac{\varpi}{2}\Big)\int\limits_0^{\varepsilon} E_2^2(\tau)\,d\tau - \frac{\varpi}{2}\int\limits_0^{\varepsilon} E_2(\varepsilon-\tau)E_2(\tau)\,d\tau\right] + O(\varepsilon^3\ln^2\varepsilon)$$

$$= \varpi\Big[\varpi\Big(\frac{1}{2} - E_3(\varepsilon)\Big) + (1-\varpi)(1 - 2E_3(\varepsilon) - \varepsilon)\Big] + O(\varepsilon^3\ln^2\varepsilon)$$

$$= \varpi\varepsilon + \varpi\Big(1 - \frac{\varpi}{2}\Big)\varepsilon^2(\ln\varepsilon + \gamma - 3/2) + O(\varepsilon^3\ln^2\varepsilon).$$

Formulas (5.1) and (5.2) are proved. From these formulas we obtain (5.3). □

We recall that

$$\lambda_0 = \frac{1-\theta}{\theta(1-\varpi)}, \quad \lambda(\varepsilon) = \frac{1}{4}\frac{\lambda_0 + \dfrac{\varepsilon}{3}}{1 + \dfrac{1}{2}\varepsilon_\varpi(\ln\varepsilon + 2\lambda_0 + \gamma + 1/2)}.$$

Lemma 5.2. *The following relation holds*:

$$\rho_\varepsilon = \frac{\varepsilon}{2\lambda(\varepsilon)}\Big(1 + \frac{\varepsilon}{3\lambda_0}\Big)(1 + O(\varepsilon^2\ln^2\varepsilon)). \tag{5.6}$$

Proof. From (4.28), (5.1)–(5.3) it follows that

$$\rho_\varepsilon = \frac{1}{2(1-\theta)\alpha_\varepsilon}[\theta(1 - \beta_\varepsilon + \alpha_\varepsilon) + (1-\theta)(1 - \alpha_\varepsilon + \beta_\varepsilon)](1 - \alpha_\varepsilon - \beta_\varepsilon)$$

$$= \frac{2\theta(1-\varepsilon) + 2(1-\theta)\varepsilon}{2(1-\theta)(1-(2-\varpi)\varepsilon)}(2\varepsilon_\varpi + \varepsilon_\varpi^2(\ln\varepsilon + \gamma - 3/2))(1 + O(\varepsilon^2\ln^2\varepsilon))$$

$$= \frac{2\theta[1 + (1/\theta - 2)\varepsilon]}{(1-\theta)(1-(2-\varpi)\varepsilon)}\varepsilon_\varpi\Big[1 + \frac{1}{2}\varepsilon_\varpi(\ln\varepsilon + \gamma - 3/2)\Big](1 + O(\varepsilon^2\ln^2\varepsilon))$$

$$= \frac{2\theta\varepsilon_\varpi}{1-\theta}[1 + (1/\theta - 2)\varepsilon][1 + (2-\varpi)\varepsilon]\Big[1 + \frac{1}{2}\varepsilon_\varpi(\ln\varepsilon + \gamma - 3/2)\Big](1 + O(\varepsilon^2\ln^2\varepsilon))$$

$$= \frac{2\varepsilon}{\lambda_0}\Big[1 + (1/\theta - 2)\varepsilon + (2-\varpi)\varepsilon + \frac{1}{2}\varepsilon_\varpi(\ln\varepsilon + \gamma - 3/2)\Big](1 + O(\varepsilon^2\ln^2\varepsilon))$$

$$= \frac{2\varepsilon}{\lambda_0}\Big[1 + \frac{1-\theta}{\theta}\varepsilon + \frac{1}{2}\varepsilon_\varpi(\ln\varepsilon + \gamma + 1/2)\Big](1 + O(\varepsilon^2\ln^2\varepsilon))$$

$$= \frac{2\varepsilon}{\lambda_0}\Big[1 + \frac{1}{2}\varepsilon_\varpi(\ln\varepsilon + 2\lambda_0 + \gamma + 1/2)\Big](1 + O(\varepsilon^2\ln^2\varepsilon))$$

$$= \frac{\varepsilon}{2\lambda(\varepsilon)}\Big(1 + \frac{\varepsilon}{3\lambda_0}\Big)(1 + O(\varepsilon^2\ln^2\varepsilon)).$$

The lemma is proved. □

Lemma 5.3. *The following relations hold*:

$$q_\varepsilon = 1 - 2\sqrt{\frac{\varepsilon}{\lambda_0}} + \frac{2\varepsilon}{\lambda_0} + O(\varepsilon^{3/2}\ln\varepsilon), \tag{5.7}$$

$$\ln q_\varepsilon = -\sqrt{\frac{\varepsilon}{\lambda(\varepsilon)}}\,(1 + O(\varepsilon^2\ln^2\varepsilon)). \tag{5.8}$$

Proof. From (5.6) it follows that

$$\rho_\varepsilon = \frac{2\varepsilon}{\lambda_0}(1 + O(\varepsilon \ln \varepsilon)),$$

$$q_\varepsilon = 1 + \rho_\varepsilon - \sqrt{2\rho_\varepsilon + \rho_\varepsilon^2} = 1 - \sqrt{2\rho_\varepsilon} + \rho_\varepsilon + O(\rho_\varepsilon^{3/2}) = 1 - 2\sqrt{\frac{\varepsilon}{\lambda_0}} + \frac{2\varepsilon}{\lambda_0} + O(\varepsilon^{3/2}\ln\varepsilon),$$

$$\begin{aligned}\ln q_\varepsilon &= -\ln[1 + \rho_\varepsilon + \sqrt{2\rho_\varepsilon + \rho_\varepsilon^2}] = -(\rho_\varepsilon + \sqrt{2\rho_\varepsilon + \rho_\varepsilon^2}) + \frac{1}{2}(\rho_\varepsilon + \sqrt{2\rho_\varepsilon + \rho_\varepsilon^2})^2 \\ &\quad - \frac{1}{3}(\rho_\varepsilon + \sqrt{2\rho_\varepsilon + \rho_\varepsilon^2})^3 + \frac{1}{4}(\rho_\varepsilon + \sqrt{2\rho_\varepsilon + \rho_\varepsilon^2})^4 + O(\rho_\varepsilon^{5/2}) \\ &= -\sqrt{2\rho_\varepsilon + \rho_\varepsilon^2} + \frac{1}{3}\rho_\varepsilon\sqrt{2\rho_\varepsilon + \rho_\varepsilon^2} + O(\rho_\varepsilon^{5/2}) = -\Big(1 - \frac{1}{3}\rho_\varepsilon\Big)\sqrt{2\rho_\varepsilon + \rho_\varepsilon^2} + O(\rho_\varepsilon^{5/2}) \\ &= -\sqrt{2\rho_\varepsilon}\Big(1 - \frac{1}{12}\rho_\varepsilon\Big)(1 + O(\varepsilon^2)) = -\sqrt{\frac{\varepsilon}{\lambda(\varepsilon)}}\sqrt{1 + \frac{\varepsilon}{3\lambda_0}}\Big(1 - \frac{\varepsilon}{6\lambda_0}\Big)(1 + O(\varepsilon^2\ln^2\varepsilon)) \\ &= -\sqrt{\frac{\varepsilon}{\lambda(\varepsilon)}}(1 + O(\varepsilon^2\ln^2\varepsilon)).\end{aligned}$$

The lemma is proved. □

Corollary 5.1. *The following relation holds*:

$$q_\varepsilon^j = e^{-\tau_j/\sqrt{\varepsilon\lambda(\varepsilon)}}(1 + O(\varepsilon^{3/2}\ln^2\varepsilon)), \quad 1 \leqslant j \leqslant m. \tag{5.9}$$

Proof. From (5.8) it follows that

$$q_\varepsilon^j = \exp(j\ln q_\varepsilon) = \exp\left[-\tau_j/\sqrt{\varepsilon\lambda(\varepsilon)}(1 + O(\varepsilon^2\ln^2\varepsilon))\right] = e^{-\tau_j/\sqrt{\varepsilon\lambda(\varepsilon)}}(1 + O(\varepsilon^{3/2}\ln^2\varepsilon)).$$

The lemma is proved. □

Lemma 5.4. *The following relations hold*:

$$\sigma_\varepsilon = 1 - 2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi} + O(\varepsilon^{3/2}\ln\varepsilon), \tag{5.10}$$

$$\sigma_\varepsilon^{-1} = 1 + 2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi} + O(\varepsilon^{3/2}\ln\varepsilon). \tag{5.11}$$

Proof. From (5.7) it follows that

$$q_\varepsilon = 1 - \frac{2\theta}{1-\theta}\sqrt{\varepsilon_{\theta,\varpi}} + \frac{2\theta}{1-\theta}\varepsilon_\varpi + O(\varepsilon^{3/2}\ln\varepsilon). \tag{5.12}$$

Using (5.1)–(5.3), (5.12), we find

$$\begin{aligned}1 - \sigma_\varepsilon &= 1 - \frac{\theta\alpha_\varepsilon + (1-\theta)\beta_\varepsilon q_\varepsilon}{1 - \theta\beta_\varepsilon - (1-\theta)\alpha_\varepsilon q_\varepsilon} = \frac{(1-\theta)(\alpha_\varepsilon + \beta_\varepsilon)(1 - q_\varepsilon) + (1 - \beta_\varepsilon - \alpha_\varepsilon)}{1 - \theta\beta_\varepsilon - (1-\theta)\alpha_\varepsilon q_\varepsilon} \\ &= \frac{2\theta(\sqrt{\varepsilon_{\theta,\varpi}} - \varepsilon_\varpi) + 2\varepsilon_\varpi + O(\varepsilon^{3/2}\ln\varepsilon)}{\theta + 2\theta\sqrt{\varepsilon_{\theta,\varpi}} + O(\varepsilon)} = \frac{2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi}}{1 + 2\sqrt{\varepsilon_{\theta,\varpi}}}(1 + O(\varepsilon\ln\varepsilon)) \\ &= (2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi})(1 - 2\sqrt{\varepsilon_{\theta,\varpi}})(1 + O(\varepsilon\ln\varepsilon)) = (2\sqrt{\varepsilon_{\theta,\varpi}} - 2\varepsilon_{\theta,\varpi})(1 + O(\varepsilon\ln\varepsilon)).\end{aligned}$$

Formula (5.10) is proved. As a consequence,

$$\sigma_\varepsilon^{-1} - 1 = \frac{1-\sigma_\varepsilon}{\sigma_\varepsilon} = \frac{2\sqrt{\varepsilon_{\theta,\varpi}} - 2\varepsilon_{\theta,\varpi} + O(\varepsilon^{3/2}\ln\varepsilon)}{1-2\sqrt{\varepsilon_{\theta,\varpi}} + O(\varepsilon)} = \frac{2\sqrt{\varepsilon_{\theta,\varpi}} - 2\varepsilon_{\theta,\varpi}}{1-2\sqrt{\varepsilon_{\theta,\varpi}}}(1+O(\varepsilon\ln\varepsilon))$$

$$= (2\sqrt{\varepsilon_{\theta,\varpi}} - 2\varepsilon_{\theta,\varpi})(1+2\sqrt{\varepsilon_{\theta,\varpi}})(1+O(\varepsilon\ln\varepsilon)) = (2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi})(1+O(\varepsilon\ln\varepsilon)).$$

The lemma is proved. □

Lemma 5.5. *The following relations hold*:

$$p_\varepsilon = 1 - 2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi} - 2\varepsilon_\varpi + O(\varepsilon^{3/2}\ln\varepsilon), \tag{5.13}$$

$$r_\varepsilon = 1 - 2\varepsilon_\varpi + O(\varepsilon^{3/2}). \tag{5.14}$$

Proof. From (5.1), (5.2), (5.10) it follows that

$$p_\varepsilon = \alpha_\varepsilon\sigma_\varepsilon + \beta_\varepsilon = (1-(2-\varpi)\varepsilon)(1-2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi}) + \varpi\varepsilon + O(\varepsilon^{3/2}\ln\varepsilon)$$

$$= 1 - 2\sqrt{\varepsilon_{\theta,\varpi}} + 2\varepsilon_{\theta,\varpi} - 2\varepsilon_\varpi + O(\varepsilon^{3/2}\ln\varepsilon),$$

$$r_\varepsilon = \alpha_\varepsilon + \beta_\varepsilon\sigma_\varepsilon = 1-(2-\varpi)\varepsilon + \varpi\varepsilon + O(\varepsilon^{3/2}) = 1 - 2\varepsilon_\varpi + O(\varepsilon^{3/2}).$$

The lemma is proved. □

6 Derivation of Asymptotics of Solution to Problem P^I

We recall that Problem P^I is the problem (1.1)–(1.5) with $J_\ell = 1$, $J_r = 0$, and $F = 0$. We note (cf. Section 4) that for this problem

$$\mathbf{I}_{\varepsilon,j} = \mathbf{z}_{\varepsilon,j} = \begin{pmatrix} z^+_{\varepsilon,j} \\ z^-_{\varepsilon,j} \end{pmatrix}, \quad 1 \leqslant j \leqslant m,$$

is a solution to the system (4.38)–(4.40), i.e.,

$$z^+_{\varepsilon,j} = C_{\varepsilon,\ell} q_\varepsilon^{j-1}(1 - \zeta_{\varepsilon,r}\sigma_\varepsilon q_\varepsilon^{2(m-j)}), \tag{6.1}$$

$$z^-_{\varepsilon,j} = C_{\varepsilon,\ell} q_\varepsilon^{j-1}(\sigma_\varepsilon - \zeta_{\varepsilon,r} q_\varepsilon^{2(m-j)}), \tag{6.2}$$

where

$$C_{\varepsilon,\ell} = \frac{1-\theta_\ell}{(1-\theta_\ell p_\varepsilon)(1-\zeta_{\varepsilon,\ell}\zeta_{\varepsilon,r} q_\varepsilon^{2(m-1)})},$$

whereas the solution I_ε for all $1 \leqslant j \leqslant m$ is defined by

$$I^+_{\varepsilon,j}(\mu,\tau) = z^+_{\varepsilon,j} e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu} S^I_{\varepsilon,j}(\tau')\,d\tau', \quad (\mu,\tau) \in \overline{D}^+_j, \tag{6.3}$$

$$I^-_{\varepsilon,j}(\mu,\tau) = z^-_{\varepsilon,j} e^{-(\tau_j-\tau)/|\mu|} + \frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|} S^I_{\varepsilon,j}(\tau')\,d\tau', \quad (\mu,\tau) \in \overline{D}^-_j. \tag{6.4}$$

where

$$S^I_{\varepsilon,j}(\tau) = \frac{1}{2}\int\limits_{-1}^{1} I_{\varepsilon,j}(\mu,\tau)\,d\mu$$

is a solution to the equation

$$S^I_{\varepsilon,j}(\tau) = \varpi\Lambda_{\varepsilon,j}(S^I_{\varepsilon,j})(\tau) + \frac{1}{2}z^+_{\varepsilon,j}E_2(\tau-\tau_{j-1}) + \frac{1}{2}z^-_{\varepsilon,j}E_2(\tau_j-\tau), \quad \tau\in[\tau_{j-1},\tau_j],$$

and has the form

$$S^I_{\varepsilon,j}(\tau) = \frac{1}{2}z^+_{\varepsilon,j}\Psi^{E_2}_{\varepsilon}(\tau-\tau_{j-1}) + \frac{1}{2}z^-_{\varepsilon,j}\Psi^{E_2}_{\varepsilon}(\tau_j-\tau), \quad \tau\in[\tau_{j-1},\tau_j]. \tag{6.5}$$

6.1. Proof of Theorem 2.2. By (5.9), (5.13), and (5.10),

$$C_{\varepsilon,\ell} = \frac{1}{\Big[1+\dfrac{\theta_\ell}{1-\theta_\ell}(1-p_\varepsilon)\Big](1-\zeta_{\varepsilon,\ell}\zeta_{\varepsilon,r}q_\varepsilon^{2(m-1)})} = \frac{1+O(\varepsilon)}{1+\dfrac{2\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}} = a^+_\ell(\varepsilon)(1+O(\varepsilon)),$$

$$C_{\varepsilon,\ell}\sigma_\varepsilon = \frac{1-2\sqrt{\varepsilon_{\theta,\varpi}}}{1+\dfrac{2\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}}(1+O(\varepsilon)) = \frac{1+O(\varepsilon)}{\Big(1+\dfrac{2\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}\Big)(1+2\sqrt{\varepsilon_{\theta,\varpi}})}$$

$$= \frac{1+O(\varepsilon)}{1+\dfrac{2}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}} = a^-_\ell(\varepsilon)(1+O(\varepsilon)),$$

$$C_{\varepsilon,\ell}\frac{1+\sigma_\varepsilon}{2} = \frac{1-\sqrt{\varepsilon_{\theta,\varpi}}}{1+\dfrac{2\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}}(1+O(\varepsilon)) = \frac{1+O(\varepsilon)}{\Big(1+\dfrac{2\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}\Big)(1+\sqrt{\varepsilon_{\theta,\varpi}})}$$

$$= \frac{1+O(\varepsilon)}{1+\dfrac{1+\theta_\ell}{1-\theta_\ell}\sqrt{\varepsilon_{\theta,\varpi}}} = a^0_\ell(\varepsilon)(1+O(\varepsilon)).$$

From (5.10), (5.13), (5.14) it follows that

$$\zeta_{\varepsilon,r} = \frac{\sigma_\varepsilon-\theta_r r_\varepsilon}{1-\theta_r p_\varepsilon} = \frac{1-\theta_r+\sigma_\varepsilon-1+\theta_r(1-r_\varepsilon)}{1-\theta_r+\theta_r(1-p_\varepsilon)} = \frac{1-\dfrac{1}{1-\theta_r}(1-\sigma_\varepsilon)+\dfrac{\theta_r}{1-\theta_r}(1-r_\varepsilon)}{1+\dfrac{\theta_r}{1-\theta_r}(1-p_\varepsilon)}$$

$$= \frac{1-\dfrac{2}{1-\theta_r}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}-\theta_r\varepsilon_\varpi)}{1+\dfrac{2\theta_r}{1-\theta_r}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}+\varepsilon_\varpi)}(1+O(\varepsilon^{3/2}\ln\varepsilon))$$

$$= \frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{\Big[1+\dfrac{2\theta_r}{1-\theta_r}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}+\varepsilon_\varpi)\Big]\Big[1+\dfrac{2}{1-\theta_r}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}-\theta_r\varepsilon_\varpi)+\dfrac{4}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}\Big]}$$

$$= \frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{1+\dfrac{2(1+\theta_r)}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{2(1+\theta_r)^2}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}} = b^0_r(\varepsilon)(1+O(\varepsilon^{3/2}\ln\varepsilon)). \tag{6.6}$$

From (6.6) and (5.10), (5.11) it follows that

$$\frac{\zeta_{\varepsilon,r}}{\sigma_\varepsilon} = \frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{\Big[1+\dfrac{2(1+\theta_r)}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{2(1+\theta_r)^2}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}\Big](1-2\sqrt{\varepsilon_{\theta,\varpi}}+2\varepsilon_{\theta,\varpi})}$$

$$= \frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{1+\dfrac{4\theta_r}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{8\theta_r^2}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}} = b_r^-(\varepsilon)(1+O(\varepsilon^{3/2}\ln\varepsilon)), \tag{6.7}$$

$$\sigma_\varepsilon\zeta_{\varepsilon,r} = \frac{\zeta_{\varepsilon,r}}{\sigma_\varepsilon^{-1}} = \frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{\Big[1+\dfrac{2(1+\theta_r)}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{2(1+\theta_r)^2}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}\Big](1+2\sqrt{\varepsilon_{\theta,\varpi}}+2\varepsilon_{\theta,\varpi})}$$

$$= \frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{1+\dfrac{4}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}}+\dfrac{8}{(1-\theta_r)^2}\varepsilon_{\theta,\varpi}} = b_r^+(\varepsilon)(1+O(\varepsilon^{3/2}\ln\varepsilon)). \tag{6.8}$$

Using (6.7), (6.8), (5.9), (5.13) and taking into account the relations

$$1-b_r^+(\varepsilon)\sim\frac{4}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}},\quad 1-b_r^-(\varepsilon)\sim\frac{4\theta_r}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}},\quad 1-b_r^0(\varepsilon)\sim\frac{2(1+\theta_r)}{1-\theta_r}\sqrt{\varepsilon_{\theta,\varpi}},\quad \varepsilon\to 0,$$

we get

$$\begin{aligned}
1-\sigma_\varepsilon\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)} &= 1-b_r^+(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon))\\
&= [1-b_r^+(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}](1+O(\varepsilon\ln^2\varepsilon)),\\
1-\frac{\zeta_{\varepsilon,r}}{\sigma_\varepsilon}q_\varepsilon^{2(m-j)} &= 1-b_r^-(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon))\\
&= [1-b_r^-(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}](1+O(\varepsilon\ln^2\varepsilon)),\\
1-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)} &= 1-b_r^0(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon))\\
&= [1-b_r^0(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}](1+O(\varepsilon\ln^2\varepsilon)).
\end{aligned}$$

Thus,

$$\begin{aligned}
z_{\varepsilon,j}^+ &= C_{\varepsilon,\ell}q_\varepsilon^{j-1}(1-\zeta_{\varepsilon,r}\sigma_\varepsilon q_\varepsilon^{2(m-j)}) = a_\ell^+(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}\\
&\times(1-b_r^+(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}})(1+O(\varepsilon\ln^2\varepsilon)) = Z_{\varepsilon,j}^+(1+O(\varepsilon\ln^2\varepsilon)),
\end{aligned}\tag{6.9}$$

$$\begin{aligned}
z_{\varepsilon,j}^- &= C_{\varepsilon,\ell}\sigma_\varepsilon q_\varepsilon^{j-1}\Big(1-\frac{\zeta_{\varepsilon,r}}{\sigma_\varepsilon}q_\varepsilon^{2(m-j)}\Big) = a_\ell^-(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}\\
&\times(1-b_r^-(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}})(1+O(\varepsilon\ln^2\varepsilon)) = Z_{\varepsilon,j}^-(1+O(\varepsilon\ln^2\varepsilon)).
\end{aligned}\tag{6.10}$$

By (3.9), from (6.5), (6.9), (6.10) it follows that

$$\begin{aligned}
S_{\varepsilon,j}^I(\tau) &= \frac{1}{2}(z_{\varepsilon,j}^+ + z_{\varepsilon,j}^-)(1+O(\varepsilon\ln\varepsilon)) = C_{\varepsilon,\ell}\frac{\sigma_\varepsilon+1}{2}q_\varepsilon^{j-1}(1-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)})(1+O(\varepsilon\ln\varepsilon))\\
&= a_\ell^0(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1-b_r^0(\varepsilon)e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}})(1+O(\varepsilon\ln^2\varepsilon)) = Z_{\varepsilon,j}(1+O(\varepsilon\ln^2\varepsilon)).
\end{aligned}\tag{6.11}$$

Now, (2.7), (2.8) follow from (6.3), (6.4) and (6.9), (6.10), whereas (2.9) is a consequence of (6.11). Theorem 2.2 is proved.

Remark 6.1. Since

$$\frac{1}{2}(z_{\varepsilon,j}^{+}+z_{\varepsilon,j}^{-})=C_{\varepsilon,\ell}\frac{1+\sigma_\varepsilon}{2}q_\varepsilon^{j-1}(1-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)})$$
$$=a_\ell^0(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1-\zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)})(1+O(\varepsilon))=a_\ell^0(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}+O(\varepsilon),$$

from (6.5) and (3.7) it follows that

$$S_{\varepsilon,j}^{I}(\tau_{j-1/2})=\frac{1}{2}(z_{\varepsilon,j}^{+}+z_{\varepsilon,j}^{-})\Psi_\varepsilon^{E_2}(\varepsilon/2)=\varkappa(\varepsilon)a_\ell^0(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}+O(\varepsilon),\qquad(6.12)$$

where $\varkappa(\varepsilon)=\varpi+(1-\varpi)E_2(\varepsilon/2)$.

6.2. Proof of Theorem 2.3. IF $\tau_*-\tau_j\geqslant\delta(\varepsilon)=\sqrt{\varepsilon\lambda(\varepsilon)}\ln(1/\varepsilon)$, then

$$\zeta_{\varepsilon,\ell}q_\varepsilon^{2(m-j)}\sim e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}=O(\varepsilon^2),$$
$$\frac{\zeta_{\varepsilon,\ell}}{\sigma_\varepsilon}q_\varepsilon^{2(m-j)}\sim e^{-2(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}=O(\varepsilon^2).$$

Therefore, (5.9), (6.1), and (6.2) imply

$$z_{\varepsilon,j}^{+}=C_{\varepsilon,\ell}e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon)),$$
$$z_{\varepsilon,j}^{-}=C_{\varepsilon,\ell}\sigma_\varepsilon e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon)).$$

Taking into account (5.13) and (5.11), we have

$$C_{\varepsilon,\ell}=\frac{1+O(\varepsilon^2)}{1+\dfrac{\theta_\ell}{1-\theta_\ell}(1-p_\varepsilon)}=\frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{1+\dfrac{2\theta_\ell}{1-\theta_\ell}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}+\varepsilon_\varpi)}=\widetilde{a}_\ell^{+}(\varepsilon)\big(1+O(\varepsilon^{3/2}\ln\varepsilon)\big),$$

$$C_{\varepsilon,\ell}\sigma_\varepsilon=\frac{C_{\varepsilon,\ell}}{\sigma_\varepsilon^{-1}}=\frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{\Big[1+\dfrac{2\theta_\ell}{1-\theta_\ell}(\sqrt{\varepsilon_{\theta,\varpi}}-\varepsilon_{\theta,\varpi}+\varepsilon_\varpi)\Big][1+2\sqrt{\varepsilon_{\theta,\varpi}}+2\varepsilon_{\theta,\varpi}]}$$
$$=\frac{1+O(\varepsilon^{3/2}\ln\varepsilon)}{1+\dfrac{2}{1-\theta_\ell}(\sqrt{\varepsilon_{\theta,\varpi}}+\varepsilon_{\theta,\varpi})+\dfrac{2\theta_\ell}{1-\theta_\ell}\varepsilon_\varpi}=\widetilde{a}_\ell^{-}(\varepsilon)(1+O(\varepsilon^{3/2}\ln\varepsilon)).$$

Thus, in view of (5.9), we have

$$z_{\varepsilon,j}^{+}=\widetilde{a}_\ell^{+}(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon))=\widetilde{Z}_{\varepsilon,j}^{+}(1+O(\varepsilon^{3/2}\ln^2\varepsilon)),\qquad(6.13)$$
$$z_{\varepsilon,j}^{-}=\widetilde{a}_\ell^{-}(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon^{3/2}\ln^2\varepsilon))=\widetilde{Z}_{\varepsilon,j}^{-}(1+O(\varepsilon^{3/2}\ln^2\varepsilon)).\qquad(6.14)$$

Taking into account (3.7) and (3.8), from (6.13), (6.14), and (6.5) we find

$$S_{\varepsilon,j}^{I}(\tau)=\frac{1}{2}z_{\varepsilon,j}^{+}\Psi^{E_2}(\tau-\tau_{j-1})+\frac{1}{2}z_{\varepsilon,j}^{-}\Psi^{E_2}(\tau_j-\tau)=\widetilde{Z}_{\varepsilon,j}(\tau)(1+O(\varepsilon^{3/2}\ln^2\varepsilon)).\qquad(6.15)$$

Now, (2.10) and (2.11) follow from (6.3), (6.4) and (6.13), (6.14), whereas (2.12) is obtained from (6.15). Theorem 2.3 is proved.

6.3. Justification of asymptotics of the solution to Problem P^{II}. We recall that Problem P^{II} is the problem (1.1)–(1.5) with $J_\ell = 0$, $J_r = 1$, and $F = 0$. We note (cf. Section 4) that for this problem

$$\mathbf{I}_{\varepsilon,j} = \mathbf{y}_{\varepsilon,j} = \begin{pmatrix} y^+_{\varepsilon,j} \\ y^-_{\varepsilon,j} \end{pmatrix}, \quad 1 \leqslant j \leqslant m,$$

is a solution to the system (4.35)–(4.37)

$$\begin{aligned} y^+_{\varepsilon,j} &= C_{\varepsilon,r} q_\varepsilon^{m-j}(\sigma_\varepsilon - \zeta_{\varepsilon,\ell} q_\varepsilon^{2(j-1)}), \\ y^-_{\varepsilon,j} &= C_{\varepsilon,r} q_\varepsilon^{m-j}(1 - \zeta_{\varepsilon,\ell}\sigma_\varepsilon q_\varepsilon^{2(j-1)}), \\ C_{\varepsilon,r} &= \frac{1-\theta_r}{(1-\theta_r p_\varepsilon)(1-\zeta_{\varepsilon,\ell}\zeta_{\varepsilon,r} q_\varepsilon^{2(m-1)})}, \end{aligned}$$

whereas the solution I_ε is defined for all $1 \leqslant j \leqslant m$ by

$$I^+_{\varepsilon,j}(\mu,\tau) = y^+_{\varepsilon,j} e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu} S^{II}_{\varepsilon,j}(\tau')\,d\tau', \ (\mu,\tau) \in \overline{D}^+_{\varepsilon,j},$$

$$I^-_{\varepsilon,j}(\mu,\tau) = y^-_{\varepsilon,j} e^{-(\tau_j-\tau)/|\mu|} + \frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|} S^{II}_{\varepsilon,j}(\tau')\,d\tau', \ (\mu,\tau) \in \overline{D}^-_{\varepsilon,j},$$

where

$$S^{II}_{\varepsilon,j}(\tau) = \frac{1}{2} y^+_{\varepsilon,j} \Psi^{E_2}_\varepsilon(\tau - \tau_{j-1}) + \frac{1}{2} y^-_{\varepsilon,j} \Psi^{E_2}_\varepsilon(\tau_j - \tau), \quad \tau \in [\tau_{j-1}, \tau_j].$$

Theorems 2.4 and 2.5 are analogues of Theorems 2.2 and 2.3 and are proved in a similar way. Using the natural symmetry of Problems P^I and P^{II}, it is possible to obtain the asymptotics for Problem P^{II} from the asymptotics of the solution to Problem P^I by replacing τ, θ_ℓ, θ_r, I^+_ε, I^-_ε with $\tau_* - \tau$, θ_r, θ_ℓ, I^-_ε I^+_ε respectively.

Remark 6.2. We write the following analogue of formula (6.12):

$$S^{II}_{\varepsilon,j}(\tau_{j-1/2}) = \varkappa(\varepsilon) a^0_r(\varepsilon) e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}} + O(\varepsilon), \quad 1 \leqslant j \leqslant m. \tag{6.16}$$

7 Justification of Asymptotics of Solution to Problem P^{III}

We recall that Problem P^{III} is the problem (1.1)–(1.5) with $J_\ell = 0$ and $J_r = 0$.

7.1. Asymptotics of the solution to Problem $\overline{P}^{III}$. We set $F_{j-1/2} = F(\tau_{j-1/2})$, $1 \leqslant j \leqslant m$. By Problem $\overline{P}^{III}$ we understand the problem different from Problem P^{III} only by function F replaced with a piecewise constant function $\overline{F}$ taking the values $\overline{F}(\tau) = F_{j-1/2}$ for $\tau \in (\tau_{j-1}, \tau_j)$, $1 \leqslant j \leqslant m$. Let $\overline{I}_\varepsilon$ be a solution to this problem.

The following relations hold (cf. Section 4):

$$\overline{I}^{+}_{\varepsilon,j}(\mu,\tau) = \overline{I}^{+}_{\varepsilon,j,\ell} e^{-(\tau-\tau_{j-1})/\mu} + \frac{1}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}\overline{\Phi}_{\varepsilon,j}(\tau')\,d\tau', \quad (\mu,\tau)\in \overline{D}^{+}_{\varepsilon,j}, \tag{7.1}$$

$$\overline{I}^{-}_{\varepsilon,j}(\mu,\tau) = \overline{I}^{-}_{\varepsilon,j,r} e^{-(\tau_j-\tau)/|\mu|} + \frac{1}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}\overline{\Phi}_{\varepsilon,j}(\tau')\,d\tau', \quad (\mu,\tau)\in \overline{D}^{-}_{\varepsilon,j}, \tag{7.2}$$

where

$$\overline{\Phi}_{\varepsilon,j}(\tau) = \varpi \overline{S}^{III}_{\varepsilon,j}(\tau) + (1-\varpi)F_{j-1/2}, \tag{7.3}$$

and the function

$$\overline{S}^{III}_{\varepsilon,j}(\tau) = \frac{1}{2}\int\limits_{-1}^{1} \overline{I}_{\varepsilon,j}(\mu,\tau)\,d\mu$$

is a solution to the equation

$$\begin{aligned}\overline{S}^{III}_{\varepsilon,j}(\tau) &= \varpi\Lambda_{\varepsilon,j}(\overline{S}^{III}_{\varepsilon,j})(\tau) + \frac{1}{2}\overline{I}^{+}_{\varepsilon,j,\ell}E_2(\tau-\tau_{j-1}) \\ &+ \frac{1}{2}\overline{I}^{-}_{\varepsilon,j,r}E_2(\tau_j-\tau) + (1-\varpi)\Lambda_{\varepsilon,j}(F_{j-1/2})(\tau).\end{aligned} \tag{7.4}$$

Furthermore,

$$\overline{\mathbf{I}}_{\varepsilon,j} = \begin{pmatrix}\overline{I}^{+}_{\varepsilon,j,\ell}\\ \overline{I}^{-}_{\varepsilon,j,r}\end{pmatrix}, \quad 1\leqslant j\leqslant m,$$

is a solution to the problem (4.22)–(4.24) with $J_\ell = 0$, $J_r = 0$, $F = \overline{F}$

$$\begin{aligned}&\mathbf{A}_\varepsilon \overline{\mathbf{I}}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \overline{\mathbf{I}}_{\varepsilon,j} = \overline{\mathbf{f}}_{\varepsilon,j}, \quad 1\leqslant j< m,\\ &\mathbf{A}_{\varepsilon,\ell}\overline{\mathbf{I}}_{\varepsilon,1} = \theta_\ell \overline{f}_{\varepsilon,1,\ell},\\ &\mathbf{B}_{\varepsilon,r}\overline{\mathbf{I}}_{\varepsilon,m} = \theta_r \overline{f}_{\varepsilon,m,r},\end{aligned}$$

where

$$\overline{\mathbf{f}}_{\varepsilon,j} = \begin{pmatrix}(1-\theta)\overline{f}_{\varepsilon,j,r} + \theta\overline{f}_{\varepsilon,j+1,\ell}\\ \theta\overline{f}_{\varepsilon,j,r} + (1-\theta)\overline{f}_{\varepsilon,j+1,\ell}\end{pmatrix}, \quad 1\leqslant j<m,$$

$$\overline{f}_{\varepsilon,j,r} = 2(1-\varpi)\int\limits_{\tau_{j-1}}^{\tau_j} \Psi^{\overline{F}}_{\varepsilon,j}(\tau)E_2(\tau_j-\tau)\,d\tau, \quad 1\leqslant j\leqslant m,$$

$$\overline{f}_{\varepsilon,j+1,\ell} = 2(1-\varpi)\int\limits_{\tau_j}^{\tau_{j+1}} \Psi^{\overline{F}}_{\varepsilon,j+1}(\tau')E_2(\tau-\tau_j)\,d\tau, \quad 0\leqslant j<m.$$

Using (4.15) and (4.16), we find

$$\overline{f}_{\varepsilon,j,r} = 2(1-\varpi)F_{j-1/2}\int\limits_0^{\varepsilon}\Psi^1_{\varepsilon}(\tau)E_2(\varepsilon-\tau)\,d\tau = (1-\alpha_\varepsilon-\beta_\varepsilon)F_{j-1/2}, \quad 1\leqslant j\leqslant m,$$

$$\overline{f}_{\varepsilon,j+1,\ell} = 2(1-\varpi)F_{j+1/2}\int\limits_0^{\varepsilon}\Psi^1_{\varepsilon}(\tau)E_2(\tau)\,d\tau = (1-\alpha_\varepsilon-\beta_\varepsilon)F_{j+1/2}, \quad 0\leqslant j< m,$$

$$\overline{\mathbf{f}}_{\varepsilon,j} = (1-\alpha_\varepsilon-\beta_\varepsilon)\begin{pmatrix}(1-\theta)F_{j-1/2}+\theta F_{j+1/2}\\ \theta F_{j-1/2}+(1-\theta)F_{j+1/2}\end{pmatrix}, \quad 1\leqslant j< m.$$

Lemma 7.1. *For all* $1\leqslant j\leqslant m$

$$\overline{\mathbf{I}}_{\varepsilon,j} = \widetilde{\mathbf{I}}_{\varepsilon,j} + \|F'\|_{C^1[0,\tau_*]}O(\varepsilon), \tag{7.5}$$

where $\widetilde{\mathbf{I}}_{\varepsilon,j} = F_{j-1/2}\mathbf{1} - F_{1/2}\mathbf{z}_{\varepsilon,j} - F_{m-1/2}\mathbf{y}_{\varepsilon,j}$, $1\leqslant j\leqslant m$.

Proof. Taking into account that

$$\mathbf{A}_\varepsilon\mathbf{z}_{\varepsilon,j+1}+\mathbf{B}_\varepsilon\mathbf{z}_{\varepsilon,j}=\mathbf{0}, \quad \mathbf{A}_\varepsilon\mathbf{y}_{\varepsilon,j+1}+\mathbf{B}_\varepsilon\mathbf{y}_{\varepsilon,j}=\mathbf{0}, \quad 1\leqslant j<m,$$

$$\mathbf{A}_{\varepsilon,\ell}\mathbf{z}_{\varepsilon,1}=1-\theta_\ell, \quad \mathbf{A}_{\varepsilon,\ell}\mathbf{y}_{\varepsilon,1}=0, \quad \mathbf{B}_{\varepsilon,r}\mathbf{z}_{\varepsilon,m}=0, \quad \mathbf{B}_{\varepsilon,r}\mathbf{y}_{\varepsilon,m}=1-\theta_r,$$

we have

$$\begin{aligned}
&\mathbf{A}_\varepsilon\widetilde{\mathbf{I}}_{\varepsilon,j+1}+\mathbf{B}_\varepsilon\widetilde{\mathbf{I}}_{\varepsilon,j} = F_{j+1/2}\mathbf{A}_\varepsilon\mathbf{1}+F_{j-1/2}\mathbf{B}_\varepsilon\mathbf{1}\\
&\quad = (1-\theta\alpha_\varepsilon-\theta\beta_\varepsilon)[F_{j+1/2}\mathbf{e}_1+F_{j-1/2}\mathbf{e}_2] - (1-\theta)(\alpha_\varepsilon+\beta_\varepsilon)[F_{j+1/2}\mathbf{e}_2+F_{j-1/2}\mathbf{e}_1]\\
&\quad = (1-\alpha_\varepsilon-\beta_\varepsilon)[((1-\theta)F_{j-1/2}+\theta F_{j+1/2})\mathbf{e}_1+(\theta F_{j-1/2}+(1-\theta)F_{j+1/2})\mathbf{e}_2]\\
&\quad + (1-\theta)(F_{j+1/2}-F_{j-1/2})(\mathbf{e}_1-\mathbf{e}_2), \quad 1\leqslant j<m,\\
&\mathbf{A}_{\varepsilon,\ell}\widetilde{\mathbf{I}}_{\varepsilon,1} = (1-\theta_\ell\alpha_\varepsilon-\theta_\ell\beta_\varepsilon)F_{1/2}-(1-\theta_\ell)F_{1/2} = (1-\alpha_\varepsilon-\beta_\varepsilon)\theta_\ell F_{1/2},\\
&\mathbf{B}_{\varepsilon,r}\widetilde{\mathbf{I}}_{\varepsilon,m} = (1-\theta_r\alpha_\varepsilon-\theta_r\beta_\varepsilon)F_{m-1/2}-(1-\theta_r)F_{m-1/2} = (1-\alpha_\varepsilon-\beta_\varepsilon)\theta_r F_{m-1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{A}_\varepsilon(\widetilde{\mathbf{I}}_{\varepsilon,j+1}-\overline{\mathbf{I}}_{\varepsilon,j+1})+\mathbf{B}_\varepsilon(\widetilde{\mathbf{I}}_{\varepsilon,j}-\overline{\mathbf{I}}_{\varepsilon,j}) = (1-\theta)(F_{j+1/2}-F_{j-1/2})(\mathbf{e}_1-\mathbf{e}_2), \quad 1\leqslant j<m,\\
&\mathbf{A}_{\varepsilon,\ell}(\widetilde{\mathbf{I}}_{\varepsilon,1}-\overline{\mathbf{I}}_{\varepsilon,1})=0,\\
&\mathbf{B}_{\varepsilon,r}(\widetilde{\mathbf{I}}_{\varepsilon,m}-\overline{\mathbf{I}}_{\varepsilon,m})=0.
\end{aligned}$$

Consequently,

$$\widetilde{\mathbf{I}}_{\varepsilon,j}-\overline{\mathbf{I}}_{\varepsilon,j} = (1-\theta)\sum_{i=1}^{m-1}(F_{i+1/2}-F_{i-1/2})(\mathbf{g}^{i,1}_{\varepsilon,j}-\mathbf{g}^{i,2}_{\varepsilon,j}), \quad 1\leqslant j\leqslant m. \tag{7.6}$$

We set

$$\mathbf{w}_{\varepsilon,j} = \mathbf{x}_\varepsilon - (\mathbf{A}_{\varepsilon,\ell}\mathbf{x}_\varepsilon)\frac{1}{1-\theta_\ell}\mathbf{z}_{\varepsilon,j} - (\mathbf{B}_{\varepsilon,r}\mathbf{x}_\varepsilon)\frac{1}{1-\theta_r}\mathbf{y}_{\varepsilon,j}, \quad 1 \leqslant j \leqslant m, \tag{7.7}$$

where

$$\mathbf{x}_\varepsilon = \frac{1}{1-(1-2\theta)(\alpha_\varepsilon - \beta_\varepsilon)}(\mathbf{e}_1 - \mathbf{e}_2), \tag{7.8}$$

Since

$$\begin{aligned}
&\mathbf{A}_\varepsilon \mathbf{w}_{\varepsilon,j+1} + \mathbf{B}_\varepsilon \mathbf{w}_{\varepsilon,j} = \mathbf{e}_1 - \mathbf{e}_2, \quad 1 \leqslant j < m,\\
&\mathbf{A}_{\varepsilon,\ell}\mathbf{w}_{\varepsilon,1} = \mathbf{0},\\
&\mathbf{B}_{\varepsilon,r}\mathbf{w}_{\varepsilon,m} = \mathbf{0},
\end{aligned}$$

we get

$$\mathbf{w}_{\varepsilon,j} = \sum_{i=1}^{m-1}(\mathbf{g}^{i,1}_{\varepsilon,j} - \mathbf{g}^{i,2}_{\varepsilon,j}), \quad 1 \leqslant j \leqslant m.$$

Thus, (7.6) implies

$$\begin{aligned}
\widetilde{\mathbf{I}}_{\varepsilon,j} - \overline{\mathbf{I}}_{\varepsilon,j} &= (1-\theta)(F_{j+1/2} - F_{j-1/2})\mathbf{w}_{\varepsilon,j}\\
&+ (1-\theta)\sum_{i=1}^{m-1}[F_{i+1/2} - F_{i-1/2} - F_{j+1/2} + F_{j-1/2}](\mathbf{g}^{i,1}_{\varepsilon,j} - \mathbf{g}^{i,2}_{\varepsilon,j}), \quad 1 \leqslant j \leqslant m.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&|F_{j+1/2} - F_{j-1/2}| \leqslant \|F'\|_{C[0,\tau_*]}\varepsilon,\\
&|F_{i+1/2} - F_{i-1/2} - F_{j+1/2} + F_{j-1/2}| \leqslant \|F''\|_{C[0,\tau_*]}|i-j|\varepsilon^2,
\end{aligned}$$

we arrive at the inequality

$$|\widetilde{\mathbf{I}}_{\varepsilon,j} - \overline{\mathbf{I}}_{\varepsilon,j}| \leqslant \|F'\|_{C[0,\tau_*]}\varepsilon|\mathbf{w}_{\varepsilon,j}| + \|F''\|_{C[0,\tau_*]}\varepsilon^2\sum_{i=1}^{m-1}|i-j||\mathbf{g}^{i,1}_{\varepsilon,j} - \mathbf{g}^{i,2}_{\varepsilon,j}|, \quad 1 \leqslant j \leqslant m. \tag{7.9}$$

It is easy to see that (7.7) and (7.8) imply

$$\max_{1\leqslant j\leqslant m}|\mathbf{w}_{\varepsilon,j}| = O(1). \tag{7.10}$$

From (4.52) and (4.53) it follows that

$$\mathbf{g}^{i,1}_{\varepsilon,j} - \mathbf{g}^{i,2}_{\varepsilon,j} = \frac{1-p_\varepsilon}{\Delta_\varepsilon}\begin{cases} -[1+\zeta_{\varepsilon,r}q_\varepsilon^{2(m-i)-1}]q_\varepsilon^{i-j}(\mathbf{v}_{\varepsilon,2} - \zeta_{\varepsilon,\ell}q_\varepsilon^{2(j-1)}\mathbf{v}_{\varepsilon,1}), & 1 \leqslant j \leqslant i < m,\\ [1+\zeta_{\varepsilon,\ell}q_\varepsilon^{2i-1}]q_\varepsilon^{j-i-1}(\mathbf{v}_{\varepsilon,1} - \zeta_{\varepsilon,r}q_\varepsilon^{2(m-j)}\mathbf{v}_{\varepsilon,2}), & 1 \leqslant i < j \leqslant m.\end{cases}$$

Taking into account (4.51), we have

$$\frac{1-p_\varepsilon}{\Delta_\varepsilon} = \frac{1}{(1-\zeta_{\varepsilon,\ell}\zeta_{\varepsilon,r}q_\varepsilon^{2(m-1)})(1-\theta p_\varepsilon + (1-\theta)p_\varepsilon)} \sim \frac{1}{2(1-\theta)}.$$

Taking also into account that $1-q_\varepsilon \sim \sqrt{\varepsilon/\lambda_0}$, we have

$$\sum_{i=1}^{m-1}|i-j||\mathbf{g}^{i,1}_{\varepsilon,j}-\mathbf{g}^{i,2}_{\varepsilon,j}| = O(1)\Bigg[\sum_{i=1}^{j-1}|i-j|q_\varepsilon^{j-i-1}+\sum_{i=j+1}^{m-1}|i-j|q_\varepsilon^{i-j}\Bigg]$$

$$= O(1)\sum_{k=1}^{\infty}kq_\varepsilon^k = O(1)\frac{1}{(1-q_\varepsilon)^2} = O(\varepsilon^{-1}). \tag{7.11}$$

Now, from (7.9)–(7.11) we obtain the estimate

$$\max_{1\leqslant j\leqslant m}|\widetilde{\mathbf{I}}_{\varepsilon,j}-\overline{\mathbf{I}}_{\varepsilon,j}| = \|F'\|_{C^1[0,\tau_*]}O(\varepsilon).$$

The lemma is proved. □

Lemma 7.2. *For all* $1\leqslant j\leqslant m$

$$\overline{S}^{III}_{\varepsilon,j}(\tau) = F_{j-1/2}-F_{1/2}S^{I}_{\varepsilon,j}(\tau)-F_{m-1/2}S^{II}_{\varepsilon,j}(\tau)+\|F'\|_{C^1[0,\tau_*]}O(\varepsilon),\quad \tau\in(\tau_{j-1},\tau_j), \tag{7.12}$$

where

$$S^{I}_{\varepsilon,j}(\tau) = \frac{1}{2}z^{+}_{\varepsilon,j}\Psi^{E_2}(\tau-\tau_{j-1})+\frac{1}{2}z^{-}_{\varepsilon,j}\Psi^{E_2}(\tau_j-\tau),$$

$$S^{II}_{\varepsilon,j}(\tau) = \frac{1}{2}y^{+}_{\varepsilon,j}\Psi^{E_2}(\tau-\tau_{j-1})+\frac{1}{2}y^{-}_{\varepsilon,j}\Psi^{E_2}(\tau_j-\tau).$$

Proof. From (7.4) and the relation

$$1 = \varpi\Lambda_{\varepsilon,j}(1)(\tau)+\frac{1}{2}E_2(\tau-\tau_{j-1})+\frac{1}{2}E_2(\tau_j-\tau)+(1-\varpi)\Lambda_{\varepsilon,j}(1)(\tau)$$

it follows that

$$\overline{S}^{III}_{\varepsilon,j}(\tau)-F_{j-1/2} = \varpi\Lambda_{\varepsilon,j}(\overline{S}^{III}_{\varepsilon,j}-F_{j-1/2})(\tau)$$
$$+\frac{1}{2}(\overline{I}^{+}_{\varepsilon,j,\ell}-F_{j-1/2})E_2(\tau-\tau_{j-1})+\frac{1}{2}(\overline{I}^{-}_{\varepsilon,j,r}-F_{j-1/2})E_2(\tau_j-\tau).$$

Therefore,

$$\overline{S}^{III}_{\varepsilon,j}(\tau)-F_{j-1/2} = \frac{1}{2}(\overline{I}^{+}_{\varepsilon,j,\ell}-F_{j-1/2})\Psi^{E_2}(\tau-\tau_{j-1})+\frac{1}{2}(\overline{I}^{-}_{\varepsilon,j,r}-F_{j-1/2})\Psi^{E_2}(\tau_j-\tau).$$

Using formula (7.5), we arrive at (7.12). □

Lemma 7.3. *If* $\tau_{j-1}<\delta(\varepsilon)$, *then*

$$\overline{I}^{+}_{\varepsilon,j}(\mu,\tau) = F_{j-1/2}-F_{1/2}\Bigg[z^{+}_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu}+\frac{\varpi}{\mu}\int_{\tau_{j-1}}^{\tau}e^{-(\tau-\tau')/\mu}S^{I}_{\varepsilon,j}(\tau')\,d\tau'\Bigg]$$
$$+\|F\|_{C^2[0,\tau_*]}\varepsilon,\quad (\mu,\tau)\in\overline{D}^{+}_{\varepsilon,j}, \tag{7.13}$$

$$\overline{I}^{-}_{\varepsilon,j}(\mu,\tau) = F_{j-1/2}-F_{1/2}\Bigg[z^{-}_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|}+\frac{\varpi}{|\mu|}\int_{\tau}^{\tau_j}e^{-(\tau'-\tau)/|\mu|}S^{I}_{\varepsilon,j}(\tau')\,d\tau'\Bigg]$$
$$+\|F\|_{C^2[0,\tau_*]}\varepsilon,\quad (\mu,\tau)\in\overline{D}^{-}_{\varepsilon,j} \tag{7.14}$$

$$\overline{S}^{III}_{\varepsilon,j}(\tau) = F_{j-1/2}-F_{1/2}S^{I}_{\varepsilon,j}(\tau)+\|F'\|_{C^1[0,\tau_*]}O(\varepsilon),\quad \tau\in(\tau_{j-1},\tau_j). \tag{7.15}$$

If $\tau_j > \tau_* - \delta(\varepsilon)$, *then*

$$\overline{I}^{+}_{\varepsilon,j}(\mu,\tau) = F_{j-1/2} - F_{m-1/2}\left[y^{+}_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}S^{II}_{\varepsilon,j}(\tau')\,d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}\varepsilon, \quad (\mu,\tau) \in \overline{D}^{+}_{\varepsilon,j}, \tag{7.16}$$

$$\overline{I}^{-}_{\varepsilon,j}(\mu,\tau) = F_{j-1/2} - F_{m-1/2}\left[y^{-}_{\varepsilon,j}e^{-(\tau_j-\tau)/|\mu|} + \frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}S^{II}_{\varepsilon,j}(\tau')\,d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}\varepsilon, \quad (\mu,\tau) \in \overline{D}^{-}_{\varepsilon,j}, \tag{7.17}$$

$$\overline{S}^{III}_{\varepsilon,j}(\tau) = F_{j-1/2} - F_{m-1/2}S^{II}_{\varepsilon,j}(\tau) + \|F'\|_{C^1[0,\tau_*]}O(\varepsilon), \quad \tau \in (\tau_{j-1},\tau_j), \tag{7.18}$$

If $\delta(\varepsilon) \leqslant \tau_{j-1} < \tau_j \leqslant \tau_* - \delta(\varepsilon)$, *then*

$$\overline{I}_{\varepsilon,j}(\mu,\tau) = F_{j-1/2} + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}_{\varepsilon,j}. \tag{7.19}$$

$$\overline{S}^{III}_{\varepsilon,j}(\tau) = F_{j-1/2} + \|F'\|_{C^1[0,\tau_*]}O(\varepsilon), \quad \tau \in (\tau_{j-1},\tau_j). \tag{7.20}$$

Proof. If $\tau_{j-1} < \delta(\varepsilon)$, then from (7.5), (7.12), (7.3) and the relations

$$\mathbf{y}_{\varepsilon,j} = O\big(e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}\big) = O(\varepsilon),$$

we obtain (7.15) and

$$\overline{\mathbf{I}}_{\varepsilon,j} = F_{j-1/2}\mathbf{1} - F_{1/2}\mathbf{z}_{\varepsilon,j} + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \tag{7.21}$$

$$\overline{\Phi}_{\varepsilon,j}(\tau) = F_{j-1/2} - F_{1/2}\varpi S^{I}_{\varepsilon,j}(\tau) + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad \tau \in [\tau_{j-1},\tau_j]. \tag{7.22}$$

Substituting (7.21), (7.22) into (7.1) and (7.2), we find

$$\overline{I}^{+}_{\varepsilon,j}(\mu,\tau) = [F_{j-1/2} - F_{1/2}z^{+}_{\varepsilon,j}]e^{-(\tau-\tau_{j-1})/\mu} + \frac{1}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}[F_{j-1/2} - F_{1/2}\varpi S^{I}_{\varepsilon,j}(\tau')]\,d\tau'$$

$$+ \|F\|_{C^2[0,\tau_*]}O(\varepsilon) = F_{j-1/2} - F_{1/2}\left[z^{+}_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}S^{I}_{\varepsilon,j}(\tau')\,d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}^{+}_{\varepsilon,j},$$

$$\overline{I}^{-}_{\varepsilon,j}(\mu,\tau) = [F_{j-1/2} - F_{1/2}z^{-}_{\varepsilon,j}]e^{-(\tau_j-\tau)/|\mu|} + \frac{1}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}\big[F_{j-1/2} - F_{1/2}\varpi S^{I}_{\varepsilon,j}(\tau')\big]\,d\tau$$

$$+ \|F\|_{C^2[0,\tau_*]}O(\varepsilon) = F_{j-1/2} - F_{1/2}\left[z^{-}_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/\mu}S^{I}_{\varepsilon,j}(\tau')\,d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}^{-}_{\varepsilon,j}.$$

Thus, we have proved (7.13)–(7.15). In the case $\tau_j > \tau_* - \delta(\varepsilon)$, formulas (7.16)–(7.18) are proved in a similar way.

If $\delta(\varepsilon) \leqslant \tau_{j-1} < \tau_j \leqslant \tau_* - \delta(\varepsilon)$, then (7.5), (7.12), and (7.3) imply (7.20) and

$$\begin{aligned} &\mathbf{I}_{\varepsilon,j} = F_{j-1/2}\mathbf{1} + \|F\|_{C^2[0,\tau_*]}O(\varepsilon),\\ &\overline{\Phi}_{\varepsilon,j}(\tau) = F_{j-1/2} + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad \tau \in [\tau_{j-1}, \tau_j]. \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{I}^{+}_{\varepsilon,j}(\mu,\tau) &= F_{j-1/2}e^{-(\tau-\tau_{j-1})/\mu} + \frac{1}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}F_{j-1/2}\,d\tau' + \|F\|_{C^2[0,\tau_*]}O(\varepsilon)\\ &= F_{j-1/2} + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}^{+}_{\varepsilon,j},\\ \overline{I}^{-}_{\varepsilon,j}(\mu,\tau) &= F_{j-1/2}e^{-(\tau_j-\tau)/|\mu|} + \frac{1}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|}F_{j-1/2}\,d\tau' + \|F\|_{C^2[0,\tau_*]}O(\varepsilon)\\ &= F_{j-1/2} + \|F\|_{C^2[0,\tau_*]}O(\varepsilon), \quad (\mu,\tau) \in \overline{D}^{-}_{\varepsilon,j}. \end{aligned}$$

The lemma is proved. □

7.2. Proof of Theorem 2.6 (continued). Let I_ε be a solution to Problem P^{III}. We note that formulas (4.2), (4.3), (7.1), (7.2) imply $I_{\varepsilon,j} - \overline{I}_{\varepsilon,j} \in C(\overline{D}_{\varepsilon,j})$, $1 \leqslant j \leqslant m$. Consequently, formula (2.1) implies

$$\max_{1\leqslant j\leqslant m} \sup_{(\mu,\tau)\in\overline{D}_{\varepsilon,j}} |I_{\varepsilon,j}(\mu,\tau) - \overline{I}_{\varepsilon,j}(\mu,\tau)| \leqslant \|F - \overline{F}\|_{L^\infty(0,\tau_*)} \leqslant \|F'\|_{C[0,\tau_*]}\varepsilon, \tag{7.23}$$

$$\max_{1\leqslant j\leqslant m} \|S^{III}_{\varepsilon,j} - \overline{S}^{III}_{\varepsilon,j}\|_{C[\tau_{j-1},\tau_j]} \leqslant \|F'\|_{C[0,\tau_*]}\varepsilon, \tag{7.24}$$

where

$$S^{III}_{\varepsilon,j}(\tau) = \frac{1}{2}\int\limits_{-1}^{1} I_{\varepsilon,j}(\mu,\tau)\,d\mu.$$

Let $\tau_{j-1} < \delta(\varepsilon)$. Using the estimate (7.23) and relations

$$\begin{aligned} &|F_{j-1/2} - F(\tau)| \leqslant \|F'\|_{C[0,\tau_*]}\varepsilon, \quad \tau \in [\tau_{j-1}, \tau_j],\\ &|F_{1/2} - F(0)| \leqslant \|F'\|_{C[0,\tau_*]}\varepsilon, \end{aligned}$$

from (7.13) and (7.14) we get

$$\begin{aligned} I^{+}_{\varepsilon,j}(\mu,\tau) = F(\tau) - F(0)\Bigg[z^{+}_{\varepsilon,j}e^{-(\tau-\tau_{j-1})/\mu} + \frac{\varpi}{\mu}\int\limits_{\tau_{j-1}}^{\tau} e^{-(\tau-\tau')/\mu}S^{I}_{\varepsilon,j}(\tau')\,d\tau'\Bigg]&\\ + \|F\|_{C^2[0,\tau_*]}\varepsilon, \quad (\mu,\tau) \in \overline{D}^{+}_{\varepsilon,j},& \end{aligned} \tag{7.25}$$

$$I^-_{\varepsilon,j}(\mu,\tau) = F(\tau) - F(0)\left[z^-_{\varepsilon,j} e^{-(\tau_j-\tau)/|\mu|} + \frac{\varpi}{|\mu|}\int\limits_{\tau}^{\tau_j} e^{-(\tau'-\tau)/|\mu|} S^I_{\varepsilon,j}(\tau')\, d\tau'\right]$$

$$+ \|F\|_{C^2[0,\tau_*]}\varepsilon, \quad (\mu,\tau) \in \overline{D}^-_{\varepsilon,j}. \tag{7.26}$$

We note that $\widetilde{a}^+_\ell(\varepsilon) = a^+_\ell(\varepsilon)(1+O(\varepsilon))$ and $\widetilde{a}^-_\ell(\varepsilon) = a^-_\ell(\varepsilon)(1+O(\varepsilon))$ and (6.13)–(6.15) imply

$$\begin{aligned} z^+_{\varepsilon,j} &= a^+_\ell(\varepsilon) e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon)) = \widehat{Z}^+_{\varepsilon,j}(1+O(\varepsilon)),\\ z^-_{\varepsilon,j} &= a^-_\ell(\varepsilon) e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}(1+O(\varepsilon)) = \widehat{Z}^-_{\varepsilon,j}(1+O(\varepsilon)),\\ S^I_{\varepsilon,j}(\tau) &= \widehat{Z}_{\varepsilon,j}(\tau)(1+O(\varepsilon)). \end{aligned} \tag{7.27}$$

Substituting these expressions into (7.25) and (7.26), we find (2.13) and (2.14). Now, (2.15) follows from (7.27).

Similarly, in the case $\tau_j > \tau_* - \delta(\varepsilon)$, (7.16)–(7.18) imply (2.16)–(2.18) and, in the case $\delta(\varepsilon) \leqslant \tau_{j-1} < \tau_j \leqslant \tau_* - \delta(\varepsilon)$, (7.19) and (7.20) imply (2.19) and (2.20). Theorem 2.6 is proved.

Remark 7.1. From (7.12) it follows that

$$S^{III}_{\varepsilon,j}(\tau_{j-1/2}) = F_{j-1/2} - F_{1/2} S^I_{\varepsilon,j}(\tau_{j-1/2}) - F_{m-1/2} S^{II}_{\varepsilon,j}(\tau_{j-1/2})$$

$$+ \|F\|_{C^2[0,\tau_*]} O(\varepsilon), \quad 1 \leqslant j \leqslant m. \tag{7.28}$$

8 Justification of Diffuse Approximation

We prove Theorem 2.7. Let u_ε be a solution to the problem

$$-\varepsilon\lambda(\varepsilon)\frac{d^2}{d\tau^2}u_\varepsilon + u_\varepsilon = F, \quad \tau \in (\varepsilon/2, \tau_* - \varepsilon/2) \tag{8.1}$$

$$B_{\varepsilon,\ell}[u_\varepsilon](\varepsilon/2) = \frac{2a^0_\ell(\varepsilon)}{1-b^0_\ell(\varepsilon)} J_\ell, \quad B_{\varepsilon,r}[u_\varepsilon](\tau_* - \varepsilon/2) = \frac{2a^0_r(\varepsilon)}{1-b^0_r(\varepsilon)} J_r, \tag{8.2}$$

where

$$B_{\varepsilon,\ell}[u_\varepsilon] = -\sqrt{\varepsilon\lambda(\varepsilon)}\frac{d}{d\tau}u_\varepsilon + \frac{1+b^0_\ell(\varepsilon)}{1-b^0_\ell(\varepsilon)}u_\varepsilon,$$

$$B_{\varepsilon,r}[u_\varepsilon] = \sqrt{\varepsilon\lambda(\varepsilon)}\frac{d}{d\tau}u_\varepsilon + \frac{1+b^0_r(\varepsilon)}{1-b^0_r(\varepsilon)}u_\varepsilon.$$

We set $M = \max\big\{\|F\|_{C[\varepsilon/2,\tau_*-\varepsilon/2]}, |J_\ell|, |J_r|\big\}$.

Lemma 8.1. *The solution to the problem* (8.1), (8.2) *satisfies the estimate*

$$\|u_\varepsilon\|_{C[\varepsilon/2,\tau_*-\varepsilon/2]} \leqslant M. \tag{8.3}$$

Proof. Let $\tau_{\max}$ be a positive maximum point of the function u_ε. If $\tau_* \in (\varepsilon/2, \tau_* - \varepsilon/2)$, then

$$\frac{d^2}{d\tau^2}u_\varepsilon(\tau_{\max}) \leqslant 0,$$

$$u_\varepsilon(\tau_{\max}) \leqslant -\frac{d^2}{d\tau^2}u_\varepsilon(\tau_{\max}) + u_\varepsilon(\tau_{\max}) = F(\tau_{\max}) \leqslant \|F\|_{C[\varepsilon/2,\tau_*-\varepsilon/2]}.$$

If $\tau_{\max} = \varepsilon/2$, then

$$\frac{d}{d\tau} u_\varepsilon(\varepsilon/2) \leqslant 0,$$
$$\frac{1+b_\ell^0(\varepsilon)}{1-b_\ell^0(\varepsilon)} u_\varepsilon(\varepsilon/2) \leqslant B_{\varepsilon,\ell}[u_\varepsilon](\varepsilon/2) = \frac{2a_\ell^0(\varepsilon)}{1-b_\ell^0(\varepsilon)} J_\ell.$$

Taking into account that $b_\ell^0(\varepsilon) < 1$ and $2a_\ell^0(\varepsilon) < 1 + b_\ell^0(\varepsilon)$, we have $u_\varepsilon(\varepsilon/2) \leqslant |J_\ell|$.

If $\tau_{\max} = \tau_* - \varepsilon/2$, then

$$\frac{d}{d\tau} u_\varepsilon(\tau_* - \varepsilon/2) \geqslant 0,$$
$$\frac{1+b_r^0(\varepsilon)}{1-b_r^0(\varepsilon)} u_\varepsilon(\tau_* - \varepsilon/2) \leqslant B_{\varepsilon,r}[u_\varepsilon](\tau_* - \varepsilon/2) = \frac{2a_r^0(\varepsilon)}{1-b_r^0(\varepsilon)} J_r,$$

which implies $u_\varepsilon(\tau_* - \varepsilon/2) \leqslant |J_r|$. Hence $u_\varepsilon(\tau_{\max}) \leqslant M$.

Arguing in a similar way, we see that the inequality $u_\varepsilon(\tau_{\min}) \geqslant -M$ holds at the negative maximum point $\tau_{\min}$. □

We represent the solution to the problem (8.1, (8.2) as

$$u_\varepsilon = J_\ell y_\varepsilon + J_r z_\varepsilon + v_\varepsilon, \tag{8.4}$$

where z_ε is a solution to the problem

$$-\varepsilon\lambda(\varepsilon) \frac{d^2}{d\tau^2} z_\varepsilon + z_\varepsilon = 0, \quad \tau \in (\varepsilon/2, \tau_* - \varepsilon/2),$$
$$B_{\varepsilon,\ell}[z_\varepsilon](\varepsilon/2) = \frac{2a_\ell^0(\varepsilon)}{1-b_\ell^0(\varepsilon)}, \quad B_{\varepsilon,r}[z_\varepsilon](\tau_* - \varepsilon/2) = 0,$$

y_ε is a solution to the problem

$$-\varepsilon\lambda(\varepsilon) \frac{d^2}{d\tau^2} y_\varepsilon + y_\varepsilon = 0, \quad \tau \in (\varepsilon/2, \tau_* - \varepsilon/2),$$
$$B_{\varepsilon,\ell}[y_\varepsilon](\varepsilon/2) = 0, \quad B_{\varepsilon,r}[y_\varepsilon](\tau_* - \varepsilon/2) = \frac{2a_r^0(\varepsilon)}{1-b_r^0(\varepsilon)},$$

and v_ε is a solution to the problem

$$-\varepsilon\lambda(\varepsilon) \frac{d^2}{d\tau^2} v_\varepsilon + v_\varepsilon = F, \quad \tau \in (\varepsilon/2, \tau_* - \varepsilon/2), \tag{8.5}$$
$$B_{\varepsilon,\ell}[v_\varepsilon](\varepsilon/2) = 0, \quad B_{\varepsilon,r}[v_\varepsilon](\tau_* - \varepsilon/2) = 0. \tag{8.6}$$

It is easy to see that

$$\begin{aligned} z_\varepsilon(\tau) &= \frac{a_\ell^0(\varepsilon) e^{-(\tau-\varepsilon/2)/\sqrt{\varepsilon\lambda(\varepsilon)}}}{1 - b_\ell^0(\varepsilon) b_r^0(\varepsilon) e^{-2(\tau_*-\varepsilon)/\sqrt{\varepsilon\lambda(\varepsilon)}}} [1 - b_r^0(\varepsilon) e^{-2(\tau_*-\varepsilon/2-\tau)/\sqrt{\varepsilon\lambda(\varepsilon)}}] \\ &= a_\ell^0(\varepsilon) e^{-(\tau-\varepsilon/2)/\sqrt{\varepsilon\lambda(\varepsilon)}} + O(\varepsilon), \end{aligned} \tag{8.7}$$

$$
\begin{aligned}
y_\varepsilon(\tau) &= \frac{a_r^0(\varepsilon)e^{-(\tau_*-\varepsilon/2-\tau)/\sqrt{\varepsilon\lambda(\varepsilon)}}}{1-b_\ell^0(\varepsilon)b_r^0(\varepsilon)e^{-2(\tau_*-\varepsilon)/\sqrt{\varepsilon\lambda(\varepsilon)}}}[1-b_\ell^0(\varepsilon)e^{-2(\tau-\varepsilon/2)/\sqrt{\varepsilon\lambda(\varepsilon)}}]\\
&= a_r^0(\varepsilon)e^{-(\tau_*-\varepsilon/2-\tau)/\sqrt{\varepsilon\lambda(\varepsilon)}} + O(\varepsilon).
\end{aligned}
\tag{8.8}
$$

Lemma 8.2. *The solution to the problem* (8.5), (8.6) *admits the representation*

$$
v_\varepsilon(\tau) = F(\tau) - F_{1/2}z_\varepsilon(\tau) - F_{m-1/2}y_\varepsilon(\tau) + \|F\|_{C^2[\varepsilon/2,\tau_*-\varepsilon/2]}O(\varepsilon), \quad \tau\in[\varepsilon/2,\tau_*-\varepsilon/2]. \tag{8.9}
$$

Proof. We recall that $F_{1/2} = F(\varepsilon/2)$, $F_{m-1/2} = F(\tau_*-\varepsilon/2)$ and set $F'_{1/2} = F'(\varepsilon/2)$, $F'_{m-1/2} = F'(\tau_*-\varepsilon/2)$. We note that the function $\psi_\varepsilon = v_\varepsilon - F + F_{1/2}z_\varepsilon + F_{m-1/2}y_\varepsilon$ is a solution to the problem

$$
\begin{aligned}
&-\varepsilon\lambda(\varepsilon)\frac{d^2}{d\tau^2}\psi_\varepsilon + \psi_\varepsilon = \varepsilon\lambda(\varepsilon)\frac{d^2}{d\tau^2}F, \quad \tau\in(\varepsilon/2,\tau_*-\varepsilon/2),\\
&B_{\varepsilon,\ell}[\psi_\varepsilon](\varepsilon/2) = \sqrt{\varepsilon\lambda(\varepsilon)}F'_{1/2} - \frac{1+b_\ell^0(\varepsilon)-2a_\ell^0(\varepsilon)}{1-b_\ell^0(\varepsilon)}F_{1/2},\\
&B_{\varepsilon,r}[\psi_\varepsilon](\tau_*-\varepsilon/2) = -\sqrt{\varepsilon\lambda(\varepsilon)}F'_{m-1/2} - \frac{1+b_r^0(\varepsilon)-2a_r^0(\varepsilon)}{1-b_r^0(\varepsilon)}F_{m-1/2}.
\end{aligned}
$$

By Lemma 8.1 applied to ψ_ε for u_ε, we obtain the estimate

$$
\begin{aligned}
\|\psi_\varepsilon\|_{C[\varepsilon/2,\tau_*-\varepsilon/2]} \leqslant \max\bigg\{&\varepsilon\lambda(\varepsilon)\|F''\|_{C[\varepsilon/2,\tau_*-\varepsilon/2]}, \frac{1-b_\ell^0(\varepsilon)}{2a_\ell^0(\varepsilon)}\bigg|\sqrt{\varepsilon\lambda(\varepsilon)}F'_{1/2}\\
&-\frac{1+b_\ell^0(\varepsilon)-2a_\ell^0(\varepsilon)}{1-b_\ell^0(\varepsilon)}F_{1/2}\bigg|, \frac{1-b_\ell^0(\varepsilon)}{2a_\ell^0(\varepsilon)}\bigg|-\sqrt{\varepsilon\lambda(\varepsilon)}F'_{m-1/2} - \frac{1+b_r^0(\varepsilon)-2a_r^0(\varepsilon)}{1-b_r^0(\varepsilon)}F_{m-1/2}\bigg|\bigg\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&1-b_\ell^0(\varepsilon) = O(\sqrt{\varepsilon}), \quad 1+b_\ell^0(\varepsilon)-2a_\ell(\varepsilon) = O(\varepsilon), \quad a_\ell^0(\varepsilon)\sim 1,\\
&1-b_r^0(\varepsilon) = O(\sqrt{\varepsilon}), \quad 1+b_r^0(\varepsilon)-2a_r(\varepsilon) = O(\varepsilon), \quad a_r^0(\varepsilon)\sim 1,
\end{aligned}
$$

we obtain the estimate

$$
\|\psi\|_{C[\varepsilon/2,\tau_*-\varepsilon/2]} \leqslant \|F\|_{C^2[\varepsilon/2,\tau_*-\varepsilon/2]}O(\varepsilon).
$$

The lemma is proved. □

We recall that $\varkappa(\varepsilon) = \varpi + (1-\varpi)E_2(\varepsilon)$ and set

$$
Z^0_{\varepsilon,j} = a_\ell^0(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}}, \quad Y^0_{\varepsilon,j} = a_r^0(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}}.
$$

Corollary 8.1. *For all* $1\leqslant j\leqslant m$

$$
u_\varepsilon(\tau_{j-1/2}) = F_{j-1/2} + (J_\ell - F_{1/2})Z^0_{\varepsilon,j} + (J_r - F_{m-1/2})Y^0_{\varepsilon,j} + MO(\varepsilon). \tag{8.10}
$$

Proof. From (8.7) and (8.8) it follows that

$$z_\varepsilon(\tau_{j-1/2}) = a_\ell^0(\varepsilon)e^{-\tau_{j-1}/\sqrt{\varepsilon\lambda(\varepsilon)}} + O(\varepsilon) = Z_{\varepsilon,j}^0 + O(\varepsilon),$$

$$y_\varepsilon(\tau_{j-1/2}) = a_r^0(\varepsilon)e^{-(\tau_*-\tau_j)/\sqrt{\varepsilon\lambda(\varepsilon)}} + O(\varepsilon) = Y_{\varepsilon,j}^0 + O(\varepsilon).$$

Therefore, from (8.4) and (8.9) we have

$$\begin{aligned} u_\varepsilon(\tau_{j-1/2}) &= J_\ell z_\varepsilon(\tau_{j-1/2}) + J_r y_\varepsilon(\tau_{j-1/2}) + v_\varepsilon(\tau_{j-1/2}) \\ &= F_{j-1/2} + (J_\ell - F_{1/2})Z_{\varepsilon,j}^0 + (J_r - F_{m-1/2})Y_{\varepsilon,j}^0 + MO(\varepsilon). \end{aligned}$$

The corollary is proved. □

Lemma 8.3. *Let I_ε be a solution to the problem* (1.1)–(1.5). *Then for all* $1 \leqslant j \leqslant m$

$$U_\varepsilon(\tau_{j-1/2}) = \frac{4\pi}{c}\big[F_{j-1/2} + \varkappa(\varepsilon)(J_\ell - F_{1/2})Z_{\varepsilon,j}^0 + \varkappa(\varepsilon)(J_r - F_{m-1/2})Y_{\varepsilon,j}^0 + MO(\varepsilon)\big]. \quad (8.11)$$

Proof. From the representation of I_ε in the form (1.6) and formulas (7.28), (6.12), (6.16) it follows that

$$\begin{aligned} U_\varepsilon(\tau_{j-1/2}) &= \frac{4\pi}{c}\big[J_\ell S_\varepsilon^I(\tau_{j-1/2}) + J_r S_\varepsilon^{II}(\tau_{j-1/2}) + S_\varepsilon^{III}(\tau_{j-1/2})\big] \\ &= \frac{4\pi}{c}\big[F_{j-1/2} + (J_\ell - F_{1/2})S^I(\tau_{j-1/2}) + (J_r - F_{m-1/2})S^{II}(\tau_{j-1/2}) + MO(\varepsilon)\big] \\ &= \frac{4\pi}{c}\big[F_{j-1/2} + \varkappa(\varepsilon)(J_\ell - F_{1/2})Z_{\varepsilon,j}^0 + \varkappa(\varepsilon)(J_r - F_{m-1/2})Y_{\varepsilon,j}^0 + MO(\varepsilon)\big]. \end{aligned}$$

The lemma is proved. □

To complete the proof of Theorem 2.7, it remains to note that (2.23) is obtained from (8.10) and (8.11), whereas (2.24) is obtained by roughening (2.23) and taking into account that $\varkappa(\varepsilon) = 1 + O(\varepsilon \ln \varepsilon)$. Theorem 2.7 is proved.

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