DOI 10.1007/s10958-019-04521-5

Journal of Mathematical Sciences, Vol. 243, No. 1, November, 2019

CONVERGENCE OF THE NEWTON–KURCHATOV METHOD UNDER WEAK CONDITIONS

S. M. Shakhno and H. P. Yarmola

UDC 519.6

We study the semilocal convergence of the combined Newton–Kurchatov method to a locally unique solution of the nonlinear equation under weak conditions imposed on the derivatives and first-order divided differences. The radius of the ball of convergence is established and the rate of convergence of the method is estimated. As a special case of these conditions, we consider the classical Lipschitz conditions.

Introduction

Consider an equation

$$H(x) \equiv F(x)+G(x) = 0\,, \tag{1}$$

where F and G are nonlinear operators defined on an open convex set D of the Banach space X with values in the Banach space Y. Let F be a Frèchet differentiable operator and let G be a continuous operator whose differentiability is, generally speaking, not required.

In view of the properties of the operator H, Eq. (1) cannot be solved by using the classical Newton method. As a rule, for this purpose, the researchers use either a Newton-type method [8, 14]

$$x_{n+1} = x_n - [F'(x_n)]^{-1} H(x_n), \qquad n \ge 0,$$

or difference methods, e.g., the method of chords (secants) [6, 9]

$$x_{n+1} = x_n - [H(x_n; x_{n-1})]^{-1} H(x_n), \qquad n \ge 0,$$

or the Kurchatov method of linear interpolation [1, 5, 12]

$$x_{n+1} = x_n - [H(2x_n - x_{n-1}; x_{n-1})]^{-1} H(x_n), \qquad n \ge 0,$$

or the method developed in [11]

$$x_{n+1} = x_n - [F(2x_n - x_{n-1}; x_{n-1}) + G(x_n; x_{n-1})]^{-1} H(x_n), \qquad n \ge 0,$$

I. Franko Lviv National University, Lviv, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 60, No. 2, pp. 7–13, April–June, 2017. Original article submitted March 9, 2017.

or the combined method [7, 10]

$$x_{n+1} = x_n - [F'(x_n) + G(x_n; x_{n-1})]^{-1} H(x_n), \qquad n \ge 0.$$

Here, $G(x; y)$ is the divided difference of the first order for the operator G at the points x and y.

In [3, 4, 13], the authors proposed a one-step iterative process

$$x_{n+1} = x_n - [F'(x_n) + G(2x_n - x_{n-1}; x_{n-1})]^{-1} H(x_n), \qquad n \ge 0, \tag{2}$$

and studied the properties of local and semilocal convergence of this method for the classical and generalized Lipschitz conditions. Method (2) is, in fact, a combination of the Newton method [2] and the difference method of linear interpolation [1, 5, 12].

In the present work, we study the convergence of method (2) under weak conditions [8, 9, 11]. In the case of conditions of type ω, we assume that the derivatives of the operator F and the first-order divided differences of the operator G satisfy the conditions

$$\left\| A_0^{-1}(F'(x) - F'(y)) \right\| \le \omega_1(\|x - y\|), \qquad x, y \in D, \tag{3}$$

$$\left\| A_0^{-1}(G(x; y) - G(u; v)) \right\| \le \omega_2(\|x - u\|, \|y - v\|), \qquad x, y, u, v \in D. \tag{4}$$

Here, ω_1 is a nondecreasing positive function on the segment $[0, R]$ and $\omega_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function of two arguments. In addition, the function ω_1 satisfies the condition

$$\omega_1(tr) \le h(t)\omega_1(r), \ t \in [0,1], \ r \in [0, R],$$

where $h : [0,1] \to \mathbb{R}$. The properties of the function h were described in [8].

Note that conditions (3) and (4) generalize the classical Lipschitz and Hölder conditions and, generally speaking, do not require the differentiability of the operator G.

Another option is given by conditions of type ε. For the Frèchet derivative and divided differences of the first order, they take the form

$$\left\| A_0^{-1}(F'(x) - F'(y)) \right\| \le \varepsilon_1, \qquad x, y \in D, \tag{5}$$

$$\left\| A_0^{-1}(G(x; y) - G(u; v)) \right\| \le \varepsilon_2, \qquad x, y, u, v \in D. \tag{6}$$

1. Semilocal Convergence of the Combined Method (2)

Let $\bar{x} \in D$. We denote

$$B(\bar{x}, R) = \{x \in X : \|x - \bar{x}\| < R\}, \qquad \overline{B(\bar{x}, R)} = \{x \in X : \|x - \bar{x}\| \le R\},$$

$$\Phi = \int_0^1 h(t)\,dt.$$

Theorem 1. *Assume that* F *and* G *are nonlinear operators defined on an open convex set* D *of the Banach space* X *with values in the Banach space* Y, F' *is the Frèchet derivative of the operator* F, *and* $G(\cdot;\cdot)$ *is the first-order divided difference of the operator* G *on the set* D. *Suppose that*

- *the linear operator* $A_0 = F'(x_0)+G(2x_0 - x_{-1};x_{-1})$, *where* x_{-1} *and* x_0 *are points from* D, *is invertible;*
- *the numbers* $\eta > 0$ *and* $\alpha > 0$ *are such that*

$$\left\|A_0^{-1}(F(x_0)+G(x_0))\right\| \le \eta, \qquad \|x_0 - x_{-1}\| \le \alpha; \tag{7}$$

- *conditions (3) and (4) are satisfied on* D;
- *the equation*

$$u\left(1-\frac{m}{1-\omega_1(u)-\omega_2(3u+\alpha,u+\alpha)}\right)-\eta = 0,$$

where $m = \Phi\omega_1(\eta)+\max\{\omega_2(\eta+\alpha,\alpha),\omega_2(2\eta,\eta)\}$, *has at least one positive root and* R *is its least positive root*.

If

$$\omega_1(R)+\omega_2(3R+\alpha,R+\alpha) < 1, \qquad M = \frac{m}{1-\omega_1(R)-\omega_2(3R+\alpha,R+\alpha)} < 1,$$

$$B(x_0,3R)\subset D, \qquad \alpha < R,$$

then the sequence $\{x_n\}_{n\ge 0}$ *generated by the iteration process (2) is well defined. Moreover, it is contained in* $B(x_0,R)$ *and converges to the unique solution* $x^* \in \overline{B(x_0,R)}$ *of Eq. (1).*

The theorem is proved by induction. Thus, we denote

$$A_n = F'(x_n)+ G(2x_n - x_{n-1};x_{n-1}).$$

Note that if $x_n, x_{n-1} \in B(x_0,R)$, $n \ge 0$, then the inequality

$$\|(2x_n - x_{n-1})-x_0\| \le \|2x_n - 2x_0\|+\|x_{n-1}-x_0\| < 3R$$

implies that $2x_n - x_{n-1} \in B(x_0,3R)\subset D$. Thus, we can show that all A_n, $n \ge 1$, are invertible operators.

In view of (2) and (7), for $n = 0$, we get

$$\|x_1 - x_0\| \le \left\|A_0^{-1}(F(x_0)+G(x_0))\right\| \le \eta < R.$$

Hence, $x_1 \in B(x_0,R)$.

By using conditions (3) and (4), we obtain

$$\left\|I - A_0^{-1}A_1\right\| = \left\|A_0^{-1}(A_0 - A_1)\right\|$$

$$\le \omega_1(\|x_1 - x_0\|) + \omega_2(\|2x_0 - x_{-1} - 2x_1 + x_0\|, \|x_{-1} - x_0\|)$$

$$\le \omega_1(\eta) + \omega_2(2\eta + \alpha, \alpha) \le \omega_1(R) + \omega_2(2R + \alpha, \alpha)$$

$$\le \omega_1(R) + \omega_2(3R + \alpha, R + \alpha) < 1.$$

By the Banach inverse theorem, we conclude that $A_1^{-1}A_0$ exists and

$$\left\|A_1^{-1}A_0\right\| \le \frac{1}{1 - \omega_1(R) - \omega_2(3R + \alpha, R + \alpha)}.$$

Thus, we can write

$$A_0^{-1}(F(x_1) + G(x_1)) = A_0^{-1}(F(x_1) - F(x_0) - F'(x_0)(x_1 - x_0))$$

$$+ A_0^{-1}(G(x_1) - G(x_0) - G(2x_0 - x_{-1}; x_{-1})(x_1 - x_0))$$

$$= \int_0^1 A_0^{-1}(F'(x_0 + t(x_1 - x_0)) - F'(x_0))\,dt\,(x_1 - x_0)$$

$$+ A_0^{-1}(G(x_1; x_0) - G(2x_0 - x_{-1}; x_{-1}))(x_1 - x_0),$$

whence, in view of conditions (3) and (4), we find

$$\|x_2 - x_1\| = \left\|A_1^{-1}(F(x_1) + G(x_1))\right\|$$

$$\le \left\|A_1^{-1}A_0\right\| \left\|A_0^{-1}(F(x_1) + G(x_1))\right\|$$

$$\le \frac{\Phi\omega_1(\|x_1 - x_0\|) + \omega_2(\|2x_0 - x_{-1} - x_1\|, \|x_{-1} - x_0\|)}{1 - \omega_1(R) - \omega_2(3R + \alpha, R + \alpha)} \|x_1 - x_0\|$$

$$\le \frac{\Phi\omega_1(\eta) + \omega_2(\eta + \alpha, \alpha)}{1 - \omega_1(R) - \omega_2(3R + \alpha, R + \alpha)} \|x_1 - x_0\| \le M\|x_1 - x_0\| < \eta.$$

On the other hand,

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\|$$

$$\le (M+1)\|x_1 - x_0\| \le (M+1)\eta = \frac{1-M^2}{1-M}\eta < \frac{1}{1-M}\eta = R.$$

Hence, $x_2 \in B(x_0, R)$.

Assume that, for $k = 1,\ldots,n-1$, the following assertions are true:

— $A_k^{-1}A_0$ exists and

$$\|A_k^{-1}A_0\| \le \frac{1}{1-\omega_1(R)-\omega_2(3R+\alpha, R+\alpha)};$$

— $\|x_{k+1} - x_k\| \le M\|x_k - x_{k-1}\| \le M^k\|x_1 - x_0\| < \eta$;

— $x_{k+1} \in B(x_0, R)$.

Thus, by virtue of conditions (3) and (4), for $k = n$, we obtain

$$\|I - A_0^{-1}A_n\| = \|A_0^{-1}(A_0 - A_n)\|$$

$$\le \omega_1(\|x_0 - x_n\|) + \omega_2(\|2x_0 - x_{-1} - 2x_n + x_{n-1}\|, \|x_{-1} - x_{n-1}\|)$$

$$\le \omega_1(R) + \omega_2(3R+\alpha, R+\alpha) < 1.$$

By the Banach theorem, $A_n^{-1}A_0$ exists and, in addition,

$$\|A_n^{-1}A_0\| \le \frac{1}{1-\omega_1(R)-\omega_2(3R+\alpha, R+\alpha)}.$$

In view of the equality

$$A_0^{-1}(F(x_n)+G(x_n)) = A_0^{-1}(F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n - x_{n-1}))$$

$$+ A_0^{-1}(G(x_n)-G(x_{n-1})-G(2x_{n-1}-x_{n-2}; x_{n-2})(x_n - x_{n-1}))$$

$$= \int_0^1 A_0^{-1}(F'(x_{n-1}+t(x_n - x_{n-1})) - F'(x_{n-1}))\,dt\,(x_n - x_{n-1})$$

$$+ A_0^{-1}(G(x_n; x_{n-1}) - G(2x_{n-1}-x_{n-2}; x_{n-2}))(x_n - x_{n-1})$$

and conditions (3) and (4), we find

$$\|x_{n+1} - x_n\| = \|A_n^{-1}(F(x_n)+G(x_n))\|$$

$$\le \left\|A_n^{-1}A_0\right\|\left\|A_0^{-1}(F(x_n)+G(x_n))\right\|$$

$$\le \frac{1}{1-\omega_1(R)-\omega_2(3R+\alpha,R+\alpha)}\big(\Phi\omega_1(\|x_n-x_{n-1}\|)$$

$$+\omega_2(\|2x_{n-1}-x_{n-2}-x_n\|,\|x_{n-1}-x_{n-2}\|)\big)\|x_n-x_{n-1}\|$$

$$\le \frac{\Phi\omega_1(\eta)+\omega_2(2\eta,\eta)}{1-\omega_1(R)-\omega_2(3R+\alpha,R+\alpha)}\|x_n-x_{n-1}\|$$

$$\le M\|x_n-x_{n-1}\| \le M^n\|x_1-x_0\| < \eta.$$

We now show that $x_{n+1}\in B(x_0,R)$. Indeed,

$$\|x_{n+1}-x_0\| \le \|x_{n+1}-x_n\|+\|x_n-x_{n-1}\|+\ldots+\|x_1-x_0\|$$

$$\le (M^n+M^{n-1}+\ldots+M+1)\|x_1-x_0\| \le \frac{1-M^{n+1}}{1-M}\eta < \frac{1}{1-M}\eta = R,$$

and $x_{n+1}\in B(x_0,R)$.

Let us show that $\{x_n\}_{n\ge 0}$ is the Cauchy sequence. Indeed, for $p\ge 1$, we get

$$\left\|x_{n+p}-x_n\right\| \le \left\|x_{n+p}-x_{n+p-1}\right\|+\left\|x_{n+p-1}-x_{n+p-2}\right\|+\ldots+\|x_{n+1}-x_n\|$$

$$\le (M^{p-1}+M^{p-2}+\ldots+1)\|x_{n+1}-x_n\|$$

$$=\frac{1-M^p}{1-M}\|x_{n+1}-x_n\| \le \frac{1-M^p}{1-M}M^n\eta < \frac{M^n}{1-M}\eta.$$

Hence, $\{x_n\}_{n\ge 0}$ is a fundamental sequence that converges to $x^*\in\overline{B(x_0,R)}$.

We now show that x^* is a unique solution of Eq. (1). Since

$$\left\|A_0^{-1}H(x_n)\right\| \le (\Phi\omega_1(\eta)+\omega_2(2\eta,\eta))\|x_n-x_{n-1}\|$$

and $\|x_n-x_{n-1}\|\to 0$ as $n\to\infty$, we conclude that $H(x^*)=0$.

The uniqueness of solution is proved by contradiction. Assume that there exists $y^*\in\overline{B(x_0,R)}$, $y^*\ne x^*$, and $H(y^*)=0$. We denote

$$A = \int_0^1 F'(x^*+t(y^*-x^*))\,dt+G(y^*;x^*).$$

In view of conditions (3) and (4), we get

$$\left\|A_0^{-1}(A_0 - A)\right\| \le \int_0^1 \omega_1\left(\left\|x_0 - x^* - t(y^* - x^*)\right\|\right) dt + \omega_2\left(\left\|2x_0 - x_{-1} - y^*\right\|, \left\|x_{-1} - x^*\right\|\right)$$

$$\le \int_0^1 \omega_1\left((1-t)\left\|x_0 - x^*\right\| + t\left\|x_0 - y^*\right\|\right) dt + \omega_2\left(\left\|x_0 - x_{-1}\right\| + \left\|x_0 - y^*\right\|, \left\|x_{-1} - x_0\right\| + \left\|x_0 - x^*\right\|\right)$$

$$\le \omega_1(R) + \omega_2(R+\alpha, R+\alpha) < 1.$$

By the Banach theorem, the operator A^{-1} exists. Since A is invertible, the identity

$$A(y^* - x^*) = H(y^*) - H(x^*)$$

implies that $y^* = x^*$.

The theorem is proved.

Theorem 2. *Let F and G be nonlinear operators defined on an open convex set D of the Banach space X with values in the Banach space Y, let F' be the Frèchet derivative of the operator F, and let $G(\cdot;\cdot)$ be the first-order divided difference of the operator G on D. Assume that*

- *the linear operator $A_0 = F'(x_0) + G(2x_0 - x_{-1}; x_{-1})$, where x_{-1} and x_0 are points from D, is invertible;*
- *conditions (5) and (6) are satisfied on D;*
- *the numbers $\eta > 0$, $\gamma > 0$, and $R > 0$ are such that*

$$\left\|A_0^{-1}(F(x_0) + G(x_0))\right\| \le \eta, \qquad \left\|x_0 - x_{-1}\right\| < R,$$

$$\gamma = \frac{\varepsilon_1 + \varepsilon_2}{1 - (\varepsilon_1 + \varepsilon_2)} < 1, \qquad \frac{\eta}{1-\gamma} < R, \qquad B(x_0, 3R) \subset D.$$

Then the iterative process (2) is well defined and the sequence $\{x_n\}_{n\ge 0}$ generated by this process is contained in $B(x_0, R)$ and converges to the unique solution $x^* \in \overline{B(x_0, R)}$ of Eq. (1). Furthermore, the following estimate is true:

$$\|x_n - x^*\| \le \frac{\gamma^n}{1-\gamma}\eta. \tag{8}$$

The proof of convergence of method (2) to the unique solution x^* of Eq. (1) is carried out by induction as in Theorem 1.

We now show that estimate (8) is true. For $n, p \in \mathbb{N}$, we obtain

$$\|x_{n+p} - x_n\| \le \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \ldots + \|x_{n+1} - x_n\|$$

$$\le (\gamma^{p-1} + \gamma^{p-2} + \ldots + 1)\|x_{n+1} - x_n\|$$

$$\le \frac{1-\gamma^p}{1-\gamma}\gamma^n \eta .$$

Thus, passing to the limit as $p \to \infty$, we arrive at (8).

The theorem is proved.

Let

$$\omega_1(\|x-y\|) = 2\ell\|x-y\| \quad \text{and} \quad \omega_2(\|x-u\|, \|y-v\|) = p(\|x-u\| + \|y-v\|).$$

Hence, Theorem 1 implies the convergence of the method under the Lipschitz conditions.

Corollary 1. *Let F and G be nonlinear operators defined on an open convex set D of the Banach space X with values in the Banach space Y; let F' be the Frèchet derivative of the operator F, and let $G(\cdot;\cdot)$ be the first-order divided difference of the operator G on D. Assume that*

— *the linear operator $A_0 = F'(x_0) + G(2x_0 - x_{-1}; x_{-1})$, where x_{-1} and x_0 are points from D, is invertible;*

— *the numbers $\eta > 0$ and $\alpha > 0$ are such that*

$$\|A_0^{-1}(F(x_0) + G(x_0))\| \le \eta, \quad \|x_0 - x_{-1}\| \le \alpha;$$

— *the Lipschitz conditions*

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2\ell\|x-y\|,$$

$$\|A_0^{-1}(G(x;y) - G(u;v))\| \le p(\|x-u\| + \|y-v\|)$$

hold on the set D;

— *the equation*

$$u\left(1-\frac{m}{1-2\ell u-p(4u+2\alpha)}\right)-\eta = 0,$$

where

$$m=\ell\eta+\max\{p(\eta+2\alpha),3p\eta\},$$

has at least one positive root, and R *is its least positive root.*

If

$$2\ell R+p(4R+2\alpha) < 1, \qquad M = \frac{m}{1-2\ell R-p(4R+2\alpha)}<1,$$

$$B(x_0,3R)\subset D, \quad \textit{and} \quad \alpha<R,$$

then the sequence $\{x_n\}_{n\geq 0}$ *generated by the iterative process (2) is well defined. Moreover, it is contained in* $B(x_0,R)$ *and converges to the unique solution* $x^* \in \overline{B(x_0,R)}$ *of Eq. (1).*

Note that, for $F(x)=0$, method (2) turns into the Kurchatov method. At the same time, if $G(x)=0$, then we get the Newton method. Hence, by virtue of Theorems 1 and 2 and Corollary 1, we get the semilocal convergence theorems for the base methods. These results do not contradict the results obtained earlier.

The results of numerical investigations of method (2) can be found in [3].

CONCLUSIONS

We apply the combined Newton–Kurchatov method to the solution of nonlinear equations with nondifferentiable operator. The semilocal convergence of the method under weak conditions that do not require the differentiability of the nonlinear operator is analyzed and the rate of convergence of this method is estimated.

REFERENCES

1. V. A. Kurchatov, "A method of linear interpolation for the solution of functional equations," *Dokl. Akad. Nauk SSSR,* **198**, No. 3, 524–526 (1971).
2. J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables,* Academic Press, New York (1970).
3. S. M. Shakhno and G. P. Yarmola, "A two-point method for the solution of nonlinear equations with nondifferentiable operator," *Mat. Studii,* **36**, No. 2, 213–220 (2011).
4. S. M. Shakhno and G. P. Yarmola, "Convergence of the Newton–Kurchatov method under the classical Lipschitz conditions," *Zh. Obchysl. Prykl. Mat.,* No. 1 (121), 89–97 (2016).
5. S. M. Shakhno, "On the difference method with quadratic convergence for the solution of nonlinear operator equations," *Mat. Studii,* **26**, No. 1, 105–110 (2006).
6. S. Shakhno, "The method of chords under the generalized Lipschitz conditions for the first-order divided differences," *Mat. Visn. NTSh.,* **4**, 296–303 (2007).
7. I. K. Argyros, "A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space," *J. Math. Anal. Appl.,* **298**, No. 2, 374–397 (2004).
8. I. K. Argyros and S. Hilout, "Newton–Kantorovich approximations under weak continuity conditions," *J. Appl. Math. Comput.,* **37**, Nos. 1-2, 361–375 (2011).

9. I. K. Argyros and H. Ren, "On the convergence of a Newton-like method under weak conditions," *Comm. Korean Math. Soc.,* **26**, No. 4, 575–584 (2011).
10. E. Cătinaş, "On some iterative methods for solving nonlinear equations," *Rev. d'Anal. Numér. Théor. de l'Appr.,* **23**, No. 1, 47–53 (1994).
11. H. Ren and I. K. Argyros, "A new semilocal convergence theorem for a fast iterative method with nondifferentiable operators," *J. Appl. Math. Comput.,* **34**, Nos. 1-2, 39–46 (2010).
12. S. M. Shakhno, "On a Kurchatov's method of linear interpolation for solving nonlinear equations," *Proc. Appl. Math. Mech.,* **4**, No. 1, 650–651 (2004). https://doi.org/10.1002/pamm.200410306.
13. S. M. Shakhno, "Combined Newton–Kurchatov method under the generalized Lipschitz conditions for the derivatives and divided differences," *Zh. Obchysl. Prykl. Mat.,* No. 2 (119), 78–89 (2015).
14. P. P. Zabrejko and D. F. Nguen, "The majorant method in the theory of Newton–Kantorovich approximations and the Pták error estimates," *Numer. Funct. Anal. Optim.,* **9**, Nos. 5-6, 671–684 (1987).