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NON-KEPLERIAN BEHAVIOR AND INSTABILITY OF MOTION OF TWO BODIES CAUSED BY THE FINITE VELOCITY OF GRAVITATION

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It is shown that the motion of two bodies described with regard for the finite velocity of gravitation does not obey the Kepler laws and that this motion is unstable.

1. Introduction

The present paper is devoted to the investigation of motion of two bodies under the action of the law of gravitation with regard for the finite velocity of gravitation.

Since the influence of gravitation of one body upon another body cannot be instantaneous and a certain time is required for the gravitational field to travel the distance between the bodies, it is quite natural that the mathematical models of motion of the systems of two bodies should be described by the systems of equations with aftereffect. For the investigation of these systems, it seems to be most convenient to use the mathematical tools based on the theory of differential equations with delayed argument.

By using these equations, it becomes possible to show that the motion of two bodies in the real space does not obey the Kepler laws and that this motion is unstable.

2. Problem of Two Bodies in the Classical Celestial Mechanics

Since the discovery of the law of gravitation by Newton (1643–1727) published in his famous "Philosophiae Naturalis Principia Mathematica" in 1687, the investigation of motion of the bodies was carried out by using ordinary differential equations because it was assumed that the velocity of gravitation is infinite and the gravitational field instantaneously propagates from the source independently of the distance.

The problem of two bodies is the simplest problem of classical celestial mechanics. By the second Newton's law and Newton's law of gravitation, the differential equations of motion of the bodies used in this problem take the following form in a fixed Cartesian coordinate system:

$$\begin{cases} m_1\ddot{\vec{r}}_1(t) = -\dfrac{Gm_1m_2}{\left|\vec{r}_2(t)-\vec{r}_1(t)\right|^3}\,\left(\vec{r}_1(t)-\vec{r}_2(t)\right), \\ m_2\ddot{\vec{r}}_2(t) = -\dfrac{Gm_1m_2}{\left|\vec{r}_2(t)-\vec{r}_1(t)\right|^3}\,\left(\vec{r}_2(t)-\vec{r}_1(t)\right), \end{cases} \tag{1}$$

where G is the gravitational constant, m_1 and m_1 are the masses of bodies, and $|\vec{r}_2(t)-\vec{r}_1(t)|$ is the Euclidean length of the vector $\vec{r}_2(t)-\vec{r}_1(t)$. It is clear that the equations of this system can be divided by m_1 and m_2, respectively.

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Passing from the coordinates $\vec{r}_1$ to $\vec{r}_2$ to the absolute coordinates of the Newton center of inertia (center of mass) of the system of bodies and the coordinates of the first body relative to the second body, i.e.,

$$\vec{r}_0 = \frac{m_1}{m_1 + m_2}\vec{r}_1 + \frac{m_2}{m_1 + m_2}\vec{r}_2, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2,$$

we reduce system (1) to the following two simple systems

$$\ddot{\vec{r}}_0 = 0, \tag{2}$$

$$\ddot{\vec{r}} = -\frac{G(m_1 + m_2)}{|\vec{r}|^3}\vec{r}. \tag{3}$$

As follows from Eq. (2), the motion of the center of inertia of the system of two bodies is uniform and rectilinear. System (3) describes the motion of a body with mass m_1 relative to the central body with mass m_2. This system was studied in numerous works (see, e.g., [1–3]).

The general solution of the problem of two bodies was found by Newton. He also gave the geometric interpretation of the solution (the trajectories of motion of one body relative to the other and relative to the center of mass are canonical sections).

3. Kepler Laws

The kinematic behavior of motion of the bodies (in particular, planets) in the classical celestial mechanics is described by the following three Kepler laws [4, p. 138]:

1. The orbit of each planet is an ellipse and the Sun is located at one of its foci.
2. The line joining a planet and the Sun sweeps out identical areas for the identical time intervals.
3. The ratio of the squared periods of rotation of the planets around the Sun is equal to the ratio of the cubes of major semiaxes of their orbits.

Note that Newton established his law of gravity by analyzing these regularities [3, p. 42].

The application of Kepler's laws to the investigation of motion of celestial bodies with regard for the finite velocity of gravitation leads to certain errors.

4. Principle of Delay of the Gravitational Field and the Mathematical Model of Motion of Two Bodies with Regard for the Finite Velocity of Gravitation

In the real world, the velocity of gravitation cannot be infinite as in Newton's theory. This statement agrees with the Einstein theory of relativity according to which the velocity of gravitation is equal to the velocity of light and with the Kopeikin and Fomalont results concerning the fundamental limit of the velocity of gravitation [5]. On the basis of this property of gravitation, it is possible to construct a mathematical model of motion of two bodies based on the differential equations with delayed argument instead of the ordinary differential equations, as in the classical celestial mechanics, and establish new properties of motion of these bodies.

To explain the influence of delay of the gravitational field, we consider the interaction of two points M_1 and M_2 with masses m_1 and m_2, respectively. The motion of these points is described in the inertial rectangular coordinate system $x,\ y,\ z$ centered at the point O. The locations of points M_1 and M_2 at time t are specified by the radius vectors $\vec{r}_i(t),\ i = \overline{1,2}$.

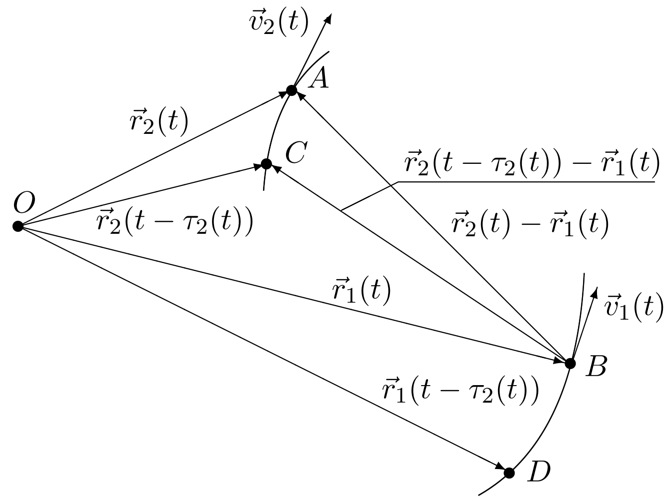

Fig. 1. Locations of the points M_1 and M_2 at the times t and $t-\tau_2(t)$.

If the velocity of gravitation is infinite (as in the Newton theory), then, according to the law of gravity, a point M_2 attracts a point M_1 at time t with a force

$$\vec{F}(t) = \frac{Gm_1m_2}{\left|\vec{r}_2(t)-\vec{r}_1(t)\right|^3}\left(\vec{r}_2(t)-\vec{r}_1(t)\right) \tag{4}$$

(the direction of this force coincides with the direction of the vector $\vec{r}_2(t)-\vec{r}_1(t)$ [2, 3]).

However, in view of the finite velocity of gravitation, the force applied to the point M_1 is different

$$\vec{F}_1(t) = \frac{Gm_1m_2}{|\vec{r}_2(t-\tau_2(t))-\vec{r}_1(t)|^3}\left(\vec{r}_2(t-\tau_2(t))-\vec{r}_1(t)\right). \tag{5}$$

The delay of gravitation $\tau_2(t)$ in (5) is determined as follows:

$$c\tau_2(t) = \left|\vec{r}_2(t-\tau_2(t))-\vec{r}_1(t)\right|, \tag{6}$$

where c is the velocity of gravitation. Indeed, assume that the points M_2 and M_1 move along the corresponding trajectories (parts of these trajectories are depicted in Fig. 1) with velocities $\vec{v}_2(t) = \dot{\vec{r}}_2(t)$ and $\vec{v}_1(t) = \dot{\vec{r}}_1(t)$, respectively, and that, at the time $t-\tau_2(t)$, where $\tau_2(t)$ satisfies (6), they are located at the points C and D, respectively. For the time $[t-\tau_2(t), t]$, the point M_2 moves from the point C to the point A, while the point M_1 moves from the point D to the point B. This time interval is sufficient for the gravitational field to propagate with velocity c from the point C to the point B. Hence, at time t, the force acting upon the point B is described by relation (5) but not by (4).

Similarly, in view of the finite velocity of gravitation, the point M_2 is subjected to the action of the force

$$\vec{F}_2(t) = \frac{Gm_1m_2}{\left|\vec{r}_1(t-\tau_1(t))-\vec{r}_2(t)\right|^3}\left(\vec{r}_1(t-\tau_1(t))-\vec{r}_2(t)\right). \tag{7}$$

The delay of gravitation $\tau_1(t)$ in (7) is determined as follows:

$$c\tau_1(t) = \left|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right|. \tag{8}$$

The existence of the functions $\tau_2(t)$ and $\tau_1(t)$ satisfying relations (6) and (7) was proved in [6]. By the theorems on implicit function [7, pp. 449–453], these functions are continuous and differentiable.

In view of this reasoning, the second Newton's law, the law of gravity, and relations (5) and (7), we arrive at the system of equations

$$\begin{cases} m_1\ddot{\vec{r}}_1(t) = \dfrac{Gm_1m_2}{\left|\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right|^3}\left(\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right), \\ m_2\ddot{\vec{r}}_2(t) = \dfrac{Gm_2m_1}{\left|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right|^3}\left(\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right), \end{cases} \tag{9}$$

which describes the motion of the points M_1 and M_2 with masses m_1 and m_2, respectively. This is a system of differential equations with delayed argument (the delays $\tau_2(t)$ and $\tau_1(t)$ depend on the locations of the points M_1 and M_2 in the space and on the velocity of gravitation) and noticeably differs from system (1).

Elements of the theory of differential equations with delayed argument can be found in [8–11].

Dividing the equations of system (9) by the nonzero masses m_1 and m_2, respectively, we get

$$\begin{cases} \ddot{\vec{r}}_1(t) = \dfrac{Gm_2}{\left|\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right|^3}\left(\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right), \\ \ddot{\vec{r}}_2(t) = \dfrac{Gm_1}{\left|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right|^3}\left(\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right). \end{cases} \tag{10}$$

For the complete description of motion of the points M_1 and M_2, system (10) should be supplemented with additional initial or boundary conditions, as this is done in the problems presented in what follows.

Problem 1. We fix arbitrary values of time t_0 and vector functions $\vec{\varphi}_{0,i}(s)$ and $\vec{\varphi}_{1,i}(s)$, $i = \overline{1,2}$, continuous on the segments $[t_0 - \tau_i(t_0), t_0]$, $i = \overline{1,2}$, respectively. It is necessary to find the solutions $\vec{r}_i(t)$, $i = \overline{1,2}$, of system (10) satisfying the initial conditions

$$\begin{cases} \vec{r}_i(s) = \vec{\varphi}_{0,i}(s), \quad s \in [t_0 - \tau_i(t_0), t_0], \\ \dot{\vec{r}}_i(s) = \vec{\varphi}_{1,i}(s), \quad s \in [t_0 - \tau_i(t_0), t_0], \end{cases} \qquad i = \overline{1,2}. \tag{11}$$

Problem 2. Let t_1 and t_2 be arbitrary times such that $t_1 < t_2 - \tau_i(t_2)$ for $i = \overline{1,2}$. We consider twice continuously differentiable vector functions $\vec{\psi}_{1,i}(s)$ and $\vec{\psi}_{2,i}(s)$, $i = \overline{1,2}$, on the segments $[t_1 - \tau_i(t_1), t_1]$ and $[t_2 - \tau_i(t_2), t_2]$, $i = \overline{1,2}$, respectively. It is necessary to find the solutions $\vec{r}_i(t)$, $i = \overline{1,2}$, of system (10) satisfying the conditions

$$\begin{cases} \vec{r}_i(s_1) = \vec{\psi}_{1,i}(s_1), \quad s_1 \in [t_1 - \tau_i(t_1), t_1], \\ \vec{r}_i(s_2) = \vec{\psi}_{2,i}(s_2), \quad s_2 \in [t_2 - \tau_i(t_2), t_2], \end{cases} \qquad i = \overline{1,2}. \tag{12}$$

The system of equations (6), (8), and (10), together with conditions (11) or (12), can be regarded as a mathematical model of motion of two bodies with regard for the finite velocity of gravitation.

Thus, the *principle of delay of the gravitational field* considered above is very important for the construction of the mathematical model of two bodies. According to this principle, at time t, the point M_1 (point B) is attracted not to the point M_2 (point A) but to the point C that coincides with the point M_2 at the time $t - \tau_2(t)$, where $\tau_2(t)$ satisfies (6). The force acting upon the point M_1 is given by relation (5).

5. Some Consequences of the Principle of Delay of Gravitational Field and the Main Aim of the Present Paper

The formulations of Kepler's laws in celestial mechanics should be corrected if we take into account the finite velocity of gravitation.

Indeed, according to the principle of delay of the gravitational field, in the course of motion of a planet around the Sun, the attracting point for a planet at time t (t is an arbitrary point of time) is not Sun's center as in the first Kepler law but a different point of the space, where Sun's center was located at time $t - \tau(t)$, where $\tau(t)$ is a delay depending on the locations of the planet and the Sun and similar to the delays $\tau_2(t)$ and $\tau_1(t)$ given by relations (6) and (8), respectively. Hence, the revolution of the planet at time t is realized not around the current Sun's center but around the point where it was located at time $t - \tau(t)$.

Thus, *the motion of planets around the Sun does not obey the first Kepler law*, which has already been indicated in [6].

The results of simple calculations carried out in [6] show that each planet moves (revolves) around "its own" attracting point that does not coincide with Sun's center and all these points are pairwise different.

In this connection, it is important to analyze applicability of Kepler's laws in the nonclassical celestial mechanics.

Thus, the aim of the present paper is to show that:

1. The motion of two bodies also does not obey the second and third Kepler laws.

2. For the distance $d(M_1, M_2)$ between the points M_1 and M_2, we have either

$$\liminf_{t \to +\infty} d(M_1, M_2) = 0$$

(in this case, we observe a collision of two bodies with nonzero sizes for a finite period of time) or

$$\limsup_{t \to +\infty} d(M_1, M_2) = +\infty.$$

3. The trajectories of motion of the bodies are unstable.

These statements are substantiated in Secs. 7–11.

6. Law of Increase in the Sector Velocity

We now consider one property of motion of the material points important for our subsequent representation.

We use the system of equations (10) that describes the motion of the points M_1 and M_2 with masses m_1 and m_2, respectively, with regard for the finite velocity of gravitation. The locations of these points are specified by the vector functions $\vec{r}_1(t)$ and $\vec{r}_2(t)$ and their velocities are given by the functions $\vec{v}_1(t) = \dot{\vec{r}}_1(t)$ and $\vec{v}_2(t) = \dot{\vec{r}}_2(t)$, respectively.

Assume that trajectories of motion of the points M_1 and M_2 lie in a certain plane E.

We describe the behavior of a vector function

$$\vec{v}_\sigma(t) = \frac{1}{2}\left(\vec{r}_1(t) - \vec{r}_2(t)\right) \times \left(\dot{\vec{r}}_1(t) - \dot{\vec{r}}_2(t)\right),$$

where $\times$ is the vector product of the corresponding vectors. Note that this function is the *sector velocity of motion of the point* M_1 *relative to the point* M_2 *at time* t. In the classical mechanics, we have $\vec{v}_\sigma(t) \equiv \vec{v}_\sigma(t_0)$ (i.e., the sector velocity is constant; see [4, p. 134]). Here, t_0 is an arbitrary (fixed) time.

By using the properties of the vector product and the equations of system (10), we conclude that

$$\begin{aligned}
2\,\frac{d\vec{v}_\sigma(t)}{dt} &= \left(\dot{\vec{r}}_1(t) - \dot{\vec{r}}_2(t)\right) \times \left(\dot{\vec{r}}_1(t) - \dot{\vec{r}}_2(t)\right) + \left(\vec{r}_1(t) - \vec{r}_2(t)\right) \times \left(\ddot{\vec{r}}_1(t) - \ddot{\vec{r}}_2(t)\right) \\
&= \left(\vec{r}_1(t) - \vec{r}_2(t)\right) \times \left(\ddot{\vec{r}}_1(t) - \ddot{\vec{r}}_2(t)\right) \\
&= \left(\vec{r}_1(t) - \vec{r}_2(t)\right) \times \left(\frac{Gm_2}{\left|\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right|^3}\left(\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right)\right. \\
&\qquad \left. - \frac{Gm_1}{\left|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right|^3}\left(\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right)\right) \\
&= \left(\vec{r}_1(t) - \vec{r}_2(t)\right) \\
&\qquad \times \left(\frac{Gm_2}{\left|\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right|^3}\left((\vec{r}_2(t-\tau_2(t)) - \vec{r}_2(t)) - (\vec{r}_1(t) - \vec{r}_2(t))\right)\right. \\
&\qquad \left. - \frac{Gm_1}{\left|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)\right|^3}\left((\vec{r}_1(t-\tau_1(t)) - \vec{r}_1(t)) + (\vec{r}_1(t) - \vec{r}_2(t))\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d\vec{v}_\sigma(t)}{dt} &= \frac{1}{2}\left(\vec{r}_1(t) - \vec{r}_2(t)\right) \times \left(\frac{Gm_2}{\left|\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)\right|^3}\left(\vec{r}_2(t-\tau_2(t)) - \vec{r}_2(t)\right)\right. \\
&\qquad \left. + \frac{Gm_1}{|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)|^3}\left(\vec{r}_1(t) - \vec{r}_1(t-\tau_1(t))\right)\right).
\end{aligned} \tag{13}$$

By using the vectors

$$\overrightarrow{M_2M_1} = \vec{r}_1(t) - \vec{r}_2(t), \qquad \overrightarrow{M_1^*M_1} = \vec{r}_1(t) - \vec{r}_1(t-\tau_1(t)), \qquad \text{and} \qquad \overrightarrow{M_2M_2^*} = \vec{r}_2(t-\tau_2(t)) - \vec{r}_2(t),$$

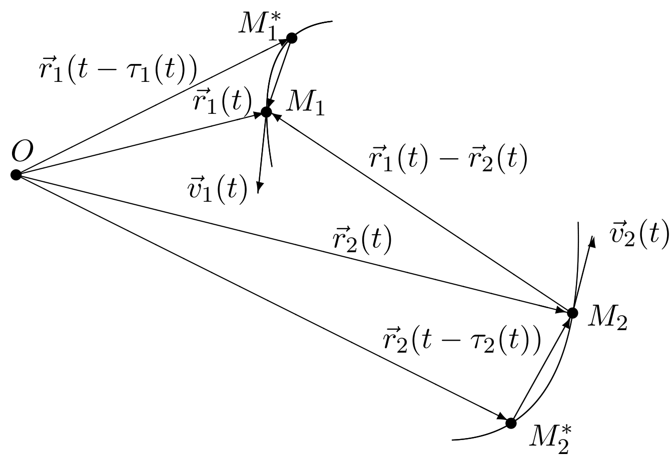

Fig. 2. Trajectories of motion of the points M_1 and M_2 and the velocities of these points.

where M_1^* and M_2^* are points of the space at which the points M_1 and M_2 are located at the times $t-\tau_1(t)$ and $t-\tau_2(t)$, respectively (Fig. 2), and relations (5)–(8), we obtain

$$\frac{d\vec{v}_\sigma(t)}{dt} = \frac{1}{2}\left(\vec{r}_1(t)-\vec{r}_2(t)\right)\times\left(\frac{1}{m_1 c\tau_2(t)}\left|\vec{F}_1(t)\right|\overrightarrow{M_2M_2^*}+\frac{1}{m_2 c\tau_1(t)}\left|\vec{F}_2(t)\right|\overrightarrow{M_1^*M_1}\right). \tag{14}$$

Let $\tau(t)=\max\{\tau_1(t),\tau_2(t)\}$. We analyze the case of validity of the relation

$$\vec{v}_\sigma(s)\neq\vec{0} \qquad \text{for all} \quad s\in[t_0-\tau(t_0),t_0]. \tag{15}$$

Note that, for the used inertial coordinate system, we can always choose another inertial coordinate system in which the angle between the vectors $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ belongs to the interval $(\pi/2,\pi)$ (as in Fig. 2). To do this, it is necessary to perform a proper transformation of the direction of constant velocity of motion of the point O. In passing to a different inertial coordinate system, the sector velocity of motion of the point M_1 relative to the point M_2 remains constant because the vectors $\vec{r}_1(t_0)-\vec{r}_2(t_0)$ and $\vec{v}_1(t_0)-\vec{v}_2(t_0)$ are invariant under transformations of inertial coordinate systems.

In Fig. 2, M_1^* and M_2^* are points of the space where the points M_1 and M_2 are located at the times $t-\tau_1(t)$ and $t-\tau_2(t)$, respectively.

For our subsequent representation, we need some notation. Let L_{M_1,M_2} be the straight line passing through the points M_1 and M_2, let $E^+_{M_1,M_2}$ be a half plane that contains the end point of the vector $\vec{v}_1(t)$ (with origin at the point M_1) but does not contain points of the straight line L_{M_1,M_2}, and let

$$E^-_{M_1,M_2}=E\setminus\left(E^+_{M_1,M_2}\bigcup L_{M_1,M_2}\right)$$

be a half plane that contains the end point of the vector $\vec{v}_2(t)$ (with origin at the point M_2) but does not contain points of the straight line L_{M_1,M_2}.

We study the motion of the points M_1 and M_2 under the assumption that the following condition is satisfied:

Condition A. *For all $t\geq t_0$, the end point of the vector $\vec{v}_1(t)$ and the point M_1^* lie on $E^+_{M_1,M_2}$ and $E^-_{M_1,M_2}\bigcup L_{M_1,M_2}$, respectively, while the end point of the vector $\vec{v}_2(t)$ and the point M_2^* lie on $E^-_{M_1,M_2}$*

and $E^{+}_{M_1,M_2} \bigcup L_{M_1,M_2}$, *respectively. Moreover, the trajectories of motion of the points* M_1 *and* M_2 *are not subsets of the same straight line.*

The requirement of validity of this condition is natural. Thus, for planets of the Solar system and the Sun, this condition is satisfied due to small velocities of the planets and the Sun as compared with the velocity of gravitation c [12] and the closeness of their trajectories of motion to elliptic trajectories.

Note that the trajectories of motion of the points M_1 and M_2 do not have points of inflection because the motion of each point is realized under the action of nonzero forces. Hence, these trajectories are convex, except the case where the points move along a straight line. This property of trajectories simplifies the investigation of motion of the points M_1 and M_2.

It is clear that, under the condition A, the pairs of vectors $\overrightarrow{M_2M_1}$ and $\vec{v}_1(t)-\vec{v}_2(t)$, $\overrightarrow{M_2M_1}$ and $\overrightarrow{M_1^*M_1}$, and $\overrightarrow{M_2M_1}$ and $\overrightarrow{M_2M_2^*}$, have the same orientation for $t \geq t_0$ (in Fig. 2, these pairs of vectors are right).

By using relations (13) and (14) and condition A, we conclude that the following statement holds:

Proposition 1 (law of increase in the sector velocity)**.** *If relation* (15) *is true and the condition A is satisfied, then the sector velocity* $\vec{v}_\sigma(t)$ *of motion of the point* M_1 *relative to the point* M_2 *is nonzero for all* $t > t_0$ *[according to* (14)*]. Moreover, this velocity is strictly increasing.*

How to describe the motion of points M_1 and M_2 if

$$\vec{v}_\sigma(s) = \vec{0} \quad \text{for all} \quad s \in [t_0 - \tau(t_0), t_0]? \tag{16}$$

In this case, $\vec{v}_1(s) - \vec{v}_2(s) = \vec{0}$ or $\vec{v}_1(s) - \vec{v}_2(s) \neq \vec{0}$ and the vectors $\vec{r}_1(s) - \vec{r}_2(s)$ and $\vec{v}_1(s) - \vec{v}_2(s)$ are collinear for any $s \in [t_0 - \tau(t_0), t_0]$. This implies that, at time t_0, the points M_1, M_1^*, M_2, and M_2^* lie on the same straight line, which passes through the end points of the vectors $\vec{r}_1(t_0)$ and $\vec{r}_2(t_0)$ whose origins coincide with the point O. Thus, in view of (13) and (14), the subsequent motion of the points M_1 and M_2 under the action of gravity forces is realized along the indicated straight line.

The following statement is true:

Proposition 2. *If relation* (16) *is true, then the sector velocity* $\vec{v}_\sigma(t)$ *of motion of the point* M_1 *relative to the point* M_2 *is nonzero at any time* $t \geq t_0$ *and the points* M_1 *and* M_2 *move along a straight line.*

Note that, for the sector velocity $\vec{v}_\sigma(t)$ of motion of the point M_1 relative to the point M_2, the relation $\vec{v}_\sigma(t) \equiv \vec{v}_\sigma(t_0) \neq \vec{0}$ cannot be true. Indeed, if this relation holds, then, according to (13) and (14),

$$\frac{d\vec{v}_\sigma(t)}{dt} \equiv \vec{0}.$$

Hence, the points M_1, M_1^*, M_2, and M_2^* lie on the same straight line for all $t \geq t_0$, which contradicts the condition A.

We now clarify the physical causes of increase in the sector velocity observed in the case where the condition A is satisfied. We assume that this velocity is nonzero at the initial time t_0 and consider Fig. 3. The point M_1 is subjected to the action of the force $\vec{F}_1(t)$ described by relation (5). This force and the vector $\overrightarrow{M_1M_2^*}$ have the same direction. In a similar way, the force $\vec{F}_2(t)$ [see relation (7)] acts upon the point M_2. The direction of this force coincides with the direction of the vector $\overrightarrow{M_2M_1^*}$. Thus,

$$\vec{F}_1(t) = \vec{F}_{1,*}(t) + \vec{F}_{1,**}(t) \qquad \text{and} \qquad \vec{F}_2(t) = \vec{F}_{2,*}(t) + \vec{F}_{2,**}(t),$$

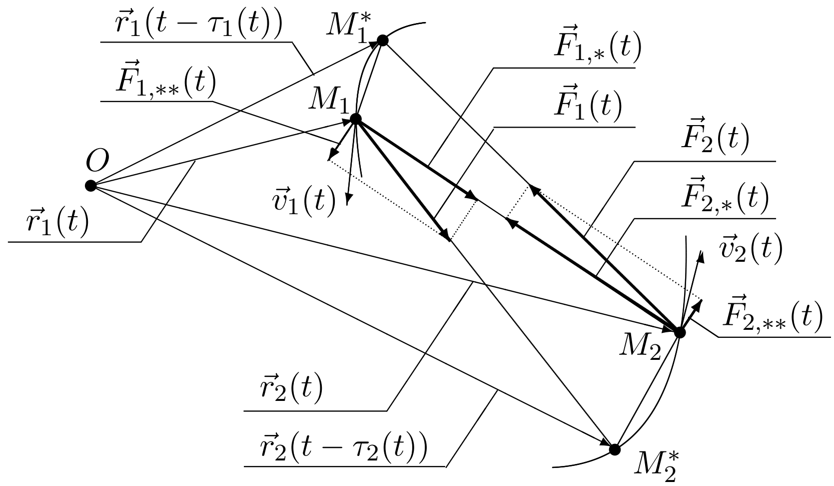

Fig. 3. Radial and transverse components of the forces $\vec{F}_1(t)$ and $\vec{F}_2(t)$.

where $\vec{F}_{1,*}(t)$, $\vec{F}_{2,*}(t)$ and $\vec{F}_{1,**}(t)$, $\vec{F}_{2,**}(t)$ are the radial and transverse components of the forces $\vec{F}_1(t)$ and $\vec{F}_2(t)$, respectively (the vectors $\vec{F}_{1,*}(t)$ and $\vec{F}_{2,*}(t)$ are collinear to the vector $\overrightarrow{M_1M_2}$, while the vectors $\vec{F}_{1,**}(t)$ and $\vec{F}_{2,**}(t)$ are orthogonal to this vector and make acute angles with the vectors $\vec{v}_1(t)$ and $\vec{v}_2(t)$, respectively).

The nonzero components $\vec{F}_{1,**}(t)$ and $\vec{F}_{2,**}(t)$ of the forces $\vec{F}_1(t)$ and $\vec{F}_2(t)$ caused by the delay of the gravitational field lead to the displacements of attracting points for the points M_1 and M_2, affect the behavior of the sector velocity of motion of the point M_1 relative to the point M_2, and are responsible for the increase in the value of this velocity.

In view of the fact that

$$\frac{d(\vec{r}_1(t)-\vec{r}_2(t))\times\left(\dot{\vec{r}}_1(t)-\dot{\vec{r}}_2(t)\right)}{dt}=(\vec{r}_1(t)-\vec{r}_2(t))\times\left(\ddot{\vec{r}}_1(t)-\ddot{\vec{r}}_2(t)\right)$$

and

$$\begin{aligned}
&(\vec{r}_1(t)-\vec{r}_2(t))\times\left(\ddot{\vec{r}}_1(t)-\ddot{\vec{r}}_2(t)\right)\\
&\quad=(\vec{r}_1(t)-\vec{r}_2(t))\times\left(\frac{1}{m_1}\,\vec{F}_1(t)-\frac{1}{m_2}\,\vec{F}_2(t)\right)\\
&\quad=(\vec{r}_1(t)-\vec{r}_2(t))\times\left(\frac{1}{m_1}\left(\vec{F}_{1,*}(t)+\vec{F}_{1,**}(t)\right)-\frac{1}{m_2}\left(\vec{F}_{2,*}(t)+\vec{F}_{2,**}(t)\right)\right)\\
&\quad=(\vec{r}_1(t)-\vec{r}_2(t))\times\left(\frac{1}{m_1}\,\vec{F}_{1,**}(t)-\frac{1}{m_2}\,\vec{F}_{2,**}(t)\right)
\end{aligned}$$

(here, we have used the collinearity of the vectors $\vec{r}_1(t)-\vec{r}_2(t)$, $\frac{1}{m_1}\vec{F}_{1,*}(t)$, and $\frac{1}{m_2}\vec{F}_{2,*}(t)$), we can represent relation (13) in the form

$$\frac{d\vec{v}_\sigma(t)}{dt}=\frac{1}{2}\left(\vec{r}_1(t)-\vec{r}_2(t)\right)\times\left(\frac{1}{m_1}\vec{F}_{1,**}(t)-\frac{1}{m_2}\vec{F}_{2,**}(t)\right).$$

The obtained relation is more convenient for applications from the physical point of view than relation (13).

Since the vectors $\vec{r}_1(t)-\vec{r}_2(t)$ and $\frac{1}{m_1}\vec{F}_{1,**}(t)-\frac{1}{m_2}\vec{F}_{2,**}(t)$ are orthogonal and the sector velocity increases, we conclude that

$$\frac{d\left|\vec{v}_\sigma(t)\right|}{dt}=\frac{\left|\vec{r}_1(t)-\vec{r}_2(t)\right|}{2}\left|\frac{1}{m_1}\vec{F}_{1,**}(t)-\frac{1}{m_2}\vec{F}_{2,**}(t)\right|.$$

Moreover, the vectors $\vec{F}_{1,**}(t)$ and $-\vec{F}_{2,**}(t)$ have the same direction (Fig. 3). This yields

$$\left|\frac{1}{m_1}\vec{F}_{1,**}(t)-\frac{1}{m_2}\vec{F}_{2,**}(t)\right|=\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right|+\frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right|$$

and

$$\frac{d\left|\vec{v}_\sigma(t)\right|}{dt}=\frac{\left|\vec{r}_1(t)-\vec{r}_2(t)\right|}{2}\left(\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right|+\frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right|\right). \tag{17}$$

These relations remain true in the case where relation (16) is valid. Thus, we get

$$\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right|+\frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right|\equiv 0$$

and

$$\vec{F}_{1,**}(t)\equiv\vec{F}_{2,**}(t)\equiv\vec{0}. \tag{18}$$

Hence, the points M_1 and M_2, are subjected solely to the action of forces $\vec{F}_{1,*}(t)$ and $\vec{F}_{2,*}(t)$, respectively, and move along the straight line passing through the end points of the vectors $\vec{r}_1(t_0)$ and $\vec{r}_2(t_0)$ (the origins of these vectors coincide with the point O). Identity (18) means that the points M_1, M_2, M_1^*, and M_2^* move along the same straight line.

7. Impossibility of Motion of Two Bodies According to the Second Kepler Law

By the law of increase in the sector velocity (under the condition A), the point M_1 cannot move relative to the point M_2 according to the second Kepler law because the motion of points obeying the Kepler laws is periodic and $\vec{v}_\sigma(t)\equiv\vec{v}_\sigma(t_0)\neq\vec{0}$ but, for each periodic motion (the periodicity of the velocities of points is also taken into account), the sector velocity cannot be an increasing function.

Hence, *the Earth cannot move around the Sun according to the second Kepler law*.

How to describe the motion of the point M_1 relative to the point M_2? The answer to this question is given in the next sections.

8. The Distance Between the Points M_1 and M_2 Approaches either 0 or $+\infty$ as $t \to +\infty$ if the Sector Velocity Is Equal to Zero

We consider the velocity $\vec{v}(t) = \vec{v}_1(t) - \vec{v}_2(t)$ of motion of the point M_1 relative to the point M_2. In the general case, this velocity can be represented in the form of a sum

$$\vec{v}(t) = \vec{v}_*(t) + \vec{v}_{**}(t), \tag{19}$$

where $\vec{v}_*(t)$ is the radial component of the velocity $\vec{v}(t)$ parallel to the vector $\overrightarrow{M_2M_1}$ and $\vec{v}_{**}(t)$ is the transverse component of the velocity orthogonal to the vector $\overrightarrow{M_2M_1}$.

We study the motion of the points M_1 and M_2 in the case where relation (16) is true.

According to Proposition 2, the definition of the sector velocity, and the equalities

$$|\vec{v}_\sigma(t)| = \frac{1}{2}\left|\left(\vec{r}_1(t) - \vec{r}_2(t)\right) \times \left(\vec{v}_1(t) - \vec{v}_2(t)\right)\right| = \frac{1}{2}\left|\vec{r}_1(t) - \vec{r}_2(t)\right|\left|\vec{v}_{**}(t)\right|, \quad t \geq t_0, \tag{20}$$

we conclude that $\vec{v}_{**}(t) \equiv \vec{0}$ for all $t \geq t_0$. Therefore, $\vec{v}_1(t) - \vec{v}_2(t) \equiv \vec{v}_*(t)$ and the points M_1 and M_2 are subjected solely to the action of the forces $\vec{F}_1(t) \equiv \vec{F}_{1,*}(t)$ and $\vec{F}_2(t) \equiv \vec{F}_{2,*}(t)$, respectively (Fig. 3), because $\vec{F}_{1,**}(t) \equiv \vec{F}_{2,**}(t) \equiv \vec{0}$. Then the points M_1, M_2, M_1^*, and M_2^* lie on the same straight line (Fig. 4).

Under the action of the forces $\vec{F}_1(t)$ and $\vec{F}_2(t)$, the following motions of the point M_1 relative to the point M_2 are possible:

(i) the distance $|\vec{r}_1(t) - \vec{r}_2(t)|$ monotonically decreases for $t \geq t_0$;

(ii) the distance $|\vec{r}_1(t) - \vec{r}_2(t)|$ increases for a certain period $[t_0, t_1]$ and decreases for $t \geq t_1$;

(iii) the distance $|\vec{r}_1(t) - \vec{r}_2(t)|$ monotonically increases for $t \geq t_0$.

The behavior of the quantity $|\vec{r}_1(t) - \vec{r}_2(t)|$ depends on the vectors $\vec{v}_1(t_0) - \vec{v}_2(t_0)$ and $\vec{r}_1(t_0) - \vec{r}_2(t_0)$.

If $\vec{v}_1(t_0) - \vec{v}_2(t_0) = \vec{0}$ or $\vec{v}_1(t_0) - \vec{v}_2(t_0) \neq \vec{0}$ and the direction of the vector $\vec{v}_1(t_0) - \vec{v}_2(t_0)$ coincides with the direction of the vector $\overrightarrow{M_1M_2}$, then the motion of the point M_1 relative to the point M_2 corresponds to the first case. In this case, the quantity $|\vec{v}_1(t) - \vec{v}_2(t)|$ monotonically increases because the direction of the vector $\vec{v}_1(t) - \vec{v}_2(t)$ coincides with the direction of the vector $\vec{F}_{1,*}(t) - \vec{F}_{2,*}(t)$. We fix an arbitrarily small number $\varepsilon \in (0, |\vec{r}_1(t_0) - \vec{r}_2(t_0)|)$. Since $|\vec{v}_1(t) - \vec{v}_2(t)|$ increases, we arrive at the equality $|\vec{r}_1(t_1) - \vec{r}_2(t_1)| = \varepsilon$ at a certain time $t_1 > t_0$. Therefore, in view of the arbitrariness of the choice of ε, this yields

$$\lim_{t \to +\infty} |\vec{r}_1(t) - \vec{r}_2(t)| = 0. \tag{21}$$

If $\vec{v}_1(t_0) - \vec{v}_2(t_0) \neq \vec{0}$ and the direction of the vector $\vec{v}_1(t_0) - \vec{v}_2(t_0)$ coincides with the direction of the vector $\overrightarrow{M_2M_1}$, then the motion of the point M_1 relative to the point M_2 corresponds to the second (or third) case. Indeed, under the action of the force $\vec{F}_{1,*}(t)$ directed opposite to the vector $\overrightarrow{M_1M_2}$, the velocity of the point M_1 decreases, whereas the velocity of the point M_2 increases. The distance $d(M_1, M_2)$ between the points M_1 and M_2 increases within a certain interval (t_0, t_1) (at time t_1, the velocity $\vec{v}_1(t_1) - \vec{v}_2(t_1)$ is equal to $\vec{0}$). This motion is possible if the quantity $|\vec{v}_1(t_0) - \vec{v}_2(t_0)|$ is small (it depends on $|\vec{r}_1(t_0) - \vec{r}_2(t_0)|$). Stating from time t_1, the point M_1 moves toward the point M_2 (as in the first case). As a result, we also get relation (21).

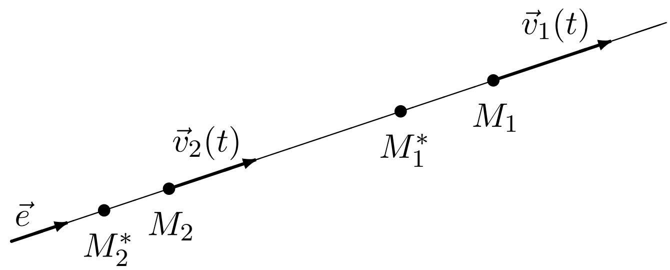

Fig. 4. Motion of the points M_1 and M_2 in the case where $\vec{v}_\sigma(t) \equiv \vec{0}$.

Note that the motion of the points M_1 and M_2 for which

$$|\vec{v}_1(t) - \vec{v}_2(t)| > 0 \quad \text{and} \quad |\vec{v}_1(t)| - |\vec{v}_2(t)| > 0 \qquad \text{for all} \quad t \in (t_0, +\infty) \tag{22}$$

(the distance between the points M_1 and M_2 monotonically increases),

$$\lim_{t\to+\infty} |\vec{v}_1(t) - \vec{v}_2(t)| = 0 \tag{23}$$

and

$$\lim_{t\to+\infty} |\vec{r}_1(t) - \vec{r}_2(t)| \in (0, +\infty), \tag{24}$$

i.e., the trajectory of motion of the point M_1 relative to the point M_2 is bounded, is impossible. Indeed, by virtue of the inequalities

$$|\vec{v}_2(t_0)| < |\vec{v}_2(t)| < |\vec{v}_1(t)| < |\vec{v}_1(t_0)| < c, \quad t > t_0, \tag{25}$$

which follow from (22) and the monotonicity of the quantities $|\vec{v}_2(t)|$ and $|\vec{v}_1(t)|$ on $[t_0, +\infty)$, the inequalities

$$\frac{c}{c - |\vec{v}_1(t_0)|} \left|\vec{r}_1(t) - \vec{r}_2(t)\right| \geq \left|\vec{r}_2(t - \tau_2(t)) - \vec{r}_1(t)\right| \geq \left|\vec{r}_1(t) - \vec{r}_2(t)\right|, \quad t \geq t_0, \tag{26}$$

$$\left|\vec{r}_1(t) - \vec{r}_2(t)\right| \geq \left|\vec{r}_1(t - \tau_1(t)) - \vec{r}_2(t)\right| \geq \frac{|\vec{r}_1(t) - \vec{r}_2(t)|}{2}, \quad t \geq t_0, \tag{27}$$

and relation (24), the following relation is true:

$$\sup_{t\geq t_0} \left\{\left|\vec{r}_1(t) - \vec{r}_2(t - \tau_2(t))\right|, \left|\vec{r}_2(t) - \vec{r}_1(t - \tau_1(t))\right|\right\} < +\infty. \tag{28}$$

Inequalities (26) follow from (25) and the fact that, for all $t \geq t_0$,

$$\left|\overrightarrow{M_1 M_2^*}\right| \geq \left|\overrightarrow{M_1 M_2}\right|$$

and

$$\left|\overrightarrow{M_1M_2^*}\right| = \left|\overrightarrow{M_1M_2}\right| + \left|\overrightarrow{M_2M_2^*}\right| \le \left|\overrightarrow{M_1M_2}\right| + \frac{|\vec{v}_2(t)|}{c}\left|\overrightarrow{M_1M_2^*}\right| \le \left|\overrightarrow{M_1M_2}\right| + \frac{|\vec{v}_1(t_0)|}{c}\left|\overrightarrow{M_1M_2^*}\right|.$$

At the same time, inequalities (27) follow from the fact that

$$\left|\overrightarrow{M_1M_2}\right| \ge \left|\overrightarrow{M_2M_1^*}\right| = \left|\overrightarrow{M_1M_2}\right| - \left|\overrightarrow{M_1M_1^*}\right| \qquad \text{and} \qquad \left|\overrightarrow{M_1M_1^*}\right| \le \tau_1(t)c = \left|\overrightarrow{M_2M_1^*}\right|$$

for all $t \ge t_0$.

By using relations (5) and (7), relation (28), and the fact that the vectors $\vec{F}_1(t)$ and $\vec{F}_2(t)$ have the opposite directions for some number $\delta > 0$, we get

$$\delta < \inf_{t\ge t_0}\left|\frac{1}{m_1}\vec{F}_1(t) - \frac{1}{m_2}\vec{F}_2(t)\right| = \inf_{t\ge t_0}\left(\frac{1}{m_1}\left|\vec{F}_1(t)\right| + \frac{1}{m_2}\left|\vec{F}_2(t)\right|\right). \tag{29}$$

In view of the equations of system (10), we conclude that, for all sufficiently large $t > t_0$,

$$\big(\vec{v}_1(t+1) - \vec{v}_2(t+1)\big) - \big(\vec{v}_1(t) - \vec{v}_2(t)\big) = \int_t^{t+1}\left(\frac{1}{m_1}\vec{F}_1(s) - \frac{1}{m_2}\vec{F}_2(s)\right)ds.$$

Therefore, in view of (29), we get

$$\left|\,\big(\vec{v}_1(t+1) - \vec{v}_2(t+1)\big) - \big(\vec{v}_1(t) - \vec{v}_2(t)\big)\,\right| > \delta$$

for all sufficiently large $t > t_0$, which is impossible according to (23).

Hence, the motion of the points M_1 and M_2 described by relations (22)–(24) is impossible.

Finally, we consider the third case of motion of the points M_1 and M_2 in which the quantity $|\vec{r}_1(t) - \vec{r}_2(t)|$ monotonically increases in the interval $[t_0, +\infty)$. By $\vec{e}$ we denote the vector whose direction coincides with the direction of the vector $\vec{v}_1(t_0) - \vec{v}_2(t_0)$ and $|\vec{e}\,| = 1$ (Fig. 4).

Further, we show that if

$$|\vec{v}_1(t_0) - \vec{v}_2(t_0)|^2 > \frac{2G(4m_1 + m_2)}{\left|\vec{r}_1(t_0) - \vec{r}_2(t_0)\right|}, \tag{30}$$

then

$$\lim_{t\to+\infty}|\vec{r}_1(t) - \vec{r}_2(t)| = +\infty. \tag{31}$$

By using the equations of system (10), we obtain

$$(\vec{v}_1(t) - \vec{v}_2(t)) \equiv (\vec{v}_1(t_0) - \vec{v}_2(t_0)) + \int_{t_0}^{t}\left(\frac{1}{m_1}\vec{F}_1(s) - \frac{1}{m_2}\vec{F}_2(s)\right)ds. \tag{32}$$

According to the restrictions imposed on $\vec{r}_1(t)-\vec{r}_2(t),\ \vec{v}_1(t)-\vec{v}_2(t),\ \vec{F}_1(t),\ \vec{F}_2(t),$ and $\vec{e},$ we find

$$\vec{r}_1(t)-\vec{r}_2(t)\equiv|\vec{r}_1(t)-\vec{r}_2(t)|\vec{e},$$

$$\vec{v}_1(t)-\vec{v}_2(t)\equiv|\vec{v}_1(t)-\vec{v}_2(t)|\vec{e},$$

and

$$\frac{1}{m_1}\vec{F}_1(t)-\frac{1}{m_2}\vec{F}_2(t)\equiv-\left|\frac{1}{m_1}\vec{F}_1(t)-\frac{1}{m_2}\vec{F}_2(t)\right|\vec{e}\equiv-\left(\frac{1}{m_1}\left|\vec{F}_1(t)\right|+\frac{1}{m_2}\left|\vec{F}_2(t)\right|\right)\vec{e}.$$

Thus, in view of (32), we get

$$|\vec{v}_1(t)-\vec{v}_2(t)|\equiv|\vec{v}_1(t_0)-\vec{v}_2(t_0)|-\int_{t_0}^{t}\left(\frac{1}{m_1}\left|\vec{F}_1(s)\right|+\frac{1}{m_2}\left|\vec{F}_2(s)\right|\right)ds.$$

Since the function $\frac{1}{m_1}\left|\vec{F}_1(t)\right|+\frac{1}{m_2}\left|\vec{F}_2(t)\right|$ is continuous on $[t_0,+\infty),$ we obtain

$$\frac{d|\vec{v}_1(t)-\vec{v}_2(t)|}{dt}\equiv-\frac{1}{m_1}\left|\vec{F}_1(t)\right|-\frac{1}{m_2}\left|\vec{F}_2(t)\right|.$$

This yields

$$\frac{d|\vec{v}_1(t)-\vec{v}_2(t)|^2}{dt}\equiv-2\left(\frac{1}{m_1}\left|\vec{F}_1(t)\right|+\frac{1}{m_2}\left|\vec{F}_2(t)\right|\right)|\vec{v}_1(t)-\vec{v}_2(t)|$$

and

$$|\vec{v}_1(t)-\vec{v}_2(t)|^2\equiv|\vec{v}_1(t_0)-\vec{v}_2(t_0)|^2 -2\int_{t_0}^{t}\left(\frac{1}{m_1}\left|\vec{F}_1(s)\right|+\frac{1}{m_2}\left|\vec{F}_2(s)\right|\right)|\vec{v}_1(s)-\vec{v}_2(s)|\,ds. \tag{33}$$

We now estimate the integral

$$\int_{t_0}^{t}\left(\frac{1}{m_1}\left|\vec{F}_1(s)\right|+\frac{1}{m_2}\left|\vec{F}_2(s)\right|\right)|\vec{v}_1(s)-\vec{v}_2(s)|\,ds$$

from above. By using relations (5) and (7), inequalities (26) and (27), and the formula

$$|\vec{v}_1(s)-\vec{v}_2(s)|\,ds=d|\vec{r}_1(s)-\vec{r}_2(s)|,\quad s\ge t_0,$$

we find

$$\frac{1}{m_1}\left|\vec{F}_1(t)\right| = \frac{Gm_2}{|\vec{r}_1(t) - \vec{r}_2(t-\tau_2(t))|^2} \le \frac{Gm_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^2},$$

$$\frac{1}{m_2}\left|\vec{F}_2(t)\right| = \frac{Gm_1}{|\vec{r}_2(t) - \vec{r}_1(t-\tau_1(t))|^2} \le \frac{4Gm_1}{|\vec{r}_1(t) - \vec{r}_2(t)|^2},$$

and

$$\int\limits_{t_0}^{t}\left(\frac{1}{m_1}\left|\vec{F}_1(s)\right| + \frac{1}{m_2}\left|\vec{F}_2(s)\right|\right)|\vec{v}_1(s) - \vec{v}_2(s)|\,ds \le \left(\frac{G(4m_1+m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|} - \frac{G(4m_1+m_2)}{|\vec{r}_1(t) - \vec{r}_2(t)|}\right).$$

By using these results and relation (33), we get

$$|\vec{v}_1(t) - \vec{v}_2(t)|^2 \ge |\vec{v}_1(t_0) - \vec{v}_2(t_0)|^2 - \frac{2G(4m_1+m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|}.$$

In the case where inequality (30) is true, we obtain

$$\inf_{s\ge t_0}|\vec{v}_1(s) - \vec{v}_2(s)| > 0.$$

Since

$$|\vec{r}_1(t) - \vec{r}_2(t)| \ge |\vec{r}_1(t_0) - \vec{r}_2(t_0))| + (t-t_0)\inf_{s\ge t_0}|\vec{v}_1(s) - \vec{v}_2(s)|, \quad t \ge t_0,$$

relation (31) is true.

Thus, in the case where inequality (30) is true, the distance between the points M_1 and M_2 tends to $+\infty$ as $t \to +\infty$.

Thus, we have proved the assertion that the distance between the points M_1 and M_2 tends either to 0 or to $+\infty$ as $t \to +\infty$ in the case where the sector velocity is equal to zero.

9. Impossibility of Bounded Motion of the Point M_1 Relative to the Point M_2 that Cannot Be Arbitrarily Close in the Case of Nonzero Sector Velocity

We fix arbitrary positive numbers $a,\ a_1,\ a_2,$ and b. Let $a_1 < a_2$ and $b < c$.

Assume that there exists a motion of the points M_1 and M_2 such that

$$|\vec{r}_1(t)| \le a, \quad t \ge t_0, \tag{34}$$

$$a_1 \le |\vec{r}_1(t) - \vec{r}_2(t)| \le a_2, \quad t \ge t_0,$$

$$|\vec{v}_1(t) - \vec{v}_2(t)| \le b, \quad t \ge t_0,$$

$$\max\{|\vec{v}_1(t)|, |\vec{v}_2(t)|\} \le b, \quad t \ge t_0, \tag{35}$$

and

$$0 \neq |\vec{v}_\sigma(t_0)|.$$

Thus, according to (20), the transverse component $\vec{v}_{**}(t)$ of the velocity of motion $\vec{v}(t) = \vec{v}_1(t) - \vec{v}_2(t)$ of the point M_1 relative to the point M_2 satisfies the relation

$$\frac{2|\vec{v}_\sigma(t_0)|}{a_2} \leq \frac{2|\vec{v}_\sigma(t)|}{a_2} \leq |\vec{v}_{**}(t)| \leq \frac{2|\vec{v}_\sigma(t)|}{a_1}, \quad t \geq t_0. \tag{36}$$

This yields

$$0 < \inf_{t \geq t_0} |\vec{v}_1(t) - \vec{v}_2(t)|. \tag{37}$$

According to inequalities (34)–(35) and (37), there exist positive numbers $\Delta,\ \Lambda_1,\ \Lambda_2\ (\Lambda_1 < \Lambda_2)$, and Υ such that

$$\max\{\tau_1(t), \tau_2(t)\} \leq \Delta, \quad t \geq t_0,$$

$$\Lambda_1 \leq \min\left\{|\vec{r}_1(t - \tau_1(t)) - \vec{r}_2(t)|, |\vec{r}_2(t - \tau_2(t)) - \vec{r}_1(t)|\right\}$$

$$\leq \max\left\{|\vec{r}_1(t - \tau_1(t)) - \vec{r}_2(t)|, |\vec{r}_2(t - \tau_2(t)) - \vec{r}_1(t)|\right\} \leq \Lambda_2, \quad t \geq t_0,$$

$$\max\left\{\left|\vec{F}_1(t)\right|, \left|\vec{F}_2(t)\right|\right\} \leq \Upsilon, \quad t \geq t_0. \tag{38}$$

Moreover, in view of the fact that the sector velocity increases, by using equality (17), we get

$$\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right| + \frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right| > 0 \quad \text{for all} \quad t \geq t_0. \tag{39}$$

Relation (39) means that the sector velocity of revolution of the point M_1 around the point M_2 is nonzero at any time $t \geq t_0$.

We now show that

$$\inf_{t \geq t_0}\left(\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right| + \frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right|\right) > 0. \tag{40}$$

Consider the set $\mathfrak{M}$ of ordered quadruples of functions $\big(\vec{r}_1(t), \vec{r}_2(t), \vec{v}_1(t), \vec{v}_2(t)\big)$ continuous on the segment $[t_0 - \Delta, t_0 + \Delta]$, where $\vec{v}_1(t) = \dot{\vec{r}}_1(t)$ and $\vec{v}_2(t) = \dot{\vec{r}}_2(t)$, each of which describes the motion of the points M_1 and M_2 on the segment $[t_0 - \Delta, t_0 + \Delta]$ and satisfies relations (34)–(38).

In view of relations (34)–(38) and the equations of system (10) used to describe the motion of the points M_1 and M_2, the quadruples $(\vec{r}_1(t), \vec{r}_2(t), \vec{v}_1(t), \vec{v}_2(t))$ of functions from the set $\mathfrak{M}$ are uniformly bounded and equicontinuous [13] on the segment $[t_0 - \Delta, t_0 + \Delta]$ and, hence, the set $\mathfrak{M}$ is closed. By virtue of the generalized

Arzela theorem [13], the bounded and closed set $\mathfrak{M}$ is compact. According to relations (5) and (7) and the equations of system (38), the scalar quantity

$$\min_{t\in[t_0,t_0+\Delta]}\left(\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right|+\frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right|\right)$$

continuously depends on $(\vec{r}_1(t),\vec{r}_2(t),\vec{v}_1(t),\vec{v}_2(t))\in\mathfrak{M}$. Hence, by the Weierstrass theorem on maximum and minimum values (see [7, p. 176] and [14, p. 34]), there exists a point $(\vec{r}_1^{\,*}(t),\vec{r}_2^{\,*}(t),\vec{v}_1^{\,*}(t),\vec{v}_2^{\,*}(t))\in\mathfrak{M}$ at which

$$\min_{t\in[t_0,t_0+\Delta]}\left(\frac{1}{m_1}\left|\vec{F}_{1,**}(t)\right|+\frac{1}{m_2}\left|\vec{F}_{2,**}(t)\right|\right)$$

takes the minimum value. According to the condition A, this value cannot be equal to zero. Hence, the motion of the points M_1 and M_2 described by the vector functions $\vec{r}_1(t)$ and $\vec{r}_2(t)$ and considered at the beginning of this section satisfies relation (40).

Thus, $\left|\vec{v}_\sigma(t)\right|\to+\infty$ as $t\to+\infty$, which contradicts (36) and the inequalities $\left|\vec{v}_{**}(t)\right|<c,\ t\geq t_0$.

Therefore, the assumption about the existence of a bounded motion of the point M_1 relative to the point M_2 that cannot be arbitrarily close is not true.

We now describe the motion of the point M_1 relative to the point M_2 under the condition A.

Proposition 3. *We have either* $\liminf_{t\to+\infty} d(M_1,M_2)=0$ *(in this case, two bodies whose sizes are not equal to zero collide within a finite time interval) or* $\limsup_{t\to+\infty} d(M_1,M_2)=+\infty$.

This implies that the motion of the point M_1 relative to the point M_2 under the condition A does not obey the third Kepler law because the trajectory of motion of the point M_1 relative to the point M_2 is not an ellipse.

Therefore, according to the results obtained in Secs. 5 and 7–9, the *motions of planets in the Solar system do not obey the Kepler laws*.

In view of relation (39) (in this case, the sector velocity of the point M_1 relative to the point M_2 is strictly increasing), the trajectories of motion of the point M_1 relative to the point M_2 in Proposition 3 are spiral on finite time intervals.

10. Existence of Unbounded Spiral Trajectories

First, we show that the set of trajectories of motion of the points M_1 and M_2 each of which does not belong to the straight line and $d(M_1,M_2)\to+\infty$ as $t\to+\infty$ is nonempty.

We now use relation (19) to represent the velocity $\vec{v}(t)=\vec{v}_1(t)-\vec{v}_2(t)$ via the radial and transverse components $\vec{v}_*(t)$ and $\vec{v}_{**}(t)$.

Consider two vectors $\vec{a}_1$ and $\vec{a}_2$ satisfying the conditions presented in what follows.

We now show that, in the case where the vectors $\vec{a}_1-\vec{a}_2$ and $\vec{r}_1(t_0)-\vec{r}_2(t_0)$ have the same direction, the quantities $|\vec{a}_1-\vec{a}_2|$ and $|\vec{r}_1(t_0)-\vec{r}_2(y_0)|$ are sufficiently large, the quantity

$$\varepsilon=\sup_{s\in[t_0-\tau_1(t_0),t_0]}|\vec{v}_1(s)-\vec{a}_1|+\sup_{s\in[t_0-\tau_2(t_0),t_0]}|\vec{v}_2(s)-\vec{a}_2|$$

$$+\sup_{s\in[t_0-\tau_1(t_0),t_0]}|\vec{r}_1(s)-\vec{r}_1(t_0)|+\sup_{s\in[t_0-\tau_2(t_0),t_0]}|\vec{r}_2(s)-\vec{r}_2(t_0)| \tag{41}$$

is sufficiently small,

$$\vec{v}_\sigma(t_0) \neq \vec{0}, \tag{42}$$

and the inequality

$$(\left|\vec{a}_1 - \vec{a}_2\right| - \varepsilon)^2 > \frac{16G(m_1 + m_2)}{\left|\vec{r}_1(t_0) - \vec{r}_2(t_0)\right|} \tag{43}$$

is true, we can write

$$\lim_{t \to +\infty} \left|\vec{r}_1(t) - \vec{r}_2(t)\right| = +\infty. \tag{44}$$

We use the relation

$$\frac{d(\vec{v}_1(t) - \vec{v}_2(t))}{dt} = \frac{1}{m_1}\,\vec{F}_1(t) - \frac{1}{m_2}\,\vec{F}_2(t), \quad t \geq t_0. \tag{45}$$

In view of (5) and (7), this relation follows from the equations of system (10). By virtue of (45), for all $t \geq t_0$, we get

$$\vec{v}(t) = \vec{v}(t_0) + \int\limits_{t_0}^{t} \left(\frac{1}{m_1}\,\vec{F}_1(s) - \frac{1}{m_2}\,\vec{F}_2(s)\right) ds. \tag{46}$$

By $\left(\vec{a}, \vec{b}\,\right)$ we denote the scalar product of the vectors $\vec{a}$ and $\vec{b}$.

We now scalarly multiply both sides of equality (46) by the vector $\vec{a}_1 - \vec{a}_2$. This yields

$$\big(\vec{a}_1 - \vec{a}_2, \vec{v}(t)\big) = \big(\vec{a}_1 - \vec{a}_2, \vec{v}(t_0)\big) + \left(\vec{a}_1 - \vec{a}_2, \int\limits_{t_0}^{t} \left(\frac{1}{m_1}\,\vec{F}_1(s) - \frac{1}{m_2}\,\vec{F}_2(s)\right) ds\right), \quad t \geq t_0.$$

This implies that

$$\big(\vec{a}_1 - \vec{a}_2, \vec{v}_*(t)\big) = \big(\vec{a}_1 - \vec{a}_2, \vec{a}_1 - \vec{a}_2\big) + \big(\vec{a}_1 - \vec{a}_2, \vec{v}(t_0) - (\vec{a}_1 - \vec{a}_2)\big)$$

$$+ \left(\vec{a}_1 - \vec{a}_2, \int\limits_{t_0}^{t} \left(\frac{1}{m_1}\,\vec{F}_1(s) - \frac{1}{m_2}\,\vec{F}_2(s)\right) ds\right), \quad t \geq t_0,$$

whence we get

$$|\vec{a}_1 - \vec{a}_2|\,|\vec{v}_*(t)| = |\vec{a}_1 - \vec{a}_2|^2 + \big(\vec{a}_1 - \vec{a}_2, \vec{v}(t_0) - (\vec{a}_1 - \vec{a}_2)\big)$$

$$+\left(\vec{a}_1-\vec{a}_2, \int_{t_0}^{t}\left(\frac{1}{m_1}\vec{F}_1(s)-\frac{1}{m_2}\vec{F}_2(s)\right)ds\right), \quad t\ge t_0. \tag{47}$$

Since the vectors $\vec{a}_1-\vec{a}_2$ and $\vec{r}_1(t_0)-\vec{r}_2(t_0)$ have the same direction, the points M_1 and M_2 attract each other by the gravity force, the quantity ε is sufficiently small, the function $\vec{v}(t)$ is continuous, and there exists an interval $[t_0,t_1)$ on which the quantity $|\vec{v}_*(t)|$ monotonically decreases and

$$|\vec{v}_*(t)|>0 \quad \text{for all} \quad t\in[t_0,t_1). \tag{48}$$

Assume that

$$\left|\vec{v}_*(t_1)\right|=0. \tag{49}$$

In view of relations (41) and (47), we find

$$|\vec{a}_1-\vec{a}_2|\,\left|\vec{v}_*(t)\right|\ge|\vec{a}_1-\vec{a}_2|^2-|\vec{a}_1-\vec{a}_2|\varepsilon$$

$$-|\vec{a}_1-\vec{a}_2|\int_{t_0}^{t}\left|\frac{1}{m_1}\vec{F}_1(s)-\frac{1}{m_2}\vec{F}_2(s)\right|ds, \quad t\in[t_0,t_1).$$

Hence,

$$|\vec{v}_*(t)|\ge|\vec{a}_1-\vec{a}_2|-\varepsilon-\int_{t_0}^{t}\left|\frac{1}{m_1}\vec{F}_1(s)-\frac{1}{m_2}\vec{F}_2(s)\right|ds, \quad t\in[t_0,t_1). \tag{50}$$

We now estimate the quantity

$$\int_{t_0}^{t}\left|\frac{1}{m_1}\vec{F}_1(s)-\frac{1}{m_2}\vec{F}_2(s)\right|ds$$

from above for $t\in[t_0,t_1)$. To this end, we use the equalities

$$\left|d\,|\vec{r}_1(s)-\vec{r}_2(s)|\right|=\left|d\sqrt{(x_1(s)-x_2(s))^2+(y_1(s)-y_2(s))^2+(z_1(s)-z_2(s))^2}\right|$$

$$=\left|\frac{\big(\vec{r}_1(s)-\vec{r}_2(s),\vec{v}_1(s)-\vec{v}_2(s)\big)}{|\vec{r}_1(s)-\vec{r}_2(s)|}\,ds\right|=\left|\vec{v}_*(s)\right|\,ds, \quad t\in[t_0,t_1), \tag{51}$$

and the fact that, according to (51),

$$\int_{t_0}^{t}\left|\frac{1}{m_1}\vec{F}_1(s)-\frac{1}{m_2}\vec{F}_2(s)\right|ds=\int_{t_0}^{t}\left|\frac{1}{m_1}\vec{F}_1(s)-\frac{1}{m_2}\vec{F}_2(s)\right|\frac{\left|d\,|\vec{r}_1(s)-\vec{r}_2(s)|\right|}{|\vec{v}_*(s)|}, \quad t\in[t_0,t_1).$$

Since

$$\left|\overrightarrow{M_1M_1^*}\right| + \left|\overrightarrow{M_1^*M_2}\right| \geq \left|\overrightarrow{M_1M_2}\right|, \qquad \left|\overrightarrow{M_2M_2^*}\right| + \left|\overrightarrow{M_2^*M_1}\right| \geq \left|\overrightarrow{M_1M_2}\right|$$

(by virtue of the triangle inequality; see Fig. 3),

$$\left|\overrightarrow{M_1M_2}\right| = |\vec{r}_1(t) - \vec{r}_2(t)|, \qquad \left|\overrightarrow{M_1^*M_2}\right| = |\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)|,$$

$$\left|\overrightarrow{M_2^*M_1}\right| = |\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)|,$$

and

$$\left|\overrightarrow{M_1M_1^*}\right| \leq \left|\overrightarrow{M_1^*M_2}\right|, \qquad \left|\overrightarrow{M_2M_2^*}\right| \leq \left|\overrightarrow{M_2^*M_1}\right|$$

(because the velocities of the points M_1 and M_2 cannot exceed the velocity of gravitation c), we obtain

$$|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)| \geq \frac{|\vec{r}_1(t) - \vec{r}_2(t)|}{2} \quad \text{and} \quad |\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)| \geq \frac{|\vec{r}_1(t) - \vec{r}_2(t)|}{2}, \quad t \geq t_0.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{m_1}\,\vec{F}_1(t) - \frac{1}{m_2}\,\vec{F}_2(t)\right| &\leq \frac{\left|\vec{F}_1(t)\right|}{m_1} + \frac{\left|\vec{F}_2(t)\right|}{m_2} \\ &= \frac{Gm_2}{|\vec{r}_2(t-\tau_2(t)) - \vec{r}_1(t)|^2} + \frac{Gm_1}{|\vec{r}_1(t-\tau_1(t)) - \vec{r}_2(t)|^2} \\ &\leq \frac{4G(m_1+m_2)}{|\vec{r}_1(t) - \vec{r}_2(t)|^2}, \quad t \geq t_0.\end{aligned}$$

By using relations (48) and (51) and the fact that the function $|\vec{v}_*(t)|$ on $[t_0, t_1)$ monotonically decreases, we get

$$\begin{aligned}\int_{t_0}^{t} \left|\frac{1}{m_1}\,\vec{F}_1(s) - \frac{1}{m_2}\,\vec{F}_2(s)\right| ds &= \int_{t_0}^{t} \left|\frac{1}{m_1}\,\vec{F}_1(s) - \frac{1}{m_2}\,\vec{F}_2(s)\right| \frac{|d(\vec{r}_1(s) - \vec{r}_2(s))|}{|\vec{v}_*(s)|} \\ &\leq \int_{t_0}^{t} \frac{4G(m_1+m_2)}{|\vec{r}_1(s) - \vec{r}_2(s)|^2}\, \frac{|d\,|\vec{r}_1(s) - \vec{r}_2(s)||}{|\vec{v}_*(s)|} \\ &\leq \frac{4G(m_1+m_2)}{|\vec{v}_*(t)|} \left|\frac{1}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|} - \frac{1}{|\vec{r}_1(t) - \vec{r}_2(t)|}\right|\end{aligned}$$

for all $t \in [t_0, t_1)$. According to (50), we find

$$|\vec{v}_*(t)| \geq |\vec{a}_1 - \vec{a}_2| - \varepsilon - \frac{1}{|\vec{v}_*(t)|} \left| \frac{4G(m_1 + m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|} - \frac{4G(m_1 + m_2)}{|\vec{r}_1(t) - \vec{r}_2(t)|} \right|, \quad t \in [t_0, t_1). \tag{52}$$

In view of (48), the quantity $|\vec{r}_1(t) - \vec{r}_2(t)|$ is strictly increasing on the segment $[t_0, t_1]$. By virtue of (52), we arrive at the following relation:

$$|\vec{v}_*(t)| \geq |\vec{a}_1 - \vec{a}_2| - \varepsilon - \frac{4G(m_1 + m_2)}{|\vec{v}_*(t)|\, |\vec{r}_1(t_0) - \vec{r}_2(t_0)|}, \quad t \in [t_0, t_1). \tag{53}$$

We now estimate the quantity $|\vec{v}_*(t)|$ on the interval $[t_0, t_1)$ from below. In view of (48) and (53), we find

$$|\vec{v}_*(t)|^2 - (|\vec{a}_1 - \vec{a}_2| - \varepsilon)|\vec{v}_*(t)| \geq -\frac{4G(m_1 + m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|}, \quad t \in [t_0, t_1).$$

Hence,

$$\left(|\vec{v}_*(t)| - \frac{|\vec{a}_1 - \vec{a}_2| - \varepsilon}{2} \right)^2 \geq \frac{(|\vec{a}_1 - \vec{a}_2| - \varepsilon)^2}{4} - \frac{4G(m_1 + m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|}, \quad t \in [t_0, t_1).$$

By using relations (43), (50), and (53), we get

$$|\vec{v}_*(t)| \geq \frac{|\vec{a}_1 - \vec{a}_2| - \varepsilon}{2} + \sqrt{\frac{(|\vec{a}_1 - \vec{a}_2| - \varepsilon)^2}{4} - \frac{4G(m_1 + m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|}}, \quad t \in [t_0, t_1). \tag{54}$$

Since the function $|\vec{v}_*(t)|$ is continuous and the number

$$\lambda = \frac{|\vec{a}_1 - \vec{a}_2| - \varepsilon}{2} + \sqrt{\frac{(|\vec{a}_1 - \vec{a}_2| - \varepsilon)^2}{4} - \frac{4G(m_1 + m_2)}{|\vec{r}_1(t_0) - \vec{r}_2(t_0)|}}$$

is positive, inequality (54) contradicts (49).

Thus, the assumption of validity of relation (49) is not true.

Hence, inequality (48) is true for $t_1 = +\infty$.

Further, since the inequality $|\vec{v}_*(t)| \geq \lambda$ is true for any $t \geq t_0$, it is clear that relation (44) takes place.

Thus, the set of trajectories of motion of the points M_1 and M_2 for each of which $d(M_1, M_2) \to +\infty$ as $t \to +\infty$ and $\vec{v}_\sigma(t) \neq \vec{0}$, $t \geq t_0$, is nonempty. In view of (42) (in this case, $|\vec{v}_{**}(t)| > 0$ for all $t \geq t_0$), all trajectories of motion of the point M_1 relative to the point M_2 from this set are spiral.

11. Instability of Unbounded Motions of the Bodies

According to the results presented above, the motion of two bodies in the real space with finite velocity of gravitation does not obey the Kepler laws. Indeed, the bodies either collide or the trajectories of motion of one body relative to the other body become unbounded.

We study the instability of motion of these bodies.

By

$$\vec{r}_i(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}), \quad i = \overline{1,2}, \tag{55}$$

we denote the solutions of the system of equations (10) satisfying the initial conditions (11).

A motion of the points M_1 and M_2 described by the vector functions (55) is called *Lyapunov stable* if, for any arbitrarily small number $\varepsilon > 0$, there exists a number $\delta > 0$ such that, for any other motion of these points described by the vector functions

$$\vec{\hat{r}}_i\left(t, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right), \quad i = \overline{1,2} \tag{56}$$

(here, $\vec{\hat{\varphi}}_{0,1}$, $\vec{\hat{\varphi}}_{0,2}$, $\vec{\hat{\varphi}}_{1,1}$, and $\vec{\hat{\varphi}}_{1,2}$ are continuous vector functions similar to the functions $\vec{\varphi}_{0,1}$, $\vec{\varphi}_{0,2}$, $\vec{\varphi}_{1,1}$, and $\vec{\varphi}_{1,2}$), the inequality

$$\sup_{s \in [t_0 - \tau_1(t_0), t_0]} \left(\left|\vec{\varphi}_{0,1}(s) - \vec{\hat{\varphi}}_{0,1}(s)\right| + \left|\vec{\varphi}_{1,1}(s) - \vec{\hat{\varphi}}_{1,1}(s)\right|\right)$$
$$+ \sup_{s \in [t_0 - \tau_2(t_0), t_0]} \left(\left|\vec{\varphi}_{0,2}(s) - \vec{\hat{\varphi}}_{0,2}(s)\right| + \left|\vec{\varphi}_{1,2}(s) - \vec{\hat{\varphi}}_{1,2}(s)\right|\right) < \delta \tag{57}$$

implies that

$$\sum_{i=1}^{2} \left|\vec{r}_i\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) - \vec{\hat{r}}_i\left(t, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right)\right| < \varepsilon, \quad t \geq t_0.$$

A motion of the points M_1 and M_2 described by the vector functions (55) is called *Lyapunov unstable* if there exists a number $\varepsilon > 0$ such that, for any arbitrarily small number $\delta > 0$, there exist motions of these points described by the vector functions (56) and time $t_1 > t_0$ for which relations (57) and

$$\sum_{i=1}^{2} \left|\vec{r}_i\left(t_1, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) - \vec{\hat{r}}_i\left(t_1, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right)\right| > \varepsilon$$

are true.

It is clear that the motion of points M_1 and M_2 described by the vector functions (55) is unstable if the motion of point M_1 relative to the point M_2 is unstable, i.e., there exists a number $\varepsilon > 0$ such that, for any arbitrarily small number $\delta > 0$, these exist a motion of these points described by the vector functions (56) and time $t_1 > t_0$ for which relations (57) are true and, for the functions

$$\vec{r}\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) = \vec{r}_1\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) - \vec{r}_2\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right),$$

$$\vec{\hat{r}}\left(t, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right) = \vec{\hat{r}}_1\left(t, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right) - \vec{\hat{r}}_2\left(t, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right),$$

the following inequality is true:

$$\left|\vec{r}\left(t_1, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) - \vec{\hat{r}}\left(t_1, t_0, \vec{\hat{\varphi}}_{0,1}, \vec{\hat{\varphi}}_{0,2}, \vec{\hat{\varphi}}_{1,1}, \vec{\hat{\varphi}}_{1,2}\right)\right| > \varepsilon.$$

In what follows, we demonstrate the instability of motion of the point M_1 relative to the point M_2 in the case where the trajectories of motion of these points are unbounded.

Assume that the vector function $\vec{r}(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2})$ is unbounded, i.e.,

$$\limsup_{t\to+\infty} d(M_1, M_2) = +\infty.$$

Without loss of generality, we can assume that the trajectory of motion of the point M_1 relative to the point M_2 is two-dimensional and lies in a certain plane E containing the center of mass of the points M_1 and M_2 (point O). In the plane E, we introduce a Cartesian coordinate system centered at the point O.

We fix an arbitrary angle $\omega \in (0, 2\pi]$ and consider the operator A_ω of rotation of points of the plane E around the center of rotation (point O) by an angle ω. This operator is given by the matrix $\begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$ [15, p. 286]. This means that if $\vec{b} = A_\omega \vec{a}$ and $\vec{a} = (a_1, a_2)$, then $\vec{b} = (a_1\cos\omega - a_2\sin\omega, a_1\sin\omega + a_2\cos\omega)$, where a_1 and a_2 are the coordinates of the vector $\vec{a}$.

Consider a function $\vec{r}(t, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2})$. Here, $A_\omega\vec{\varphi}_{0,1}$, $A_\omega\vec{\varphi}_{0,2}$, $A_\omega\vec{\varphi}_{1,1}$, and $A_\omega\vec{\varphi}_{1,2}$ are functions $A_\omega\vec{\varphi}_{0,1}(s)$, $A_\omega\vec{\varphi}_{0,2}(s)$, $A_\omega\vec{\varphi}_{1,1}(s)$, and $A_\omega\vec{\varphi}_{1,2}(s)$, respectively, such that

$$\vec{r}_i\left(s, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right) = A_\omega\vec{\varphi}_{0,i}(s), \quad s \in [t_0 - \tau_i(t_0), t_0], \quad i = \overline{1,2}, \tag{58}$$

$$\dot{\vec{r}}\left(s, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right) = A_\omega\vec{\varphi}_{1,i}(s), \quad s \in [t_0 - \tau_i(t_0), t_0], \quad i = \overline{1,2}. \tag{59}$$

It is easy to see that the vector functions

$$A_\omega\vec{r}_i\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right), \quad i = \overline{1,2},$$

such that

$$A_\omega\vec{r}_i\left(s, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) = A_\omega\vec{\varphi}_{0,i}(s), \quad s \in [t_0 - \tau_i(t_0), t_0], \quad i = \overline{1,2},$$

and

$$\left(A_\omega\vec{r}_i\left(s, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right)\right)' = A_\omega\vec{\varphi}_{1,i}(s), \quad s \in [t_0 - \tau_i(t_0), t_0], \quad i = \overline{1,2},$$

are also solutions of the system of equations (10).

In view of the uniqueness of solution of the system of equations (10) satisfying conditions (58) and (59), the equalities

$$\vec{r}_i\left(t, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right) = A_\omega\vec{r}_i\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right), \quad i = \overline{1,2}, \tag{60}$$

are true. Hence,

$$A_\omega\vec{r}\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) = \vec{r}\left(t, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right).$$

Consider the trajectories of motion of the point M_1 relative to the point M_2 corresponding to the functions

$$\vec{r}\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right) \tag{61}$$

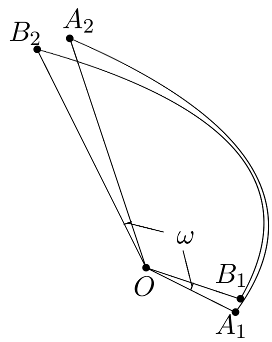

Fig. 5. Spiral motion of the point M_1 relative to the point M_2.

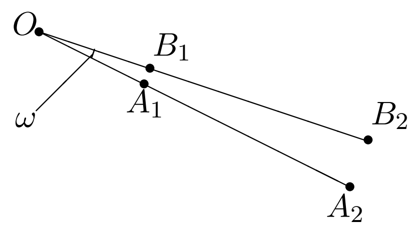

Fig. 6. Rectilinear motion of the point M_1 relative to the point M_2.

and

$$\vec{r}\left(t, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right) \tag{62}$$

(see Fig. 5 for the case of spiral motion and Fig. 6 for the case of rectilinear motion).

The trajectories with points A_1 and A_2 correspond to functions (61), while the trajectories with points B_1 and B_2 correspond to functions (62).

In view of (60), the second trajectories are obtained from the first trajectories by the anticlockwise rotation by an angle ω about the point O.

Note that

$$\begin{aligned} \overrightarrow{OA_1} &= \vec{r}\left(t_0, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right), \\ \overrightarrow{OA_2} &= \vec{r}\left(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\right), \end{aligned} \tag{63}$$

$$\begin{aligned} \overrightarrow{OB_1} &= \vec{r}\left(t_0, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right), \\ \overrightarrow{OB_2} &= \vec{r}\left(t, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\right) \end{aligned} \tag{64}$$

and the angles between the vectors $\overrightarrow{OA_1}$ and $\overrightarrow{OB_1}$ and between the vectors $\overrightarrow{OA_2}$ and $\overrightarrow{OB_2}$ are equal to ω.

Since the trajectory of motion of the point M_1 relative to the point M_2 is unbounded, for any $\omega > 0$, we have

$$\limsup_{t\to+\infty} d(A_2, B_2) = +\infty$$

and, according to (63) and (64),

$$\limsup s_{t\to+\infty}\Big|\vec{r}\big(t, t_0, \vec{\varphi}_{0,1}, \vec{\varphi}_{0,2}, \vec{\varphi}_{1,1}, \vec{\varphi}_{1,2}\big) - \vec{r}\big(t, t_0, A_\omega\vec{\varphi}_{0,1}, A_\omega\vec{\varphi}_{0,2}, A_\omega\vec{\varphi}_{1,1}, A_\omega\vec{\varphi}_{1,2}\big)\Big| = +\infty \quad \text{for all} \quad \omega > 0. \tag{65}$$

Note that $\lim\limits_{\omega\to 0} A_\omega\vec{a} = \vec{a}$ for every vector $\vec{a}$. In view of the continuity of the functions $\vec{\varphi}_{0,1}(s)$ and $\vec{\varphi}_{1,1}(s)$ on $[t_0 - \tau_1(t_0), t_0]$ and the functions $\vec{\varphi}_{0,2}(s)$ and $\vec{\varphi}_{1,2}(s)$ on $[t_0 - \tau_2(t_0), t_0]$, we get

$$\lim_{\omega\to 0}\left(\sup_{s\in[t_0-\tau_1(t_0),t_0]}\left(|A_\omega\vec{\varphi}_{0,1}(s) - \vec{\varphi}_{0,1}(s)| + |A_\omega\vec{\varphi}_{1,1}(s) - \vec{\varphi}_{1,1}(s)|\right) + \sup_{s\in[t_0-\tau_2(t_0),t_0]}\left(|A_\omega\vec{\varphi}_{0,2}(s) - \vec{\varphi}_{0,2}(s)| + |A_\omega\vec{\varphi}_{1,2}(s) - \vec{\varphi}_{1,2}(s)|\right)\right) = 0.$$

Thus, relation (65) is true for an arbitrarily small quantity

$$\delta = \sup_{s\in[t_0-\tau_1(t_0),t_0]}\left(|A_\omega\vec{\varphi}_{0,1}(s) - \vec{\varphi}_{0,1}(s)| + |A_\omega\vec{\varphi}_{1,1}(s) - \vec{\varphi}_{1,1}(s)|\right) + \sup_{s\in[t_0-\tau_2(t_0),t_0]}\left(|A_\omega\vec{\varphi}_{0,2}(s) - \vec{\varphi}_{0,2}(s)| + |A_\omega\vec{\varphi}_{1,2}(s) - \vec{\varphi}_{1,2}(s)|\right) > 0.$$

This means that the motion of the point M_1 relative to the point M_2 *is unstable if the trajectory of motion of* M_1 *relative to* M_2 *is unbounded.*

If the trajectory of motion of one body relative to another body is bounded, then the bodies collide within a finite period of time. The motion of bodies in this case can be also regarded as unstable because the states of the bodies undergo qualitative changes at the time of their collision.

12. Conclusions and Remarks

1. The results of our investigations of the motion of two bodies of arbitrary masses show that, in the real world with finite velocity of gravitation, the celestial bodies move according to the laws that differ from the laws of classical celestial mechanics.

2. For two bodies, the typical trajectories of motion are spiral. In celestial mechanics, the attention given to trajectories of this type is insufficient because they do not agree with the Kepler laws. The set of these motions is nonempty, which is confirmed, e.g., by the presence of spiral galaxies, including, in particular, our galaxy and the Andromeda Nebula. According to the results of our investigations, binary stars and the systems formed by a star and a black hole also move along spiral or rectilinear curves and do not obey the Kepler laws.

3. On the basis of the data of observations of celestial objects, it is difficult to detect the deviations of the actual trajectories of motion from the trajectories of motion given by the Kepler laws. The indicated deviations can be very small (these deviations depend on the states of celestial objects at the "initial" time). Thus, for one year, the distances between the Earth and the Sun and between the Earth and the Moon increase by about 15 cm and 3.82 cm, respectively [16].

The deviations from the elliptic (Kepler) trajectories of motion of two bodies are explained by the increase in the sector velocity of one body relative to the other body caused by the finite velocity of gravitation. Note that, in [16], the increase in the distances between the Earth and the Sun and between the Earth and the Moon is explained by the tidal interaction. However, even in the ideal case where the masses of bodies are concentrated at the centers of mass (in this case, the tidal interaction is absent), according to the presented theory of motion of the bodies, the distance between bodies may also increase (or decrease). This shows that the causes of increase in the distance between celestial objects discussed in [16] are incomplete and that it is reasonable to consider the law of increase in the sector velocity as one of these causes.

The indicated deviations (15 cm and 3.82 cm) are very small as compared with the distances between the Earth and the Sun and between the Earth and the Moon ($1.495978706960 \times 10^{11} \pm 0.1$ m and 3.844×10^{8} m, respectively). The application of Kepler's laws to actual systems with finite velocity of gravitation whose sizes are similar to the sizes of Solar system does not lead to noticeable errors for small time intervals. However, for large time intervals (first of all, in the case of unbounded trajectories of motion of the bodies), the errors can be quite large.

4. The application of the ordinary differential equations (1) for the investigation of the dynamics of motion of two bodies for large time intervals may lead to significant differences between the results of theoretical analysis and the actual data of observations. These deviations can be removed by using the differential equations with delayed argument (10). Moreover, the instability of the unbounded trajectories of motion of the bodies should also be taken into account.

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