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INTEGRABLE DYNAMICAL SYSTEMS WITH DISSIPATION ON TANGENT BUNDLES OF 2D AND 3D MANIFOLDS

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Abstract. In many problems of dynamics, one has to deal with mechanical systems whose configurational spaces are two- or three-dimensional manifolds. For such a system, the phase space naturally coincides with the tangent bundle of the corresponding manifold. Thus, the problem of a flow past a (four-dimensional) pendulum on a (generalized) spherical hinge leads to a system on the tangent bundle of a two- or three-dimensional sphere whose metric has a particular structure induced by an additional symmetry group. In such cases, dynamical systems have variable dissipation, and their complete list of first integrals consists of transcendental functions in the form of finite combinations of elementary functions. Another class of problems pertains to a point moving on a two- or three-dimensional surface with the metric induced by the encompassing Euclidean space. In this paper, we establish the integrability of some classes of dynamical systems on tangent bundles of two- and three-dimensional manifolds, in particular, systems involving fields of forces with variable dissipation and of a more general type than those considered previously.

Introduction

This paper is aimed at studying integrability cases for dynamical systems on tangent bundles of two- and three- dimensional manifolds. The problems under consideration involve dissipation of variable sign.

We examine nonconservative systems for which the methods used for studying Hamiltonian systems are inapplicable, in general. Such systems require some kind of "direct" integration of the basic equation of dynamics.

In the general case, it is hardly possible to construct an integration theory for nonconservative systems (even for lower dimensions). However, there are systems with additional symmetries for which one can find first integrals in terms of finite combinations of elementary functions [1, 11–13, 16, 17, 20].

We find cases of complete integrability of nonconservative dynamical systems with nontrivial symmetries. In some integrability cases, each of the first integrals is expressed in terms of a finite combination of elementary functions and at the same time is a transcendental function of its variables. The term "transcendental function" is understood in the sense of complex analysis: being continued to the complex region, such functions have essential singularities [21,22,26,27]. This fact can be explained by the presence of attractive and repelling sets in the system (for instance, attractive and repelling foci).

We find new integrable cases of motion of a rigid body, in particular, those generalizing the classical problem of a spherical pendulum in an incoming flow [28, 29, 32].

Many results of the present paper have been regularly presented at numerous seminars, in particular, the V. V. Trofimov seminar "Current Problems in Geometry and Mechanics" [3–10] supervised by D. V. Georgievskii and M. V. Shamolin.

1. Dynamics on the Tangent Bundle of a 2D Manifold

1.1. Equations of Geodesics on the Tangent Bundle of a 2D Manifold.

1.1.1. Some general terms. Consider a smooth 2d Riemannian manifold M^2 with a metric g_{ij}, which generates an affine connectedness in given local coordinates $x = (x^1, x^2)$.

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Consider also the tangent bundle

$$T_*M^2\{z_2, z_1; x^1, x^2\},$$

where $z = (z_2, z_1)$ are the coordinates in the tangent space.

If $z_i = \dot{x}^i$, $i = 1, 2$, then the equations of the geodesic lines take the form

$$\ddot{x}^i + \Gamma^i_{11}(x)(\dot{x}^1)^2 + 2\Gamma^i_{12}(x)(\dot{x}^1)(\dot{x}^2) + \Gamma^i_{22}(x)(\dot{x}^2)^2 = 0, \quad i = 1, 2. \tag{1.1}$$

1.1.2. Some special terms. For the sake of clarity, in the case of a 2d manifold, we denote the coordinates (x^1, x^2) by (α, β).

Then equations (1.1) on the tangent bundle $T_*M^2\{\dot{\alpha}, \dot{\beta}; \alpha, \beta\}$ take the form

$$\begin{aligned} &\ddot{\alpha} + \Gamma^\alpha_{\alpha\alpha}(\alpha, \beta)\dot{\alpha}^2 + 2\Gamma^\alpha_{\alpha\beta}(\alpha, \beta)\dot{\alpha}\dot{\beta} + \Gamma^\alpha_{\beta\beta}(\alpha, \beta)\dot{\beta}^2 = 0, \\ &\ddot{\beta} + \Gamma^\beta_{\alpha\alpha}(\alpha, \beta)\dot{\alpha}^2 + 2\Gamma^\beta_{\alpha\beta}(\alpha, \beta)\dot{\alpha}\dot{\beta} + \Gamma^\beta_{\beta\beta}(\alpha, \beta)\dot{\beta}^2 = 0. \end{aligned} \tag{1.2}$$

Example 1. In the case of spherical coordinates (α, β), with the metric on the 2d sphere induced by the Euclidean metric of the 3d space, equations (1.2) become

$$\ddot{\alpha} - \dot{\beta}^2 \sin\alpha\cos\alpha = 0, \quad \ddot{\beta} + 2\dot{\alpha}\dot{\beta}\frac{\cos\alpha}{\sin\alpha} = 0, \tag{1.3}$$

i.e., the nonzero connectedness coefficients take the form

$$\Gamma^\alpha_{\beta\beta}(\alpha, \beta) = -\sin\alpha\cos\alpha, \quad \Gamma^\beta_{\alpha\beta}(\alpha, \beta) = \frac{\cos\alpha}{\sin\alpha}.$$

Example 2. In the case of spherical coordinates (α, β), with the metric on the 2d sphere induced by the metric of some special field of forces (see [33, 34]), equations (1.2) take the form

$$\ddot{\alpha} - \dot{\beta}^2\frac{\sin\alpha}{\cos\alpha} = 0, \quad \ddot{\beta} + \dot{\alpha}\dot{\beta}\frac{1 + \cos^2\alpha}{\cos\alpha\sin\alpha} = 0, \tag{1.4}$$

i.e., the nonzero connectedness coefficients become

$$\Gamma^\alpha_{\beta\beta}(\alpha, \beta) = -\frac{\sin\alpha}{\cos\alpha}, \quad \Gamma^\beta_{\alpha\beta}(\alpha, \beta) = \frac{1 + \cos^2\alpha}{2\cos\alpha\sin\alpha}.$$

1.1.3. Changing coordinates on the tangent bundle. One of the purposes of this study is to examine the structure of equations (1.1) after changing the coordinates on the tangent bundle T_*M^2.

Consider the following transformation of the coordinates in the tangent space:

$$\begin{aligned} \dot{\alpha} &= R_1 z_1 + R_2 z_2, \\ \dot{\beta} &= R_3 z_1 + R_4 z_2, \end{aligned} \tag{1.5}$$

and its inversion

$$\begin{aligned} z_1 &= T_1\dot{\alpha} + T_2\dot{\beta}, \\ z_2 &= T_3\dot{\alpha} + T_4\dot{\beta}. \end{aligned} \tag{1.6}$$

Here, R_k, T_k, $k = 1, \ldots, 4$, are functions of α, β, and

$$RT = E,$$

where

$$R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

Equations (1.5) (or (1.6)) will be referred to as *new kinematic relations*, i.e., relations on the tangent bundle T_*M^2.

The following identities hold:

$$\begin{aligned} \dot{z}_1 &= T_{1\alpha}\dot{\alpha}^2 + T_{1\beta}\dot{\alpha}\dot{\beta} + T_{2\alpha}\dot{\alpha}\dot{\beta} + T_{2\beta}\dot{\beta}^2 + T_1\ddot{\alpha} + T_2\ddot{\beta}, \\ \dot{z}_2 &= T_{3\alpha}\dot{\alpha}^2 + T_{3\beta}\dot{\alpha}\dot{\beta} + T_{4\alpha}\dot{\alpha}\dot{\beta} + T_{4\beta}\dot{\beta}^2 + T_3\ddot{\alpha} + T_4\ddot{\beta}, \end{aligned} \tag{1.7}$$

where

$$T_{k\alpha} = \frac{\partial T_k}{\partial \alpha}, \quad T_{k\beta} = \frac{\partial T_k}{\partial \beta}, \quad k = 1, \ldots, 4.$$

Substituting (1.2) into (1.7), we get

$$\begin{aligned} \dot z_1 &= \dot\alpha^2\{T_{1\alpha} - T_1\Gamma^\alpha_{\alpha\alpha} - T_2\Gamma^\beta_{\alpha\alpha}\} + \dot\alpha\dot\beta\{T_{1\beta} + T_{2\alpha} - 2T_1\Gamma^\alpha_{\alpha\beta} - 2T_2\Gamma^\beta_{\alpha\beta}\} + \dot\beta^2\{T_{2\beta} - T_1\Gamma^\alpha_{\beta\beta} - T_2\Gamma^\beta_{\beta\beta}\}, \\ \dot z_2 &= \dot\alpha^2\{T_{3\alpha} - T_3\Gamma^\alpha_{\alpha\alpha} - T_4\Gamma^\beta_{\alpha\alpha}\} + \dot\alpha\dot\beta\{T_{3\beta} + T_{4\alpha} - 2T_3\Gamma^\alpha_{\alpha\beta} - 2T_4\Gamma^\beta_{\alpha\beta}\} + \dot\beta^2\{T_{4\beta} - T_3\Gamma^\alpha_{\beta\beta} - T_4\Gamma^\beta_{\beta\beta}\}, \end{aligned} \tag{1.8}$$

where $\dot\alpha, \dot\beta$ should be replaced by their expressions from (1.5).

Proposition 1.1. *In the domain where* $\det R(\alpha, \beta) \neq 0$*, system* (1.2) *is equivalent to the composite system* (1.5)*,* (1.8)*.*

Thus, the result of passing from the geodesic equations (1.2) to the equivalent system (1.5), (1.8) depends both on the transformation of the variables (1.5) (or (1.6)) (i.e., new kinematic relations) and on the affine connectedness Γ^i_{jk}.

Corollary 1.1. *In the case of spherical coordinates* (α, β)*, with the metric on the 2d sphere induced by that of the 3d Euclidean space* (*see also Example* 1)*, the system equivalent to the geodesic equations* (1.3) *takes the form*

$$\dot\alpha = -z_2, \quad \dot z_2 = -z_1^2 \frac{1}{\cos\alpha \sin\alpha}, \quad \dot z_1 = z_1 z_2 \frac{1}{\cos\alpha\sin\alpha}, \quad \dot\beta = z_1 \frac{1}{\cos\alpha\sin\alpha}, \tag{1.9}$$

provided that the first and the fourth equations of system (1.9) *are regarded as new kinematic relations.*

Corollary 1.2. *In the case of spherical coordinates* (α, β)*, with the metric on the 2d sphere induced by that of a cetrain special field of forces* (*see* [40, 41] *and Example* 2)*, the system equivalent to the geodesic equations* (1.4) *takes the form*

$$\dot\alpha = -z_2, \quad \dot z_2 = -z_1^2 \frac{\cos\alpha}{\sin\alpha}, \quad \dot z_1 = z_1 z_2 \frac{\cos\alpha}{\sin\alpha}, \quad \dot\beta = z_1 \frac{\cos\alpha}{\sin\alpha}, \tag{1.10}$$

provided that the first and the fourth equations of system (1.10) *are regarded as new kinematic relations.*

1.1.4. A complete list of first integrals for geodesic equations. Consider a fairly general case of kinematic relations

$$\dot\alpha = -z_2, \quad \dot\beta = z_1 f(\alpha), \tag{1.11}$$

where $f(\alpha)$ is a sufficiently smooth function.

Proposition 1.2. *In the case of kinematic relations* (1.11)*, equations* (1.8) *take the form*

$$\begin{aligned} \dot z_1 &= -\frac{1}{f(\alpha)}\Gamma^\beta_{\alpha\alpha}(\alpha,\beta) z_2^2 + \left[2\Gamma^\beta_{\alpha\beta}(\alpha,\beta) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2 - f(\alpha)\Gamma^\beta_{\beta\beta}(\alpha,\beta) z_1^2, \\ \dot z_2 &= \Gamma^\alpha_{\alpha\alpha}(\alpha,\beta) z_2^2 - 2\Gamma^\alpha_{\alpha\beta}(\alpha,\beta) f(\alpha) z_1 z_2 + f^2(\alpha)\Gamma^\alpha_{\beta\beta}(\alpha,\beta) z_1^2. \end{aligned} \tag{1.12}$$

Thus, after a suitable choice of kinematic relations, geodesic equations (1.2) become equivalent (almost everywhere) to the composite system (1.11), (1.12) on the manifold $T_*M^2\{z_2, z_1; \alpha, \beta\}$.

For the complete integration of the fourth-order system (1.11), (1.12), one should know three independent first integrals, in general.

Corollary 1.3. *If*

$$\Gamma^\alpha_{\alpha\alpha}(\alpha,\beta) \equiv \Gamma^\alpha_{\alpha\beta}(\alpha,\beta) \equiv \Gamma^\beta_{\alpha\alpha}(\alpha,\beta) \equiv \Gamma^\beta_{\beta\beta}(\alpha,\beta) \equiv 0, \tag{1.13}$$

then the system equivalent to the geodesic equations (1.2) *can be reduced to*

$$\begin{cases} \dot{\alpha} = -z_2, \\ \dot{z}_2 = \Gamma^{\alpha}_{\beta\beta}(\alpha,\beta) f^2(\alpha) z_1^2, \\ \dot{z}_1 = \left[2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2, \\ \dot{\beta} = z_1 f(\alpha). \end{cases} \tag{1.14}$$

Proposition 1.3. *If the relation*

$$\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta) f^2(\alpha) + 2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) + \frac{d\ln|f(\alpha)|}{d\alpha} \equiv 0 \tag{1.15}$$

holds everywhere, then system (1.14) *has the following analytic first integral*:

$$\Phi_1(z_2, z_1) = z_1^2 + z_2^2 = C_1^2 = \text{const}. \tag{1.16}$$

Proposition 1.4. *If the function* $\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta)$ *depends only on* α,

$$\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) = \Gamma^{\beta}_{\alpha\beta}(\alpha), \tag{1.17}$$

then system (1.14) *has the following first integral*:

$$\Phi_2(z_1;\alpha) = z_1 \Phi_0(\alpha) = C_2 = \text{const}, \tag{1.18}$$

$$\Phi_0(\alpha) = f(\alpha) \exp\left\{2 \int\limits_{\alpha_0}^{\alpha} \Gamma^{\beta}_{\alpha\beta}(b)\,db\right\}.$$

Remark 1.1. If (1.17) holds and $\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta)$, too, depends only on α,

$$\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta) = \Gamma^{\alpha}_{\beta\beta}(\alpha), \tag{1.19}$$

then system (1.14) contains an independent third-order subsystem that consists of the first three equations (the equation for $\dot{\beta}$ is separated):

$$\begin{cases} \dot{\alpha} = -z_2, \\ \dot{z}_2 = \Gamma^{\alpha}_{\beta\beta}(\alpha) f^2(\alpha) z_1^2, \\ \dot{z}_1 = \left[2\Gamma^{\beta}_{\alpha\beta}(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2, \end{cases} \tag{1.20}$$

$$\dot{\beta} = z_1 f(\alpha). \tag{1.21}$$

In particular, properties (1.15), (1.17) ensure the appearance of such an independent subsystem (1.20).

Proposition 1.5. *If conditions* (1.15), (1.17) *are satisfied, then system* (1.20), (1.21) *admits the following first integral*:

$$\Phi_3(z_2, z_1; \alpha, \beta) = \beta \pm \int\limits_{\alpha_0}^{\alpha} \frac{C_2 f(a)}{\sqrt{C_1^2 \Phi_0^2(a) - C_2^2}}\,da = C_3 = \text{const}, \tag{1.22}$$

where, after calculating the integral (1.22), *the constants* C_1, C_2 *should be replaced by the left-hand sides of* (1.16), (1.18), *respectively.*

Theorem 1.1. *If conditions* (1.15), (1.17) *are satisfied, then system* (1.20), (1.21) *has a complete set* (*three*) *of independent first integrals* (1.16), (1.18), (1.22).

2. Equations of Motion on the Tangent Bundle of a 2D Manifold in a Potential Field of Forces

2.1. Reduced System. Let us modify system (1.14) and thus obtain a conservative system. The presence of forces is characterized by the term $F(\alpha)$ in the second equation in (2.1) (in contrast to system (1.14)). The system under consideration on the tangent bundle $T_*M^2\{z_2, z_1; \alpha, \beta\}$ takes the form

$$\begin{cases} \dot{\alpha} = -z_2, \\ \dot{z}_2 = F(\alpha) + \Gamma^{\alpha}_{\beta\beta}(\alpha,\beta) f^2(\alpha) z_1^2, \\ \dot{z}_1 = \left[2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2, \\ \dot{\beta} = z_1 f(\alpha). \end{cases} \tag{2.1}$$

Remark 2.1. System (2.1) almost everywhere is equivalent to the following system:

$$\begin{cases} \ddot{\alpha} + F(\alpha) + \Gamma^{\alpha}_{\beta\beta}(\alpha,\beta)\dot{\beta}^2 = 0, \\ \ddot{\beta} + 2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta)\dot{\alpha}\dot{\beta} = 0. \end{cases} \tag{2.2}$$

2.2. A Complete List of First Integrals for the System in a Potential Field of Forces.

Proposition 2.1. *If* (1.15) *holds everywhere, then system* (2.1) *has the following analytic first integral*:

$$\Phi_1(z_2, z_1; \alpha) = z_1^2 + z_2^2 + F_1(\alpha) = C_1 = \text{const}, \quad F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(a)\, da. \tag{2.3}$$

Proposition 2.2. *If the function* $\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta)$ *depends only on* α (*condition* (1.17)), *then system* (2.1) *admits the following first integral:*

$$\Phi_2(z_1; \alpha) = z_1 \Phi_0(\alpha) = C_2 = \text{const}, \tag{2.4}$$

$$\Phi_0(\alpha) = f(\alpha) \exp\left\{2\int_{\alpha_0}^{\alpha} \Gamma^{\beta}_{\alpha\beta}(b)\,db\right\}.$$

Remark 2.2. If condition (1.17) is satisfied and the function $\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta)$, too, depends only on α (condition (1.19)), then system (2.1) contains an independent third-order subsystem that consists of the first three equations (equation for $\dot{\beta}$ is dropped):

$$\begin{cases} \dot{\alpha} = -z_2, \\ \dot{z}_2 = F(\alpha) + \Gamma^{\alpha}_{\beta\beta}(\alpha) f^2(\alpha) z_1^2, \\ \dot{z}_1 = \left[2\Gamma^{\beta}_{\alpha\beta}(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2, \end{cases} \tag{2.5}$$

$$\dot{\beta} = z_1 f(\alpha). \tag{2.6}$$

In particular, such a subsystem (2.5) can be separated if both properties (1.15) and (1.17) hold.

Proposition 2.3. *If conditions* (1.15), (1.17) *hold, then system* (2.5), (2.6) *admits the following first integral*:

$$\Phi_3(z_2, z_1; \alpha, \beta) = \beta \pm \int_{\alpha_0}^{\alpha} \frac{C_2 f(a)}{\sqrt{\Phi_0^2(a)(C_1 - F_1(a)) - C_2^2}}\, da = C_3 = \text{const}, \tag{2.7}$$

where, having calculated the integral (2.7), *one should replace the constants* C_1, C_2 *by the left-hand sides of* (2.3), (2.4), *respectively.*

Theorem 2.1. *If conditions* (1.15), (1.17) *are satisfied, then system* (2.5), (2.6) *has a complete set* (*three*) *of independent first integrals:* (2.3), (2.4), (2.7).

3. Equations of Motion on a Tangent Bundle of a 2D Manifold in a Force Field with Dissipation

3.1. A Reduced System. Let us modify system (2.1)) and thus obtain a system with dissipation. The presence of dissipation (of alternating sign, in general) is characterized by the term $bg(\alpha), b > 0$, in the first equation os system (3.1) (in contrast to system (2.1))) Our system on the tangent bundle $T_*M^2\{z_2, z_1; \alpha, \beta\}$ takes the form

$$\begin{cases} \dot\alpha = -z_2 + bg(\alpha), \\ \dot z_2 = F(\alpha) + \Gamma^\alpha_{\beta\beta}(\alpha,\beta) f^2(\alpha) z_1^2, \\ \dot z_1 = \left[2\Gamma^\beta_{\alpha\beta}(\alpha,\beta) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2, \\ \dot\beta = z_1 f(\alpha). \end{cases} \tag{3.1}$$

Remark 3.1. System (3.1) is almost everywhere equivalent to the following one:

$$\begin{cases} \ddot\alpha - bg'(\alpha)\dot\alpha + F(\alpha) + \Gamma^\alpha_{\beta\beta}(\alpha,\beta)\dot\beta^2 = 0, \\ \ddot\beta - bg(\alpha)f(\alpha)\dot\beta + 2\Gamma^\beta_{\alpha\beta}(\alpha,\beta)\dot\alpha\dot\beta = 0. \end{cases} \tag{3.2}$$

3.2. A Complete List of First Integrals for a System with Dissipation. Let us integrate the fourth-order system (3.1) under the conditions (1.15), (1.17). In this case, system (3.1) admits separation of the following independent third-order system:

$$\begin{cases} \dot\alpha = -z_2 + bg(\alpha), \\ \dot z_2 = F(\alpha) + \Gamma^\alpha_{\beta\beta}(\alpha) f^2(\alpha) z_1^2, \\ \dot z_1 = \left[2\Gamma^\beta_{\alpha\beta}(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2, \end{cases} \tag{3.3}$$

$$\dot\beta = z_1 f(\alpha). \tag{3.4}$$

First, we associate to the third-order system (3.3) the following nonautonomous second-order system:

$$\begin{aligned} \frac{dz_2}{d\alpha} &= \frac{F(\alpha) + \Gamma^\alpha_{\beta\beta}(\alpha) f^2(\alpha) z_1^2}{-z_2 + bg(\alpha)}, \\ \frac{dz_1}{d\alpha} &= \frac{\left[2\Gamma^\beta_{\alpha\beta}(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_2}{-z_2 + bg(\alpha)}. \end{aligned} \tag{3.5}$$

Then, introducing homogeneous variables

$$z_k = u_k g(\alpha), \quad k = 1, 2, \tag{3.6}$$

we reduce system (3.5) to

$$\begin{aligned} g(\alpha)\frac{du_2}{d\alpha} + g'(\alpha)u_2 &= \frac{F(\alpha) + \Gamma^\alpha_{\beta\beta}(\alpha) f^2(\alpha) g^2(\alpha) u_1^2}{-u_2 g(\alpha) + bg(\alpha)}, \\ g(\alpha)\frac{du_1}{d\alpha} + g'(\alpha)u_1 &= \frac{\left[2\Gamma^\beta_{\alpha\beta}(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] g^2(\alpha) u_1 u_2}{-u_2 g(\alpha) + bg(\alpha)}, \end{aligned} \tag{3.7}$$

which, in view of (1.15), is almost everywhere equivalent to

$$
\begin{aligned}
g(\alpha)\frac{du_2}{d\alpha} &= \frac{F_3(\alpha)+\Gamma^{\alpha}_{\beta\beta}(\alpha)f^2(\alpha)g(\alpha)u_1^2+g'(\alpha)u_2^2-bg'(\alpha)u_2}{-u_2+b},\\
g(\alpha)\frac{du_1}{d\alpha} &= \frac{-\Gamma^{\alpha}_{\beta\beta}(\alpha)f^2(\alpha)g(\alpha)u_1u_2+g'(\alpha)u_1u_2-bg'(\alpha)u_1}{-u_2+b},\\
F_3(\alpha) &= \frac{F(\alpha)}{g(\alpha)}.
\end{aligned}
\tag{3.8}
$$

Now, in order to integrate system (3.8), we impose the following two conditions.

- There is $\kappa\in\mathbf{R}$ such that

$$
\Gamma^{\alpha}_{\beta\beta}(\alpha)f^2(\alpha)=\kappa\frac{g'(\alpha)}{g(\alpha)}; \tag{3.9}
$$

- There is $\lambda\in\mathbf{R}$ such that

$$
F_3(\alpha)=\lambda g'(\alpha). \tag{3.10}
$$

Conditions (3.9) and (3.10) can be respectively rewritten as follows:

$$
\Gamma^{\alpha}_{\beta\beta}(\alpha)f^2(\alpha)=\kappa\frac{d}{d\alpha}\ln|g(\alpha)|; \tag{3.11}
$$

$$
F(\alpha)=\lambda\frac{d}{d\alpha}\frac{g^2(\alpha)}{2}. \tag{3.12}
$$

Indeed, if conditions (3.9) and (3.10) (or (3.11) and (3.12)) are fulfilled, then system (3.8) can be reduced to the first-order equation

$$
\frac{du_2}{du_1}=\frac{\lambda+\kappa u_1^2+u_2^2-bu_2}{(1-\kappa)u_1u_2-bu_1}. \tag{3.13}
$$

Equation (3.13) is an Abel equation [11,13,42]. For $\kappa=-1$, it has the following first integral:

$$
\frac{u_2^2+u_1^2-bu_2+\lambda}{u_1}=C_1=\text{const}, \tag{3.14}
$$

which, in the former variables, reads as follows:

$$
\Theta_1(z_2,z_1;\alpha)=\frac{z_2^2+z_1^2-bz_2g(\alpha)+\lambda g^2(\alpha)}{z_1g(\alpha)}=C_1=\text{const}. \tag{3.15}
$$

Remark 3.2. If α is a 2π-periodic coordinate, then system (3.3) (being a part of system (3.3), (3.4)) becomes a dynamical system with zero mean variable dissipation [14,15,23], and it turns into the following conservative system for $b=0$:

$$
\begin{cases}
\dot\alpha=-z_2,\\
\dot z_2=F(\alpha)+\Gamma^{\alpha}_{\beta\beta}(\alpha)f^2(\alpha)z_1^2,\\
\dot z_1=\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha)+\dfrac{d\ln|f(\alpha)|}{d\alpha}\right]z_1z_2.
\end{cases}
\tag{3.16}
$$

System (3.16) admits two smooth first integrals (2.3), (2.4), which we transform as follows. By virtue of (3.10) (or (3.12)), we have

$$
\Phi_1(z_2,z_1;\alpha)=z_1^2+z_2^2+2\int\limits_{\alpha_0}^{\alpha}F(a)\,da=z_1^2+z_2^2+\lambda\int\limits_{\alpha_0}^{\alpha}\frac{d}{da}g^2(a)\,da\cong z_1^2+z_2^2+\lambda g^2(\alpha), \tag{3.17}
$$

where "≅" means equal to within an additive constant.

Further, in view of (1.15), we have

$$\Phi_2(z_1;\alpha) = z_1 f(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma^{\beta}_{\alpha\beta}(b)db\right\}$$
$$= z_1 f(\alpha)\exp\left\{-\int\limits_{\alpha_0}^{\alpha}\left[\Gamma^{\alpha}_{\beta\beta}(b)f^2(b)+\frac{d\ln|f(b)|}{db}\right]db\right\} \cong z_1\exp\left\{-\int\limits_{\alpha_0}^{\alpha}\Gamma^{\alpha}_{\beta\beta}(b)f^2(b)db\right\},$$

where "$\cong$" means being equal to within a multiplicative constant.

Now, using (3.9) (or (3.11)), we can rewrite the last expression, for $\kappa=-1$, in the form

$$z_1\exp\left\{\int\limits_{\alpha_0}^{\alpha}\frac{d}{db}\ln|g(b)|db\right\} \cong z_1 g(\alpha) \tag{3.18}$$

to within a multiplicative constant.

Obviously, the ratio of the two first integrals (3.17), (3.18) (or (2.3), (2.4)) is also a first integral of system (3.16). But, for $b\neq 0$, neither

$$z_2^2+z_1^2-bz_2g(\alpha)+\lambda g^2(\alpha) \tag{3.19}$$

nor (3.18), separately, is a first integral of (3.3). However, the ratio of the functions (3.19), (3.18) is a first integral of system (3.3) (for $\kappa=-1$) for any b.

Now, let us find in explicit form an additional first integral of the third-order system (3.3) for $\kappa=-1$. To this end, we start with transforming the invariant relation (3.14) for $u_1\neq 0$ as follows:

$$\left(u_2-\frac{b}{2}\right)^2+\left(u_1-\frac{C_1}{2}\right)^2=\frac{b^2+C_1^2}{4}-\lambda. \tag{3.20}$$

We see that the parameters of this invariant relation must satisfy the condition

$$b^2+C_1^2-4\lambda\geq 0, \tag{3.21}$$

and the phase space of system (3.3) foliates into a family of surfaces defined by (3.20).

Thus, in view of (3.14), the first equation of system (3.8), under the conditions (3.9), (3.10), and $\kappa=-1$, takes the form

$$\frac{g(\alpha)}{g'(\alpha)}\frac{du_2}{d\alpha}=\frac{2(\lambda-bu_2+u_2^2)-C_1U_1(C_1,u_2)}{-u_2-b_*}, \tag{3.22}$$

where

$$U_1(C_1,u_2)=\frac{1}{2}\left\{C_1\pm\sqrt{C_1^2-4(u_2^2-bu_2+\lambda)}\right\}, \tag{3.23}$$

and the integration constant C_1 is chosen from (3.21).

Therefore, the quadrature for the additional first integral of system (3.3) takes the form

$$\int\frac{dg(\alpha)}{g(\alpha)}=\int\frac{(b-u_2)du_2}{2(\lambda-bu_2+u_2^2)-C_1\{C_1\pm\sqrt{C_1^2-4(u_2^2-bu_2+\lambda)}\}/2}. \tag{3.24}$$

The left-hand side of this relation is obviously equal to (to within an additive constant)

$$\ln|g(\alpha)|. \tag{3.25}$$

If

$$u_2-\frac{b}{2}=r_1,\quad b_1^2=b^2+C_1^2-4\lambda, \tag{3.26}$$

then the right-hand side of (3.24) takes the form

$$-\frac{1}{4}\int\frac{d(b_1^2-4r_1^2)}{(b_1^2-4r_1^2)\pm C_1\sqrt{b_1^2-4r_1^2}}-b\int\frac{dr_1}{(b_1^2-4r_1^2)\pm C_1\sqrt{b_1^2-4r_1^2}}=-\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4r_1^2}}{C_1}\pm 1\right|\mp\frac{b}{2}I_1, \tag{3.27}$$

where

$$I_1 = \int \frac{dr_3}{\sqrt{b_1^2 - r_3^2}(r_3 \pm C_1)}, \quad r_3 = \sqrt{b_1^2 - 4r_1^2}. \tag{3.28}$$

When calculating the integral (3.28), we consider the following possible cases.

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln\left| \frac{\sqrt{b^2-4} + \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}} \right| + \frac{1}{2\sqrt{b^2-4}} \ln\left| \frac{\sqrt{b^2-4} - \sqrt{b_1^2 - r_3^2}}{r_3 \mp C_1} \mp \frac{C_1}{\sqrt{b^2-4}} \right| + \text{const.} \tag{3.29}$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1 r_3 + b_1^2}{b_1(r_3 \pm C_1)} + \text{const.} \tag{3.30}$$

III. $b = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - r_3^2}}{C_1(r_3 \pm C_1)} + \text{const.} \tag{3.31}$$

Going back to the variable

$$r_1 = \frac{z_2}{g(\alpha)} - \frac{b}{2}, \tag{3.32}$$

we obtain the final expression for I_1:

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln\left| \frac{\sqrt{b^2-4} \pm 2r_1}{\sqrt{b_1^2 - 4r_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}} \right| + \frac{1}{2\sqrt{b^2-4}} \ln\left| \frac{\sqrt{b^2-4} \mp 2r_1}{\sqrt{b_1^2 - 4r_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}} \right| + \text{const.} \tag{3.33}$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1\sqrt{b_1^2 - 4r_1^2} + b_1^2}{b_1(\sqrt{b_1^2 - 4r_1^2} \pm C_1)} + \text{const.} \tag{3.34}$$

III. $b = 2$.

$$I_1 = \mp \frac{2r_1}{C_1(\sqrt{b_1^2 - 4r_1^2} \pm C_1)} + \text{const.} \tag{3.35}$$

Thus, having found an additional first integral for the third-order system (2.5) for $\kappa = -1$, we obtain a complete set of first integrals in the form of transcendental functions of their phase variables [24, 25].

Remark 3.3. In the expressions of the first integrals found above, the constant C_1 should be formally replaced by the left-hand side of the first integral (3.14).

Then, the said additional first integral acquires the following structural form (cf. [30, 31]):

$$\Theta_2(z_2, z_1; \alpha) = G\left(g(\alpha), \frac{z_2}{g(\alpha)}, \frac{z_1}{g(\alpha)} \right) = C_2 = \text{const}, \tag{3.36}$$

in other words,

$$\Theta_2(z_2, z_1; \alpha) = g(\alpha) \exp\left\{ \int\limits_{\alpha_0}^{\alpha} A(u_2; C_1)\, du_2 \right\} = C_2 = \text{const}, \tag{3.37}$$

$$A(u_2; C_1) = \frac{u_2 - b}{2(\lambda - bu_2 + u_2^2) - C_1\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + \lambda)}\}/2}.$$

Thus, in order to integrate the fourth-order system (2.5), (2.6), we have found, under certain conditions, two independent first integrals of system (2.5). For its complete integration, it suffices to find one more (additional) first integral that "attaches" equation (2.6).

Since

$$g(\alpha)\frac{du_1}{d\alpha} = \frac{u_1\big((1-\kappa)u_2 - b\big)g'(\alpha)}{b-u_2}, \quad g(\alpha)\frac{d\beta}{d\alpha} = \frac{u_1 g(\alpha)f(\alpha)}{b-u_2}, \tag{3.38}$$

it follows that

$$\frac{du_1}{d\beta} = [(1-\kappa)u_2 - b]\frac{g'(\alpha)}{g(\alpha)f(\alpha)}. \tag{3.39}$$

Omitting further calculations, we conclude that the desired additional first integral has the following structural form:

$$\Theta_3(z_2, z_1; \alpha, \beta) = G_1\left(g(\alpha), \beta, \frac{z_2}{g(\alpha)}, \frac{z_1}{g(\alpha)}\right) = C_2 = \text{const.} \tag{3.40}$$

Thus, in the case under consideration, the system of dynamic equations (3.3), (3.4) admits three first integrals (3.15), (3.36), (3.40), which are transcendental functions (in the sense of complex analysis) of phase variables.

Theorem 3.1. *System* (3.3), (3.4) *has a complete set* (*three*) *of independent first integrals. For* $\kappa = -1$, *these integrals have the form* (3.15), (3.36), (3.40).

4. On Surfaces of Revolution

We have indicated two concrete cases, (1.3) and (1.4) (for $f(\alpha)$ defining the metric on the 2d sphere), in which it is possible to obtain integrable dynamical systems with dissipation. Let us apply the above methods to the case of 2d surfaces of revolution.

In the 3d Euclidean space with cylindrical coordinates (ρ, φ, z), consider a surface of revolution defined by the equation

$$\rho = \rho(z). \tag{4.1}$$

The equations of geodesics on this surface have the form

$$\begin{cases} \ddot\alpha + \Gamma_1(\alpha)\dot\alpha^2 + \Gamma_2(\alpha)\dot\beta^2 = 0, \\ \ddot\beta + \Gamma_3(\alpha)\dot\alpha\dot\beta = 0, \end{cases} \tag{4.2}$$

$$\Gamma_1(\alpha) = \frac{\rho'(\alpha)\rho''(\alpha)}{1+\rho'^2(\alpha)}, \quad \Gamma_2(\alpha) = \frac{\rho(\alpha)\rho'(\alpha)}{1+\rho'^2(\alpha)}, \quad \Gamma_3(\alpha) = 2\frac{\rho'(\alpha)}{\rho'(\alpha)}, \tag{4.3}$$

where $\alpha = z$, $\beta = \varphi$.

Let us introduce new kinematic relations:

$$\dot\alpha = f_1(\alpha)z_2, \quad \dot\beta = f_2(\alpha)z_1. \tag{4.4}$$

Then system (4.2), (4.3) can be rewritten in the form

$$\begin{cases} \dot\alpha = f_1(\alpha)z_2, \\ \dot z_2 = -z_1^2\left\{-\Gamma_1(\alpha)f_1(\alpha) - f_1'(\alpha)\right\} + z_1^2\left\{\dfrac{-\Gamma_2(\alpha)f_2^2(\alpha)}{f_1(\alpha)}\right\}, \\ \dot z_1 = z_2 z_1\left\{-\Gamma_3(\alpha)f_1(\alpha) - f_2'(\alpha)\dfrac{f_1(\alpha)}{f_2(\alpha)}\right\}, \end{cases} \tag{4.5}$$

$$\dot\beta = f_2(\alpha)z_1, \tag{4.6}$$

where equation (4.6) can be separated from the fourth-order system (4.5), (4.6).

Sufficient conditions for the existence of the first integral

$$\Phi_1(z_2, z_1) = z_1^2 + z_2^2 = \text{const} \tag{4.7}$$

of system (4.5), (4.6) can be written as two groups of relations:

$$\begin{cases} \Gamma_1(\alpha)f_1(\alpha) + f_1'(\alpha) = 0, \\ \frac{\Gamma_2(\alpha)f_2^3(\alpha)}{f_2^2(\alpha)} + \Gamma_3(\alpha)f_2(\alpha) + f_2'(\alpha) = 0. \end{cases} \tag{4.8}$$

Equations (4.8) admit the following solutions:

$$f_1(\alpha) = \frac{A_1}{\sqrt{1+\rho'^2(\alpha)}}, \quad f_2(\alpha) = \frac{A_1}{\rho(\alpha)\sqrt{A_2A_1^2\rho^2(\alpha)-1}}, \quad A_2 > 0, \quad A_1, A_2 \in \mathbf{R}. \tag{4.9}$$

Choosing $f_1(\alpha), f_2(\alpha)$ as solutions of (4.9), we rewrite system (4.5), (4.6) in the form

$$\begin{cases} \dot{\alpha} = z_2 \frac{A_1}{\sqrt{1+\rho'^2(\alpha)}}, \\ \dot{z}_2 = -z_1^2\Gamma(\alpha), \\ \dot{z}_1 = z_2 z_1 \Gamma(\alpha), \end{cases} \tag{4.10}$$

$$\dot{\beta} = z_1 \frac{A_1}{\rho(\alpha)\sqrt{A_2A_1^2\rho^2(\alpha)-1}}, \tag{4.11}$$

where

$$\Gamma(\alpha) = \frac{A_1\rho'(\alpha)}{\rho(\alpha)\sqrt{1+\rho'^2(\alpha)}(A_2A_1^2\rho^2(\alpha)-1)}. \tag{4.12}$$

Let us introduce a conservative force field $F(\alpha)$ and dissipation $g(\alpha), b > 0$, in system (4.10), (4.11). We thus obtain the following system:

$$\begin{cases} \dot{\alpha} = z_2 f_1(\alpha) + bg(\alpha), \\ \dot{z}_2 = F(\alpha) - z_1^2\Gamma(\alpha), \\ \dot{z}_1 = z_2 z_1 \Gamma(\alpha), \end{cases} \tag{4.13}$$

$$\dot{\beta} = z_1 f_2(\alpha). \tag{4.14}$$

In order to integrate system (4.13), (4.14), we associate to the third-order system (4.13) the nonautonomous second-order system

$$\frac{dz_2}{d\alpha} = \frac{F(\alpha) - \Gamma(\alpha)z_1^2}{z_2 f_1(\alpha) + bg(\alpha)}, \quad \frac{dz_1}{d\alpha} = \frac{\Gamma(\alpha)z_1z_2}{z_2 f_1(\alpha) + bg(\alpha)}. \tag{4.15}$$

Then, introducing the homogeneous variables

$$z_k = u_k \frac{g(\alpha)}{f_1(\alpha)}, \quad k = 1, 2, \tag{4.16}$$

we reduce system (4.15) to

$$\begin{aligned} \frac{g(\alpha)}{f_1(\alpha)}\frac{du_2}{d\alpha} + \left[\frac{g(\alpha)}{f_1(\alpha)}\right]' u_2 &= \frac{F(\alpha) - \Gamma(\alpha)\left[\frac{g(\alpha)}{f_1(\alpha)}\right]^2 u_1^2}{-u_2 g(\alpha) + bg(\alpha)}, \\ \frac{g(\alpha)}{f_1(\alpha)}\frac{du_1}{d\alpha} + \left[\frac{g(\alpha)}{f_1(\alpha)}\right]' u_1 &= \frac{\Gamma(\alpha)\left[\frac{g(\alpha)}{f_1(\alpha)}\right]^2 u_1 u_2}{-u_2 g(\alpha) + bg(\alpha)}, \end{aligned} \tag{4.17}$$

which, almost everywhere, is equivalent to

$$
\begin{aligned}
&\frac{g(\alpha)}{f_1(\alpha)}\frac{du_2}{d\alpha} = \frac{F_3(\alpha) - \Gamma(\alpha)\frac{g(\alpha)}{f_1^2(\alpha)}u_1^2 - \left[\frac{g(\alpha)}{f_1(\alpha)}\right]' u_2^2 - b\left[\frac{g(\alpha)}{f_1(\alpha)}\right]' u_2}{u_2 + b},\\
&\frac{g(\alpha)}{f_1(\alpha)}\frac{du_1}{d\alpha} = \frac{\Gamma(\alpha)\frac{g(\alpha)}{f_1^2(\alpha)}u_1u_2 - \left[\frac{g(\alpha)}{f_1(\alpha)}\right]' u_1u_2 - b\left[\frac{g(\alpha)}{f_1(\alpha)}\right]' u_1}{u_2 + b},\\
&F_3(\alpha) = \frac{F(\alpha)}{g(\alpha)}.
\end{aligned}
\tag{4.18}
$$

And now, in order to integrate system (4.18), we impose the following two conditions.

- There is $\kappa \in \mathbf{R}$ such that

$$
\Gamma(\alpha)\frac{g(\alpha)}{f_1^2(\alpha)} = \kappa\left[\frac{g(\alpha)}{f_1(\alpha)}\right]'. \tag{4.19}
$$

- There is $\lambda \in \mathbf{R}$ such that

$$
F_3(\alpha) = \lambda\left[\frac{g(\alpha)}{f_1(\alpha)}\right]'. \tag{4.20}
$$

Indeed, if conditions (4.19) and (4.20) are fulfilled, system (4.18) reduces to the first-order equation

$$
\frac{du_2}{du_1} = \frac{\lambda - \kappa u_1^2 - u_2^2 - bu_2}{(\kappa - 1)u_1u_2 - bu_1}. \tag{4.21}
$$

Equation (4.21) is an Abel equation [11, 13, 42]. For $\kappa = -1$, it has the following first integral:

$$
\frac{-u_2^2 - u_1^2 - bu_2 + \lambda}{u_1} = C_1 = \text{const}, \tag{4.22}
$$

which, in former variables, has the form

$$
\Theta_1(z_2, z_1; \alpha) = \frac{-z_2^2 f_1^2(\alpha) - z_1^2 f_1^2(\alpha) - bz_2 g(\alpha) f_1(\alpha) + \lambda g^2(\alpha)}{z_1 g(\alpha) f_1(\alpha)} = C_1 = \text{const}. \tag{4.23}
$$

In a similar way, we find two other first integrals (see above and [2, 18, 19]).

In particular, property (4.19) takes the form

$$
g(\alpha) = \frac{A_1A_3}{\sqrt{1 + \rho'^2(\alpha)}}\left|\frac{\rho^2(\alpha)}{A_2A_1^2\rho^2(\alpha) - 1}\right|^{-1/2\kappa} = C_1 = \text{const}, \quad A_1 \neq 0, \quad A_2 > 0, \quad A_1, A_2, A_3, \kappa \in \mathbf{R}. \tag{4.24}
$$

5. Dynamics on the Tangent Bundle of a 3D Manifold

In dynamic problems for systems with three degrees of freedom, configuration spaces are 3d manifolds. Accordingly, phase spaces of such systems are tangent bundles of such manifolds. For instance, to study the motion of a four-dimensional rigid body such as a pendulum (the generalized spherical pendulum) in a nonconservative force field one has to consider a dynamical system on the tangent bundle of a 3d sphere, with its special metric induced by an additional group of symmetries [35]. In this case, dynamical systems describing the motion of such pendulums have variable dissipation, and the complete list of first integrals consists of transcendental functions expressed in terms of finite combination of elementary functions [36].

There is also a class of problems regarding the motion of a point on a 3d surface whose metric is induced by the Euclidean metric of the all-encompassing four-dimensional space. For some systems with variable dissipation, it is also possible to obtain a complete list of first integrals consisting of transcendental functions. The results obtained in this direction are especially important for cases with nonconservative force fields [37].

In the present paper, we establish integrability of some classes of dynamical systems on tangent bundles of 3d manifolds, with force fields having variable dissipation [38] and generalizing those considered previously.

5.1. Geodesic Equations after Changing Coordinates. For a 3d Riemannian manifold M^3 with coordinates (α, β), $\beta = (\beta_1, \beta_2)$, and affine connectedness $\Gamma^i_{jk}(x)$, the geodesic equations on the tangent bundle $T_*M^3\{\dot\alpha,\ \dot\beta_1,\ \dot\beta_2;\ \alpha, \beta_1, \beta_2\}$, $\alpha = x^1$, $\beta_1 = x^2$, $\beta_2 = x^3$, $x = (x^1, x^2, x^3)$, have the form

$$\ddot x^i + \sum_{j,k=1}^{3} \Gamma^i_{jk}(x)\dot x^j \dot x^k = 0,\ \ i = 1, 2, 3, \tag{5.1}$$

where differentiation is in the natural parameter.

Let us examine the structure of equations (5.1) after changing the coordinates on the tangent bundle T_*M^3. Consider the following transformation of the coordinates of a point in the tangent space depending on the point x on the manifold:

$$\dot x^i = \sum_{j=1}^{3} R^{ij}(x) z_j. \tag{5.2}$$

This transformation can be inverted:

$$z_j = \sum_{i=1}^{3} T_{ji}(x)\dot x^i,$$

where $R^{ij}, T_{ji}, i, j = 1, 2, 3$, are functions of x^1, x^2, x^3 and

$$RT = E, \quad R = (R^{ij}), \quad T = (T_{ji}).$$

Equations (5.2) will be called *new kinematic relations*, i.e., relations on the tangent bundle T_*M^3.

The following identities hold:

$$\dot z_j = \sum_{i=1}^{3} \dot T_{ji}\dot x^i + \sum_{i=1}^{3} T_{ji}\ddot x^i,\ \ \dot T_{ji} = \sum_{k=1}^{3} T_{ji,k}\dot x^k, \tag{5.3}$$

where

$$T_{ji,k} = \frac{\partial T_{ji}}{\partial x^k}, \quad j, i, k = 1, 2, 3.$$

Substituting (5.1) into (5.3), we obtain

$$\dot z_i = \sum_{j,k=1}^{3} T_{ij,k}\dot x^j \dot x^k - \sum_{j,p,q=1}^{3} T_{ij}\Gamma^j_{pq}\dot x^p \dot x^q, \tag{5.4}$$

where $\dot x^i,\ i = 1, 2, 3$, should be replaced by (5.2).

Thus, (5.4) can be rewritten in the form

$$\dot z_i + \sum_{j,k=1}^{3} Q_{ijk}\dot x^j \dot x^k |_{(5.2)} = 0, \tag{5.5}$$

where

$$Q_{ijk}(x) = \sum_{s=1}^{3} T_{is}(x)\Gamma^s_{jk}(x) - T_{ij,k}(x). \tag{5.6}$$

Proposition 5.1. *In the domain where* $\det R(x) \neq 0$*, system* (5.1) *is equivalent to the composite system* (5.2)*,* (5.4)*.*

Therefore, the result of passing from geodesic equations (5.1) to the equivalent system (5.2), (5.4) depends both on the transformation of the variables (5.2) (i.e., new kinematic relations) and on the affine connectedness $\Gamma^i_{jk}(x)$.

5.2. A Fairly General Case. Consider a fairly general case of kinematic relations,

$$\dot\alpha = -z_3, \quad \dot\beta_1 = z_2 f_1(\alpha), \quad \dot\beta_2 = z_1 f_2(\alpha) g(\beta_1), \tag{5.7}$$

where $f_1(\alpha)$, $f_2(\alpha)$, $g(\beta_1)$ are smooth functions in their domains. Such coordinates z_1, z_2, z_3 on the tangent bundle are introduced when one considers the following geodesic equations (in particular, on surfaces of revolution):

$$\begin{cases} \ddot\alpha + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 = 0, \\ \ddot\beta_1 + 2\Gamma^1_{\alpha 1}(\alpha,\beta)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 = 0, \\ \ddot\beta_2 + 2\Gamma^2_{\alpha 2}(\alpha,\beta)\dot\alpha\dot\beta_2 + 2\Gamma^2_{12}(\alpha,\beta)\dot\beta_1\dot\beta_2 = 0, \end{cases} \tag{5.8}$$

i.e., the other connectedness coefficients are equal to zero. In the case (5.7), equations (5.4) take the form

$$\begin{aligned} \dot z_1 &= \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_1 z_3 - \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_2, \\ \dot z_2 &= \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_2 z_3 - \Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g^2(\beta_1) z_1^2, \\ \dot z_3 &= \Gamma^\alpha_{11} f_1^2(\alpha) z_2^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g^2(\beta_1) z_1^2, \end{aligned} \tag{5.9}$$

and equations (5.8) are almost everywhere equivalent to the composite system (5.7), (5.9) on the manifold $T_*M^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$.

For the complete integration of system (5.7), (5.9), it is necessary to know five independent first integrals, in general.

Proposition 5.2. *If the relations*

$$\begin{cases} 2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha} + \Gamma^\alpha_{11}(\alpha,\beta) f_1^2(\alpha) \equiv 0, \\ 2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha} + \Gamma^\alpha_{22}(\alpha,\beta) f_2^2(\alpha) g^2(\beta_1) \equiv 0, \\ \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right] f_1^2(\alpha) + \Gamma^1_{22}(\alpha,\beta) f_2^2(\alpha) g^2(\beta_1) \equiv 0 \end{cases} \tag{5.10}$$

hold in their domain, then system (5.7), (5.9) *has the following analytic first integral*:

$$\Phi_1(z_3, z_2, z_1) = z_1^2 + z_2^2 + z_3^2 = C_1^2 = \text{const}. \tag{5.11}$$

One can prove a separate theorem on the existence of a solution $f_1(\alpha)$, $f_2(\alpha)$, $g(\beta_1)$ for the quasilinear system (5.10), in order to obtain the analytic first integral (5.11) for system (5.7), (5.9) equivalent to geodesic equations (5.8). But it will be shown below that such arguments would make sense either for systems without a force field or systems with a potential force field. However, for systems with dissipation, conditions (5.10) acquire a somewhat different sense.

Still, in what follows, we assume that

$$f_1(\alpha) = f_2(\alpha) = f(\alpha) \tag{5.12}$$

in (5.7) and $g(\beta_1)$ satisfies the transformed third relation from (5.10):

$$2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1} + \Gamma^1_{22}(\alpha,\beta) g^2(\beta_1) \equiv 0. \tag{5.13}$$

Thus, $g(\beta_1)$ is chosen in agreement with the connectedness coefficients and the constraints on $f(\alpha)$ will be specified below.

Proposition 5.3. *If conditions* (5.12), (5.13) *hold and*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma_1(\alpha), \tag{5.14}$$

then system (5.7), (5.9) *has the following smooth first integral*:

$$\Phi_2(z_2, z_1; \alpha) = \sqrt{z_1^2 + z_2^2}\;\Phi_0(\alpha) = C_2 = \text{const}, \quad \Phi_0(\alpha) = f(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}.$$

Proposition 5.4. *If condition* (5.12) *holds,*

$$\Gamma^2_{12}(\alpha,\beta) = \Gamma_2(\beta_1), \tag{5.15}$$

and $\Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma_1(\alpha)$ (*the second relation in* (5.14)), *then system* (5.7), (5.9) *has the following smooth first integral*:

$$\Phi_3(z_1; \alpha, \beta_1) = z_1\Phi_0(\alpha)\Phi(\beta_1) = C_3 = \text{const}, \quad \Phi(\beta_1) = g(\beta_1)\exp\left\{2\int\limits_{\beta_{10}}^{\beta_1}\Gamma_2(b)db\right\}.$$

Proposition 5.5. *If conditions* (5.12), (5.13), (5.14), (5.15) *are satisfied, then system* (5.7), (5.9) *has the following first integral*:

$$\Phi_4(z_2, z_1; \beta) = \beta_2 \pm \int\limits_{\beta_{10}}^{\beta_1}\frac{C_3 g(b)}{\sqrt{C_2^2\Phi^2(b) - C_3^2}}\,db = C_4 = \text{const}, \tag{5.16}$$

where, after calculating the integral (5.16), *one should replace the constants* C_2, C_3 *with the left-hand sides of* (5.15), (5.16), *respectively.*

Under the above conditions, system (5.7), (5.9) has a complete set (four) of independent first integrals: (5.11), (5.15), (5.16), (5.16).

5.3. Equations of Motion on the Tangent Bundle of a 3D Manifold in a Potential Field of Forces. Let us modify system (5.7), (5.9) under the conditions (5.12), (5.13), (5.14), (5.15) and thus obtain a conservative system. The presence of a force field is characterized by the term $F(\alpha)$ in the second equation of system (5.17). This system on the tangent bundle $T_*M^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$ takes the form

$$\begin{cases} \dot\alpha = -z_3, \\ \dot z_3 = F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)f^2(\alpha)z_2^2 + \Gamma^\alpha_{22}(\alpha,\beta)f^2(\alpha)g^2(\beta_1)z_1^2, \\ \dot z_2 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right]z_2z_3 - \Gamma^1_{22}(\beta_1)f(\alpha)g^2(\beta_1)z_1^2, \\ \dot z_1 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right]z_1z_3 - \left[2\Gamma_2(\beta_1) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right]f(\alpha)z_1z_2, \\ \dot\beta_1 = z_2 f(\alpha), \\ \dot\beta_2 = z_1 f(\alpha)g(\beta_1), \end{cases} \tag{5.17}$$

and is almost everywhere equivalent to the following system:

$$\begin{cases} \ddot\alpha + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 = 0, \\ \ddot\beta_1 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\beta_1)\dot\beta_2^2 = 0, \\ \ddot\beta_2 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 = 0. \end{cases}$$

Proposition 5.6. *Under the conditions of Proposition* 5.2, *system* (5.17) *has the following smooth first integral*:

$$\Phi_1(z_3, z_2, z_1; \alpha) = z_1^2 + z_2^2 + z_3^2 + F_1(\alpha) = C_1 = \text{const}, \quad F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(a)da. \tag{5.18}$$

Proposition 5.7. *Under the conditions of Propositions* 5.3, 5.4, *system* (5.17) *has two smooth first integrals* (5.15), (5.16).

Proposition 5.8. *Under the conditions of Proposition* 5.5, *system* (5.17) *has the first integral* (5.16).

Under the above conditions, system (5.17) has a complete set (four) of independent first integrals: (5.18), (5.15), (5.16), (5.16).

5.4. Equations of Motion on the Tangent Bundle of a 2D Manifold in a Force Field with Dissipation. Let us modify system (5.17) to include dissipation (of changing sign, in general), which is characterized by the term $b\delta(\alpha)$ in the first equation of the system

$$\begin{cases} \dot\alpha = -z_3 + b\delta(\alpha), \\ \dot z_3 = F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta) f^2(\alpha) z_2^2 + \Gamma^\alpha_{22}(\alpha,\beta) f^2(\alpha) g^2(\beta_1) z_1^2, \\ \dot z_2 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_2 z_3 - \Gamma^1_{22}(\beta_1) f(\alpha) g^2(\beta_1) z_1^2, \\ \dot z_1 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z_1 z_3 - \left[2\Gamma_2(\beta_1) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right] f(\alpha) z_1 z_2, \\ \dot\beta_1 = z_2 f(\alpha), \\ \dot\beta_2 = z_1 f(\alpha) g(\beta_1), \end{cases} \tag{5.19}$$

which is almost everywhere equivalent to

$$\begin{cases} \ddot\alpha - b\dot\alpha\delta'(\alpha) + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 = 0, \\ \ddot\beta_1 - b\dot\beta_1\delta(\alpha) f(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\beta_1)\dot\beta_2^2 = 0, \\ \ddot\beta_2 - b\dot\beta_2\delta(\alpha) f(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 = 0. \end{cases}$$

Let us integrate the 6th-order system (5.19) under the conditions (5.13) and

$$\Gamma^\alpha_{11}(\alpha,\beta) = \Gamma^\alpha_{22}(\alpha,\beta) g^2(\beta_1) = \Gamma_3(\alpha). \tag{5.20}$$

We introduce (by analogy with (5.13)) a constraint on $f(\alpha)$: this function should satisfy the first relation in (5.10):

$$2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha} + \Gamma_3(\alpha) f^2(\alpha) \equiv 0. \tag{5.21}$$

For its complete integration, it is necessary to know five independent first integrals, in general. However, after the following transformation of the variables:

$$z_1, z_2 \to z, z_*, \quad z = \sqrt{z_1^2 + z_2^2}, \quad z_* = \frac{z_2}{z_1},$$

system (5.19) splits as follows:

$$\begin{cases} \dot\alpha = -z_3 + b\delta(\alpha), \\ \dot z_3 = F(\alpha) + \Gamma_3(\alpha) f^2(\alpha) z^2, \\ \dot z = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] z z_3, \end{cases} \tag{5.22}$$

$$
\begin{cases}
\dot z_* = \pm z\sqrt{1+z_*^2}\, f(\alpha)\left[2\Gamma_2(\beta_1) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right], \\
\dot\beta_1 = \pm \dfrac{z z_*}{\sqrt{1+z_*^2}} f(\alpha),
\end{cases} \tag{5.23}
$$

$$
\dot\beta_2 = \pm \frac{z}{\sqrt{1+z_*^2}} f(\alpha) g(\beta_1). \tag{5.24}
$$

It can be seen that for the complete integration of system (5.22)–(5.24), it suffices to find two independent first integrals of system (5.22), one integral for system (5.23), and an additional first integral "attaching" equation (5.24) (four integrals altogether).

Theorem 5.1. *Suppose that for some* $\kappa, \lambda \in \mathbf{R}$ *, we have*

$$
\Gamma_3(\alpha) f^2(\alpha) = \kappa \frac{d}{d\alpha} \ln|\delta(\alpha)|, \quad F(\alpha) = \lambda \frac{d}{d\alpha}\frac{\delta^2(\alpha)}{2}. \tag{5.25}
$$

Then system (5.19) *with the conditions* (5.13), (5.20), (5.21) *has a complete set* (*four*) *independent transcendental* (*in general*) *first integrals.*

We start with associating to the third-order system (5.22) the following nonautonomous second-order system:

$$
\frac{dz_3}{d\alpha} = \frac{F(\alpha) + \Gamma_3(\alpha) f^2(\alpha) z^2}{-z_3 + bg(\alpha)}, \quad \frac{dz}{d\alpha} = \frac{\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] z z_3}{-z_3 + bg(\alpha)}. \tag{5.26}
$$

Introducing homogeneous variables

$$
z_3 = u_3 \delta(\alpha), \;\; z = u\delta(\alpha), \tag{5.27}
$$

we reduce system (5.26) to

$$
\begin{aligned}
\delta(\alpha)\frac{du_3}{d\alpha} + \delta'(\alpha) u_3 &= \frac{F(\alpha) + \Gamma_3(\alpha) f^2(\alpha)\delta^2(\alpha) u^2}{-u_3\delta(\alpha) + b\delta(\alpha)}, \\
\delta(\alpha)\frac{du}{d\alpha} + \delta'(\alpha) u &= \frac{\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]\delta^2(\alpha) u u_3}{-u_3\delta(\alpha) + b\delta(\alpha)},
\end{aligned} \tag{5.28}
$$

which, on account of (5.21), is almost everywhere equivalent to

$$
\begin{aligned}
\delta(\alpha)\frac{du_3}{d\alpha} &= \frac{F_3(\alpha) + \Gamma_3(\alpha) f^2(\alpha)\delta(\alpha) u^2 + \delta'(\alpha) u_3^2 - b\delta'(\alpha) u_3}{-u_3 + b}, \\
\delta(\alpha)\frac{du}{d\alpha} &= \frac{-\Gamma_3(\alpha) f^2(\alpha)\delta(\alpha) u u_3 + \delta'(\alpha) u u_3 - b\delta'(\alpha) u}{-u_3 + b},
\end{aligned} \tag{5.29}
$$

$$
F_3(\alpha) = \frac{F(\alpha)}{\delta(\alpha)}.
$$

Upon fulfilling the conditions (5.25), we reduce system (5.29) to the first-order equation

$$
\frac{du_3}{du} = \frac{\lambda + \kappa u^2 + u_3^2 - bu_3}{(1-\kappa) u u_3 - bu}. \tag{5.30}
$$

Equation (5.30) is an Abel equation [11, 13, 42]. For $\kappa = -1$, it has the following first integral:

$$
\frac{u_3^2 + u^2 - bu_3 + \lambda}{u} = C_1 = \text{const}, \tag{5.31}
$$

which in the former variables has the form

$$
\Theta_1(z_3, z; \alpha) = G_1\left(\frac{z_3}{\delta(\alpha)}, \frac{z}{\delta(\alpha)}\right) = \frac{z_3^2 + z^2 - bz_3\delta(\alpha) + \lambda\delta^2(\alpha)}{z\delta(\alpha)} = C_1 = \text{const}. \tag{5.32}
$$

Now, let us find an explicit expression for the additional first integral for the third-order system (5.22) for $\kappa = -1$. To this end, we transform the invariant relation (5.31) for $u \neq 0$ as follows:

$$\left(u_3 - \frac{b}{2}\right)^2 + \left(u - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - \lambda. \tag{5.33}$$

We see that the parameters of this invariant relation must satisfy the condition

$$b^2 + C_1^2 - 4\lambda \geq 0, \tag{5.34}$$

and the phase space of system (5.22) foliates into the family of surfaces defined by (5.33).

Thus, by virtue of (5.31), the first equation of system (5.29) with the conditions (5.21) and $\kappa = -1$ takes the form

$$\frac{\delta(\alpha)}{\delta'(\alpha)} \frac{du_3}{d\alpha} = \frac{2(\lambda - bu_3 + u_3^2) - C_1 U_1(C_1, u_3)}{-u_3 + b}, \tag{5.35}$$

where

$$U_1(C_1, u_3) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4(u_3^2 - bu_3 + \lambda)}\right\}, \tag{5.36}$$

and the integration constant C_1 is chosen from the condition (5.34).

Now, the additional first integral for system (5.22) has the following structural form:

$$\Theta_2(z_3, z; \alpha) = G_2\left(\delta(\alpha), \frac{z_3}{\delta(\alpha)}, \frac{z}{\delta(\alpha)}\right) = C_2 = \mathrm{const} \tag{5.37}$$

and, for $\kappa = -1$, it can be found from the quadrature

$$\ln|g(\alpha)| = \int \frac{(b - u_3)du_3}{2(\lambda - bu_3 + u_3^2) - C_1\left\{C_1 \pm \sqrt{C_1^2 - 4(u_3^2 - bu_3 + \lambda)}\right\}/2},$$

where

$$u_3 = \frac{z_3}{\delta(\alpha)}.$$

Here, after calculating the integral, C_1 should be replaced by the left-hand side of (5.32). The right-hand side of this relation can be expressed in terms of a finite combination of elementary functions and its left-hand side depends $\delta(\alpha)$. Therefore, the expressios of the first integrals (5.32), (5.37) in terms of finite linear combinations of elementary functions depends not only on the quadrature calculations, but also on the explicit form of the function $\delta(\alpha)$.

The first integral for system (5.23) has the form

$$\Theta_3(z_*; \beta_1) = \frac{\sqrt{1 + z_*^2}}{\Phi(\beta_1)} = C_3 = \mathrm{const}, \tag{5.38}$$

where $\Phi(\beta_1)$ is defined in (5.16). The additional first integral that "attaches" to equation (5.24) can be found by analogy with (5.16):

$$\Theta_4(z_*; \beta) = \beta_2 \pm \int_{\beta_{10}}^{\beta_1} \frac{g(b)}{\sqrt{C_3^2 \Phi^2(b) - 1}}\, db = C_4 = \mathrm{const},$$

where, after calculating the integral, one should replace C_3 by the left-hand side of (5.38).

6. Structure of First Integrals for Systems with Dissipation

If α is a 2π-periodic coordinate, then (5.22) becomes a dynamical system with variable zero mean dissipation [37]. If, in addition, $b = 0$, then this system is conservative and has two first integrals of the form (5.18), (5.15). In view of (5.25), we have

$$\Phi_1(z_3, z_2, z_1; \alpha) = z_1^2 + z_2^2 + z_3^2 + 2\int\limits_{\alpha_0}^{\alpha} F(a)da \cong z^2 + z_3^2 + \lambda\delta^2(\alpha), \tag{6.1}$$

where "$\cong$" means equal to within an additive constant. Relations (5.21), (5.25) ensure that

$$\Phi_2(z_2, z_1; \alpha) = \sqrt{z_1^2 + z_2^2} f(\alpha) \exp\left\{2\int\limits_{\alpha_0}^{\alpha} \Gamma_1(b)db\right\} \cong z\delta(\alpha) = C_2 = \mathrm{const}, \tag{6.2}$$

where "$\cong$" means equal to within a multiplicative constant.

Obviously, the ratio of the two first integrals (6.1), (6.2) (or (5.18), (5.15)) is also a first integral of system (5.22) with $b = 0$. For $b \neq 0$, neither the function

$$z^2 + z_3^2 - bz_3\delta(\alpha) + \lambda\delta^2(\alpha) \tag{6.3}$$

nor (6.2) separately is a first integral of system (5.22). However, the ratio of the functions (6.3) and (6.2) is a first integral of system (5.22) for $\kappa = -1$ and any b.

In general, for systems with dissipation, the fact that their first integrals are expressed by transcendental functions (in the sense that these functions have essential singularities) is due to the existence of attractive or repelling limit sets [38].

7. Conclusion

By analogy with smaller dimensional cases, we point out two essential cases of functions $f(\alpha)$ that determine the metric on the sphere:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha}, \tag{7.1}$$

$$f(\alpha) = \frac{1}{\cos\alpha\sin\alpha}. \tag{7.2}$$

Case (7.1) forms a class of systems corresponding to the motion of a dynamically symmetric four-dimensional rigid body on the zero-level of cyclic integrals in a nonconservative (in general) field of forces [39]. Case (7.2) forms a class of systems corresponding to the motion of a material point in a nonconservative (in general) field of forces [40]. In particular, for $\delta(\alpha) \equiv F(\alpha) \equiv 0$, the system under consideration describes a geodesic flow on a 3d sphere. In case (7.1), the condition

$$\delta(\alpha) = \frac{F(\alpha)}{\cos\alpha},$$

corresponds to a system describing spatial motion of a four-dimensional rigid body in a force field $F(\alpha)$ with a following force [41]. Thus, if $F(\alpha) = \sin\alpha\cos\alpha$, $\delta(\alpha) = \sin\alpha$, then the system also describes a generalized four-dimensional spherical pendulum in a nonconservative force field and has a complete set of transcendental first integrals expressed in terms of finite combinations of elementary functions.

If the function $\delta(\alpha)$ is nonperiodic, then the dissipative system under consideration is a system with variable nonzero mean dissipation (the system is dissipative in the proper sense). Nevertheless, in this case, too, one can find explicit expressions for transcendental first integrals in terms of finite combinations of elementary functions. The latter is a new nontrivial case of integrating dissipative systems in explicit form.

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