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WEAKLY NONLINEAR BOUNDARY-VALUE PROBLEMS FOR THE FREDHOLM INTEGRAL EQUATIONS WITH DEGENERATE KERNELS IN BANACH SPACES

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We consider weakly nonlinear boundary-value problems for the Fredholm integral equations with degenerate kernel in Banach spaces, establish necessary and sufficient conditions for the existence of solutions of these problems, and construct convergent iterative procedures for the determination of solutions of these boundary-value problems.

The present paper is a continuation of our investigations of the conditions of solvability and construction of the solutions of weakly nonlinear integral Fredholm equations with degenerate kernels in Banach spaces originated in [1].

Constructive methods for the analysis of weakly nonlinear boundary-value problems for systems of functional-differential and other equations traditionally occupy one of important places in the qualitative theory of differential equations and continue the development of the methods of perturbation theory and, in particular, of the methods of Lyapunov–Poincaré small parameter [2, 3].

These methods were successfully developed in [4, 5] and applied to the study of weakly nonlinear boundary-value problems for systems of ordinary differential equations [6] and to the construction of bounded solutions of weakly nonlinear differential equations in Banach spaces [7].

In finite-dimensional Euclidean spaces, weakly nonlinear integrodifferential equations and Fredholm integral equations with nondegenerate kernels, which are not always solvable, were studied in [8, 9].

A specific feature of the investigation of boundary-value problems for systems of integral equations in Banach spaces is connected with the fact that their linear part is an operator that does not have the inverse operator [10], which significantly complicates the study of boundary-value problems for equations of this kind. Therefore, the problem of investigation of the conditions of existence and construction of the general solutions of weakly nonlinear boundary-value problems for Fredholm integral equations with degenerate kernel that are not always solvable in Banach spaces is topical.

Statement of the Problem

We consider a weakly nonlinear boundary-value problem

$$(Lz)(t) := z(t) - M(t)\int_a^b N(s)z(s)ds = f(t) + \varepsilon\int_a^b K(t,s)Z(z(s,\varepsilon),s,\varepsilon)ds, \tag{1}$$

$$\ell z(\cdot) = \alpha + \varepsilon J(z(\cdot,\varepsilon),\varepsilon). \tag{2}$$

Here, the operator-valued functions $M(t)$ and $N(t)$ are defined on a finite interval $\mathcal{I} = [a,b]$, act from the Banach space $\mathbf{B}$ into the same space, and are strongly continuous with the norms

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$$|||M||| = \sup_{t\in\mathcal{I}}\|M(t)\|_{\mathbf{B}} = M_0 < \infty \qquad \text{and} \qquad |||N||| = \sup_{t\in\mathcal{I}}\|N(t)\|_{\mathbf{B}} = N_0 < \infty.$$

The operator-valued function $K(t,s)$ is defined in the square $\mathcal{I}\times\mathcal{I}$, acts from the Banach space $\mathbf{B}$ into the same space with respect to each variable, and is strongly continuous with respect to each variable with the norm

$$|||K||| = \sup_{t\in\mathcal{I}}\|K(t,s)\|_{\mathbf{B}} = K_0 < \infty.$$

Moreover, $Z(z(t,\varepsilon),t,\varepsilon)$ is a nonlinear z bounded operator function, $J(z(\cdot,\varepsilon),\varepsilon)$ is a nonlinear z vector functional, which has a strongly continuous Fréchet derivative with respect to z in a neighborhood of the generating solution $\|z - z_0\| \le q$ and is continuous in the set of variables z, t, ε, q and ε_0 (these are sufficiently small constants); $Z(0,t,0) = 0$, $Z'_z(0,t,0) = 0$, $J(0,0) = 0$, $J'_z(0,0) = 0$; $f(t)$ is a vector-valued function in the Banach space $\mathbf{C}(\mathcal{I},\mathbf{B})$ of continuous vector functions on the interval $\mathcal{I}$, and α is an element of the Banach space $\mathbf{B}_1 : \alpha \in \mathbf{B}_1$.

Parallel with the problem (1), (2) we consider a linear generating boundary-value problem

$$z_0(t) - M(t)\int_a^b N(s)z_0(s)ds = f(t), \tag{3}$$

$$\ell z_0(\cdot) = \alpha, \tag{4}$$

which is obtained from (1), (2) for $\varepsilon = 0$.

The problem is to establish necessary and sufficient conditions for the existence of solutions of the weakly nonlinear boundary-value problem (1), (2). We seek solutions in the class of vector-valued functions $z(t,\varepsilon)$ continuous both in the variable t and in the parameter ε and turning into the generating solution of the linear boundary-value problem (3), (4) for $\varepsilon = 0$.

Auxiliary Information

Suppose that a bounded linear operator

$$D = I_{\mathbf{B}} - \int_a^b N(s)M(s)\,ds, \qquad D : \mathbf{B} \to \mathbf{B}$$

is generalized invertible. Then there exist (see [11, 12]) a bounded projector $\mathcal{P}_{N(D)} : \mathbf{B} \to N(D)$ that projects a Banach space $\mathbf{B}$ onto the null space $N(D)$ of the operator D, a bounded projector $\mathcal{P}_{Y_D} : \mathbf{B} \to Y_D$ that projects a Banach space $\mathbf{B}$ onto the subspace $Y_D = \mathbf{B} \ominus R(D)$, and a bounded generalized inverse operator D^- for the operator D [5, 4, 13].

The class of bounded linear generalized invertible operators acting from the Banach space $\mathbf{B}$ into the Banach space $\mathbf{B}$ is denoted by $\mathbf{GI}(\mathbf{B},\mathbf{B})$. It is obvious that the operator belonging to $\mathbf{GI}(\mathbf{B},\mathbf{B})$ is normally solvable [14].

It is shown in [15] that if the operator $D \in \mathbf{GI}(\mathbf{B},\mathbf{B})$, then, under the condition

$$M(t)\mathcal{P}_{Y_D}\int_a^b N(s)f(s)ds = 0$$

and only under this condition, the operator gather (3) is solvable and has a family of solutions

$$z_0(t) = M(t)\mathcal{P}_{N(D)}c + (L^- f)(t), \tag{5}$$

where c is an arbitrary element of the Banach space $\mathbf{B}$ and

$$(L^- f)(t) = f(t) + M(t)D^- \int_a^b N(s)f(s)ds$$

is a bounded generalized operator inverse to the integral operator L [10].

Substituting the solution (5) of the inhomogeneous operator gather (3) in the boundary condition (4), we arrive at the operator gather

$$Qc + \ell f(\cdot) + \ell M(\cdot)D^- \int_a^b N(s)f(s)ds = \alpha,$$

where $Q = \ell M(\cdot)\mathcal{P}_{N(D)}: \mathbf{B} \to \mathbf{B}_1$ is a bounded linear operator.

Let the operator $Q \in \mathbf{GI}(\mathbf{B}, \mathbf{B}_1)$. Also let $\mathcal{P}_{N(Q)}: \mathbf{B} \to N(Q)$ be a bounded projector of the Banach space $\mathbf{B}$ onto the null space $N(Q)$ of the operator Q, let $\mathcal{P}_{Y_Q}: \mathbf{B}_1 \to Y_Q$ be a bounded projector of the Banach space $\mathbf{B}_1$ onto the subspace $Y_Q = \mathbf{B}_1 \ominus R(Q)$, and let Q^- be a bounded generalized operator inverse to the operator Q.

Theorem 1 [15]. *Let $D \in \mathbf{GI}(\mathbf{B}, \mathbf{B})$ and $Q \in \mathbf{GI}(\mathbf{B}, \mathbf{B}_1)$.*

Then the homogeneous $(f(t) = 0,\ \alpha = 0)$ boundary-value problem corresponding to (3), (4) has a family of solutions

$$z(t) = \widetilde{M}(t)c,$$

where $\widetilde{M}(t) = M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(Q)}$ and c is an arbitrary element of the Banach space $\mathbf{B}$.

The inhomogeneous boundary-value problem (3), (4) is solvable for those and only those $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B})$ and $\alpha \in \mathbf{B}_1$ that satisfy the system of conditions

$$\begin{cases} M(t)\mathcal{P}_{Y_D} \displaystyle\int_a^b N(s)f(s)ds = 0, \\ \mathcal{P}_{Y_Q}\left[\alpha - \ell f(\cdot) - \ell M(\cdot)D^- \displaystyle\int_a^b N(s)f(s)ds\right] = 0. \end{cases} \tag{6}$$

Moreover, this boundary-value problem has a family of solutions

$$z_0(t) = \widetilde{M}(t)c + (Gf)(t) + M(t)\mathcal{P}_{N(D)}Q^-\alpha, \tag{7}$$

where

$$(Gf)(t) := \left[f(t) - M(t)\mathcal{P}_{N(D)}Q^-\ell f(\cdot)\right]$$

$$+ M(t)\left[I_{\mathbf{B}} - \mathcal{P}_{N(D)} Q^{-} \ell M(\cdot)\right] D^{-} \int\limits_a^b N(s) f(s) ds \tag{8}$$

is a generalized Green operator of the semihomogeneous ($\alpha = 0$) *boundary-value problem corresponding to (3), (4).*

It is worth noting that the first condition in (6) is always satisfied if the condition

$$\mathcal{P}_{Y_D} \int\limits_a^b N(s) f(s) ds = 0$$

is satisfied.

To solve the problem, we need information about the conditions of solvability and about the representation of solutions of the operator gathers with a linear operator B_0, i.e., with an operator matrix

$$B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $B_1\colon \mathbf{B} \to \mathbf{B}$ and $B_2\colon \mathbf{B} \to \mathbf{B}_1$ are linear bounded generalized invertible operators [11].

In this case [11, 12], there are bounded projectors $\mathcal{P}_{N(B_1)}\colon \mathbf{B} \to N(B_1)$ and $\mathcal{P}_{N(B_2)}\colon \mathbf{B} \to N(B_2)$ onto the null spaces of the operators B_1 and B_2, the bounded projectors $\mathcal{P}_{Y_{B_1}}\colon \mathbf{B} \to Y_{B_1}$ and $\mathcal{P}_{Y_{B_2}}\colon \mathbf{B}_1 \to Y_{B_2}$ onto the subspaces $Y_{B_1} = \mathbf{B} \ominus R(B_1)$ and $Y_{B_2} = \mathbf{B}_1 \ominus R(B_2)$, respectively, and also the bounded generalized inverse operators B_1^- and B_2^-.

Thus, by using [16] for the system of operator equations

$$B_0 c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} c = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad b_1 \in \mathbf{B}, \quad b_2 \in \mathbf{B}_1, \tag{9}$$

we conclude that the following theorem is true:

Theorem 2 [17]**.** *Let* $B_1 \in \mathbf{GI}(\mathbf{B}, \mathbf{B})$ *and* $B_2 \in \mathbf{GI}(\mathbf{B}, \mathbf{B}_1)$. *Then the system of operator gathers (9) is solvable if and only if* $\operatorname{col}[b_1, b_2]$ *satisfies the condition*

$$\mathcal{P}_{Y_{B_0}} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0,$$

under which this system has a family of solutions

$$c = \mathcal{P}_{N(B_0)} \hat{c} + B_0^- \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where

$$\mathcal{P}_{Y_{B_0}} = \begin{bmatrix} I_{\mathbf{B}} - B_1 \mathcal{P}_{N(B_2)} B_1^- & -B_1 B_2^- \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix}$$

is a bounded projector onto the subspace $Y_{B_0} = I_{\mathbf{B}\times\mathbf{B}_1} \ominus R(B_0)$, $\mathcal{P}_{N(B_0)} = \mathcal{P}_{N(B_2)}\mathcal{P}_{N(B_1)}$ *is a bounded projector onto the null space* $N(B_0)$ *of the operator* B_0, $\hat{c}$ *is an arbitrary element of the Banach space* $\mathbf{B}$, *and*

$$B_0^- = \begin{bmatrix} \mathcal{P}_{N(B_2)} B_1^- & B_2^- \end{bmatrix}$$

is a bounded generalized operator inverse to the operator B_0.

Main Result

Using the generalized Green operator (8) of a linear semihomogeneous boundary-value problem, we seek the existence conditions for the solutions $z = z(t,\varepsilon)$ of the boundary-value problem (1), (2) defined in the class of vector functions $z(\cdot,\varepsilon) \in \mathbf{C}(\mathcal{I},\mathbf{B})$, $z(t,\cdot) \in \mathbf{C}(0,\varepsilon_0]$, which turn into one of the generating solutions $z_0(t,c)$ for $\varepsilon = 0$.

In (1), (2), we perform the change of variables

$$z(t,\varepsilon) = z_0(t,c) + x(t,\varepsilon).$$

As a result, for the deviation $x(t,\varepsilon)$ from the generating solution, we obtain the following boundary-value problem:

$$x(t) - M(t)\int_a^b N(s)x(s)ds = \varepsilon \int_a^b K(t,s)Z(z_0(s,c) + x(s,\varepsilon), s, \varepsilon)\, ds, \tag{10}$$

$$\ell x(\cdot) = \varepsilon J\big(z_0(\cdot,c) + x(\cdot,\varepsilon), \varepsilon\big). \tag{11}$$

We now establish a necessary condition for the existence of solutions $z(t,\varepsilon)$ of the boundary-value problem (1), (2), which turns, for $\varepsilon = 0$, into one of the generating solutions $z_0(t,c) \in \mathbf{C}(\mathcal{I},\mathbf{B})$ of the generating boundary-value problem (3), (4).

Suppose that the boundary-value problem (1), (2) has a solution $z(t,\varepsilon)$. Then, by Theorem 1, the following system of solvability conditions must be valid:

$$\begin{cases} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s)\left[f(t) + \varepsilon \int_a^b K(s,\tau)Z(z(\tau,\varepsilon),\tau,\varepsilon)d\tau \right] ds = 0, \\ \mathcal{P}_{Y_Q} \Bigg[\alpha + \varepsilon J(z(\cdot,\varepsilon),\varepsilon) - \ell f(\cdot) \\ \qquad - \ell M(\cdot) D^- \displaystyle\int_a^b N(s)\left[f(s) + \varepsilon \int_a^b K(s,\tau)Z(z(\tau,\varepsilon),\tau,\varepsilon)d\tau \right] ds \Bigg] = 0. \end{cases}$$

In view of (6) and the fact that $\varepsilon \neq 0$, this system takes the form

$$\begin{cases} \mathcal{P}_{Y_D} \left[\displaystyle\int_a^b N(s) \int_a^b K(s,\tau)Z(z(\tau,\varepsilon),\tau,\varepsilon)d\tau \right] ds = 0, \\ \mathcal{P}_{Y_Q} \left[J(z(\cdot,\varepsilon),\varepsilon) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)Z(z(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \right] = 0. \end{cases} \tag{12}$$

Taking into account the continuity of the operator-valued functions $Z(z,t,\varepsilon)$ and $J(z(\cdot,\varepsilon),\varepsilon)$ with respect to the totality of variables z, t, and ε and passing to the limit at $\varepsilon \to 0$ in system (12), we obtain the following necessary condition for the existence of solutions of the boundary-value problem (1), (2):

$$F(c) = \begin{cases} \mathcal{P}_{Y_D}\left[\int_a^b N(s)\int_a^b K(s,\tau)Z(z_0(\tau,c),\tau,0)d\tau\right]ds = 0, \\ \mathcal{P}_{Y_Q}\left[J(z_0(\cdot,c),0) - \ell M(\cdot)D^-\int_a^b N(s)\int_a^b K(s,\tau)Z(z_0(\tau,c),\tau,0)d\tau\, ds\right] = 0. \end{cases}$$

Thus, the following theorem is valid for the boundary-value problem (1), (2):

Theorem 3. *Suppose that, under the conditions imposed above, the boundary-value problem (1), (2) has a solution* $z(t,\varepsilon)$ *continuous on* $\varepsilon \in [0,\varepsilon_0]$, *which turns, for* $\varepsilon = 0$, *into a generating solution* $z_0(t,c)$ *of the form (7) obtained for* $c = c_0$. *Then the element* $c_0 \in \mathbf{B_1}$ *satisfies the system of gathers*

$$F(c_0) = \begin{cases} \mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)Z((z_0(\tau,c_0)),\tau,\varepsilon)d\tau ds = 0, \\ \mathcal{P}_{Y_Q}\left[J(z_0(\cdot,c_0)) - \ell M(\cdot)D^-\int_a^b N(s)\int_a^b K(s,\tau)Z(z_0(s,c_0)),\tau,\varepsilon)d\tau\, ds\right] = 0. \end{cases} \tag{13}$$

By analogy with weakly nonlinear problems for ordinary differential gathers [2, 4, 5], the system of gathers (13) is called a system of equations for generating constants.

Therefore, if the system of equations (13) has a solution $c = c_0 \in \mathbf{B}$, then the element c_0 determines the generating solution $z_0(t,c_0)$ that can be associated with the solution $z(t,\varepsilon)$ of the original nonlinear boundary-value problem (1), (2). If the system of equations (13) does not have solutions, then the boundary-value problem (1), (2) does not have the desired solution. Thus, the necessary condition for the existence of a solution of the boundary-value problem (1), (2) is satisfied by choosing the constant c in the generating solution (7) as the real root of the system of equations (13).

To prove sufficiency, by using the conditions imposed on the nonlinear operator-valued functions $Z(z,t,\varepsilon)$ and $J(z,\varepsilon)$, we separate the linear parts with respect to x and the terms of order zero with respect to ε. As a result, we get the expansions

$$Z(z_0(t,c_0) + x(t,\varepsilon),t,\varepsilon) = Z_0(t,c_0) + T(t)x(t,\varepsilon) + R(x(t,\varepsilon),t,\varepsilon),$$

$$J(z_0(\cdot,c_0) + x(\cdot,\varepsilon),\varepsilon) = J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon),$$

where

$$Z_0(t,c_0) = Z(z_0(t,c_0),t,0) \in \mathbf{C}(\mathcal{I},\mathbf{B}),$$

$$J_0(\cdot,c_0) = J_0(z_0(\cdot,c_0),0) \in \mathbf{B}_1;$$

$$T(t) = T(t,c_0) = \left.\frac{\partial Z(z,t,0)}{\partial z}\right|_{z=z(t,c_0)} \in \mathbf{C}(\mathcal{I},\mathbf{B}),$$

$$\ell_1 = \left.\frac{\partial J(z,0)}{\partial z}\right|_{z=z(\cdot,c_0)}, \qquad \ell_1\colon \mathbf{C}(\mathcal{I},\mathbf{B}) \to \mathbf{B}_1;$$

$R(x(t,\varepsilon),t,\varepsilon)$ is a nonlinear vector-valued function, and $R_1(x(\cdot,\varepsilon),\varepsilon)$ is a nonlinear vector-valued functional.

We now consider the nonlinearities in the boundary-value problem (10), (11) as inhomogeneities and apply Theorem 1 to this problem. This yields the following expression for the representation of its solution $x(t,\varepsilon)$:

$$x(t,\varepsilon) = \widetilde{M}(t)c + \bar{x}(t,\varepsilon).$$

In this case, the unknown vector $c = c(\varepsilon) \in \mathbf{B}_1$ is determined from the solvability conditions of the form (12), namely,

$$\begin{cases} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)\Big\{Z_0(\tau,c_0) + T(\tau)x(\tau,\varepsilon) \\ \qquad + R(x(\tau,\varepsilon),\tau,\varepsilon)\Big\} d\tau\, ds = 0, \\ \mathcal{P}_{Y_Q}\Bigg[J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon) \\ \qquad - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)\Bigg\{ Z_0(\tau,c_0) \\ \qquad + T(\tau)x(\tau,\varepsilon) + R(x(\tau,\varepsilon),\tau,\varepsilon)\Bigg\} d\tau ds \Bigg] = 0. \end{cases} \tag{14}$$

The unknown vector function $\bar{x}(t,\varepsilon)$ is defined by the formula

$$\bar{x}(t,\varepsilon) = \varepsilon\left(G\int_a^b K(\cdot,s)\left\{Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon)\right\} ds\right)(t)$$
$$+ M(t)Q^-\big[J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon)\big],$$

where the operator G acts upon the vector function

$$\varphi(t,\varepsilon) = \int_a^b K(t,s)\big\{Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon)\big\} ds$$

by the rule (8).

Substituting the expression $x(t,\varepsilon)$ for $\widetilde{M}(t)c + \bar{x}(t,\varepsilon)$ in (14), isolating the terms containing the constant c, and taking into account (13), we obtain the operator equation

$$B_0c = -\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds, \\ \mathcal{P}_{Y_Q}\Bigg[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) \\ \qquad -\ell M(\cdot)D^-\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds\Bigg] \end{bmatrix},$$

where

$$B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$B_1 = \mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)T(\tau)\widetilde{M}(\tau)d\tau ds,$$

$$B_2 = \mathcal{P}_{Y_Q}\left[\ell_1\widetilde{M}(\cdot) - \ell M(\cdot)D^-\int_a^b N(s)\int_a^b K(s,\tau)T(\tau)\widetilde{M}(\tau)d\tau ds\right],$$

$$\overline{R}(x(s,\varepsilon),\bar{x}(s,\varepsilon),s,\varepsilon) := T(s)\bar{x}(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon),$$

$$\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) := \ell_1\bar{x}(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon).$$

The operator B_0 acts from the Banach space $\mathbf{B}$ into the direct product of the Banach spaces $\mathbf{B}$ and $\mathbf{B}_1$, i.e., $B_0\colon \mathbf{B} \to \mathbf{B} \times \mathbf{B}_1$.

In view of the fact that the vector constant $c_0 \in \mathbf{B_1}$ satisfies the system of equations for the generating constants (13), in order to find a continuous (in ε) solution $x(\cdot,\varepsilon) \in \mathbf{C}(\mathcal{I},\mathbf{B})$, $x(t,0) = 0$, of the weakly nonlinear boundary-value problem (1), (2), we consider the equivalent operator system

$$x(t,\varepsilon) = \widetilde{M}(t)c(\varepsilon) + \bar{x}(t,\varepsilon),$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} c = -\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau ds, \\ \mathcal{P}_{Y_Q}\Bigg[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) \\ \qquad -\ell M(\cdot)D^-\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds\Bigg] \end{bmatrix}, \tag{15}$$

$$\bar{x}(t,\varepsilon) = \varepsilon\left(G \int\limits_a^b K(\cdot,s)\,[Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon)]\,ds \right)(t)$$

$$+ M(t)Q^{-}\big[J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon)\big],$$

Let $B_1 \in \mathbf{GI}(\mathbf{B},\mathbf{B})$ and $B_2 \in \mathbf{GI}(\mathbf{B},\mathbf{B}_1)$. Then, by Theorem 2, in view of the normal solvability, the second equation of the operator system (15) is solvable if and only if its right-hand side satisfies the condition

$$\begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds, \\ \mathcal{P}_{Y_Q}\left[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) - \ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau) \right. \\ \left. \times\, \overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \right] \end{bmatrix} = 0_{2\times 1}, \tag{16}$$

where $0_{2\times 1}$ is a dimensional zero matrix, $\widetilde{\mathcal{P}}_{Y_{B_1}} = I_{\mathbf{B}} - B_1 \mathcal{P}_{N(B_2)} B_1^{-}$, and $B_{12} = -B_1 B_2^{-}$.

For

$$\begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{Y_D} \\ \mathcal{P}_{Y_Q} \end{bmatrix} = 0_{2\times 1}, \tag{17}$$

condition (16) is always satisfied and, by Theorem 2, the second equation of the operator system (15) possesses a family of solutions

$$c(\varepsilon) = \mathcal{P}_{N(B_0)}\hat{c} - \big[\mathcal{P}_{N(B_2)}B_1^{-} \;\; B_2^{-}\big]$$

$$\times \begin{bmatrix} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds, \\ \mathcal{P}_{Y_Q}\left[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) \right. \\ \left. -\ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \right] \end{bmatrix},$$

where $\mathcal{P}_{N(B_0)} = \mathcal{P}_{N(B_2)}\mathcal{P}_{N(B_1)}$ is the projector onto the null space $N(B_0)$ of the operator B_0, $\hat{c}$ is an arbitrary element of the Banach space $\mathbf{B}$, and $\big[\mathcal{P}_{N(B_2)}B_1^{-} \;\; B_2^{-}\big]$ is a generalized operator inverse to the operator

$$B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

We set $\hat{c} \equiv 0$. If conditions (17) are satisfied, then one solution of the second equation in the operator system (15) takes the form

$$c(\varepsilon) = \widetilde{B}_1^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds$$

$$+ \widetilde{B}_2^- \Bigg[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon)$$

$$- \ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \Bigg],$$

where $\widetilde{B}_1^- = -\mathcal{P}_{N(B_2)} B_1^- \mathcal{P}_{Y_D}$ and $\widetilde{B}_2^- = -B_2^- \mathcal{P}_{Y_Q}$.

Thus, if conditions (17) are satisfied, then the operator system (15) takes the form

$$x(t,\varepsilon) = \widetilde{M}(t)c(\varepsilon) + \bar{x}(t,\varepsilon),$$

$$c(\varepsilon) = \widetilde{B}_1^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds$$

$$+ \widetilde{B}_2^- \Bigg[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) - \tag{18}$$

$$- \ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \Bigg],$$

$$\bar{x}(t,\varepsilon) = \varepsilon \left(G \int_a^b K(\cdot,s)\{Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon)\}\, ds \right)(t)$$

$$+ M(t)Q^-\big[J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon)\big].$$

By analogy with [1, 4, 5, 9], it can be shown that the operator system (18) belongs to the class of systems for which it is possible to apply the convergent method of simple iterations.

Theorem 4. *Suppose that the generating boundary-value problem (3), (4) with conditions (6) has a family of generating solutions (7). Then, for each element* $c_0 \in \mathbf{B_1}$ *satisfying the system of equations for the generating constants (13) with the following conditions:*

$$\mathcal{P}_{N(B_0)} \neq 0, \qquad \begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{Y_D} \\ \mathcal{P}_{Y_Q} \end{bmatrix} = 0_{2\times 1}$$

the boundary-value problem (1), (2) has at least one solution $z(t,\varepsilon) = z_0(t,c_0) + x(t,\varepsilon)$ *continuous in* ε, *which turns into a generating solution* $z_0(t,c_0)$ *for* $\varepsilon = 0$. *This solution can be found as a result of convergence to* $[0,\varepsilon_*] \subset [0,\varepsilon_0]$ *of the iterative process*

$$z_{k+1}(t,\varepsilon) = z_0(t,c_0) + x_{k+1}(t,\varepsilon),$$

$$x_{k+1}(t,\varepsilon) = \widetilde{M}(t)c_k(\varepsilon) + \bar{x}_{k+1}(t,\varepsilon), \quad k = 0,1,2,\ldots,$$

$$c_k(\varepsilon) = \widetilde{B}_1^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x_k(\tau,\varepsilon),\bar{x}_k(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds$$

$$+ \widetilde{B}_2^- \Bigg[\overline{R}_1(x_k(\cdot,\varepsilon),\bar{x}_k(\cdot,\varepsilon),\varepsilon) \tag{19}$$

$$- \ell M(\cdot)D^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x_k(\tau,\varepsilon),\bar{x}_k(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \Bigg],$$

$$\bar{x}_{k+1}(t,\varepsilon) = \varepsilon \Bigg[G\Bigg(\int_a^b K(\cdot,s) \Big\{ Z_0(s,c_0)$$

$$+ T(s)\big[\widetilde{M}(s)c_k(\varepsilon) + \bar{x}_k(s,\varepsilon)\big] + R(x_k(s,\varepsilon),s,\varepsilon)ds \Big\} \Bigg)(t)$$

$$+ M(t)Q^- \big\{ J_0(\cdot,c_0) + \ell_1\big[\widetilde{M}(\cdot)c_k(\varepsilon) + \bar{x}_k(\cdot,\varepsilon)\big] + R_1(x_k(\cdot,\varepsilon),\varepsilon) \big\} \Bigg].$$

Remark 1. If $\mathcal{P}_{N(B_0)} \neq 0$ and

$$\begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} = 0_{2\times 1},$$

then the operator B_0 is d-normal. In this case, condition (17) is always satisfied, the second equation of the operator system (15) is always solvable, and the generalized inverse operator B_0^- is a right inverse operator $(B_0)_r^{-1}$ [13]. Then the boundary-value problem (1), (2) has at least one solution determined by the convergent iterative process (19) in which $B_0^- = (B_0)_r^{-1}$.

Remark 2. If $\mathcal{P}_{N(B_0)} = 0$ and

$$\mathcal{P}_{Y_{B_0}} = \begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \neq 0_{2\times 1},$$

then the operator B_0 is n-normal. In this case, the generalized inverse operator B_0^- is the left inverse operator $(B_0)_l^{-1}$ and, under condition (17), the second equation in the operator system (15) is definitely solvable [13]. Then, for each c_0 in the system for generating constants (13), the boundary-value problem (1), (2) has a single solution determined by the convergent iterative process (19) in which $B_0^- = (B_0)_l^{-1}$.

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