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To the theory of quasiconformal mappings

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The article is dedicated to the memory of Professor Bogdan Bojarski

Abstract. The open questions of the theory of quasiconformal mappings that are adjacent to the field of studies of Professor Bogdan Bojarski are discussed.

Keywords. Quasiconformal mapping, Beltrami equation, Liouville theorem, nonlinear operator, distortion, isotopy.

1. Introduction

One of key objects of early studies of Professor B. Bojarski was the Beltrami equation in its various manifestations and applications. The interest of Bogdan Bojarski in this theme was apparenly stimulated to a large extent by I. N. Vekua.

The works by L. Ahlfors [1], I. N. Vekua [2], and B. Bojarski [3] appeared practically simultaneously and further became classical. In [4], I. N. Vekua gave the efficient proof of the theorem which is frequently called after the work by L. Ahlfors and L. Bers [5] the measurable Riemann mapping theorem, though this term is not quite correct. The compact presentation of the theorem with the analysis of the holomorphic dependence of the solution on a parameter can be found, for example, in [6].

This theorem generalizing the classical Riemann theorem of conformal mapping plays, the key apparatus role almost in all questions of the geometric theory of functions (quasiconformal mappings, Teichmüller theory, holomorphic dynamics, geometry proper...).

The history is repeated. This time, apparently under the influence of Professor B. Bojarski, Tadeusz Iwaniec has entered this circle of questions and developed them. His works are related to the Beltrami equation, general questions of equations of the elliptic type, and multidimensional quasiconformal mappings [7–9].

Below, I will consider some questions with which I was faced during the studies of multidimensional quasiconformal mappings. It seems to me that they remain still open.

2. Beltrami and Liouville

The classical Riemann theorem of conformal mapping for the domains of a plane is related to the Cauchy–Riemann equation. The same question about the conformal mapping of a domain of the surface on a plane (like the question about the conformally Euclidean metric on a surface) leads to the Beltrami equation. The required solution exists, implies local conformal flexibility (elasticity) of two-dimensional surfaces.

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For higher dimensions, the situation changes drastically. For example, in an Euclidean space whose dimension is more than two, only "linear-fractional" conformal mappings (compositions, homotheties, translations, and inversions) exist. It is the classical Liouville theorem of conformal rigidity of the domains of a space. The cause is in the following. In contrast to the Cauchy–Riemann system arising in the two-dimensional case, the condition of conformality of a mapping for higher dimensions leads to the overdetermined system of equations whose solutions were indicated above.

Recall another classical Liouville theorem telling that any bounded entire function is constant. It is valid not only for holomorphic functions, but also for the solutions of a wide class of equations sometimes without assumption that the space is finite-dimensional. For example, such theorem is valid for the quasiconformal mappings of the Euclidean space $\mathbb{R}^n$ onto itself for $n \geq 2$.

The known proofs of this theorem for quasiconformal mappings concern any, finite dimension. It is a challenge to find a proof in the infinite-dimensional case, for example, for a separable Hilbert space.

Note that the Liouville theorem for conformal mappings of domains of a space was proved (practically with the same arguments) in the infinite-dimensional case [10] as well.

Generalizing the notion of a conformal (quasiconformal) mapping of domains of Riemann manifolds of the same dimension, M. Gromov proposed to consider a mapping of metric spaces, for example, a mapping $F : \mathbb{R}^m \to \mathbb{R}^n \quad (m \geq n)$, to be conformal (quasiconformal), if the infinitely small ball at every point of the mapped domain is transformed in an infinitely small ball (respectively, in an ellipsoid whose eccentricities are bounded by a common constant) [11]. In connection with such extension of the notions of conformality and quasiconformality of a mapping, M. Gromov posed naturally the question: Which facts of the classical theory are valid for those mappings? In particular, let the mapping $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be conformal and bounded. Is it true that this mapping is a constant for $n \geq 2$?

3. Nonlinear operators

The quasiconformal mappings satisfy the theorem connecting the local and global invertibilities of a mapping. This theorem can be formulated as follows:

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally invertible quasiconformal mapping, and if $n > 2$, then the equation $f(x) = y$ has the unique solution for any right-hand side $y \in \mathbb{R}^n$.

The question about the validity of *such theorem for nonlinear operators acting in the Banach or Hilbert space with infinite dimension* remains open.

No matter of importance of nonlinear operators with indicated condition on their quasiconformality, the very statement of such question leads to a number of natural questions related already to the theory of quasiconformal mappings in the finite-dimensional case. Recall the following.

Radius of injectiveness.

O. Martio, S. Rickman, and J. Väisälä [12] found the remarkable development of the theorem of global homeomorphism, which also generalizes the quasiisometry result of F. John [13] to the case of quasiconformal mappings.

If the mapping $f : B^n \to \mathbb{R}^n$ of a unit ball is locally homeomorphic and k-quasiconformal, then, for $n > 2$, *there exists a quantity $r = r(k, n)$ which depends only on the coefficient of quasiconformality of a mapping and on the dimension of a space such that the mapping is homeomorphic in a ball $B^n(r) \subset B^n$ with the radius $r(k, n)$.*

The quantity $r(k, n)$ is called the *radius of injectivity* of a mapping.

It is natural to clarify how this quantity depends on n. Is it true that *there exists a function $\rho(k) > 0$ which depends only on the coefficient of quasiconformality of a mapping and can serve a guaranteed radius of injectivity in a space $\mathbb{R}^n$ with any dimension $n > 2$?*

The theorem of injectivity radius generalizes, as it was mentioned above, the theorem of global homeomorphism. Indeed, if the unit ball is replaced by a ball with different radius, then the ball, where the mapping is injective, is obviously changed proportionally. In particular, it will be infinite, if the input ball coincides with the whole spece. Of course, this proves only the injectivity of the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, but all the rest, i.e., the relation $f(\mathbb{R}^n) = \mathbb{R}^n$ is simpler.

We note that, for the quasiisometric mappings (when not only the ratios of local tensions, but the tensions themselves are bounded), the existence of the ingectivity radius is easily proved for any locally invertible quasiisometric mapping of a ball and in any Banach space with any (finite or infinite) dimension. It is the John theorem.

4. The theorem of distortion

The proof of the Martio–Rickman–Väisälä theorem on radius of injectivity is based on the extreme property of the Teichmüller ring and on the following theorem of distortion for quasiconformal mappings.

Let $f : D \to \mathbb{R}^n$ be a homeomorphic quasiconformal mapping of a domain D of the Euclidean space $\mathbb{R}^n$ of dimension $n \geq 2$ into the space $\mathbb{R}^n$, and let B be a ball centered at o which is contained in D. If the image fB of the ball is contained in the image fD of the domain D together with some ball which is centered at $f(o)$ and contains fB, then the ratio $\max_{x\in\partial B} |f(x) - f(o)|/ \min_{x\in\partial B} |f(x) - f(o)|$ is bounded by $\varepsilon = \varepsilon(k, n)$ depending only on the coefficient of quasiconformality of a mapping and on the dimension of a space.

(Here, as usual, ∂B is the boundary of the domain B; in this case, it is the boundary sphere S of the ball B.)

It would be interesting to study the behavior of the ingectivity radius as a function of the dimension of a space. We suspect existence of a universal radius independent of the dimension of a space. In view of proof of the Martio–Rickman–Väisälä theorem, it is natural to investigate the asymptotics in dimension for the Teichmüller ring modulus and for the quantity $\varepsilon(k, n)$ entering the theorem of distortion.

The asymptotics of the conformal capacity of a Teichmüller ring was determined in [14].

Similarly to the case of ingectivity radius, we conjecture that there exists a *universal estimate $\epsilon(k)$ in the theorem of distortion independent of the dimension of a space.*

Moreover, certain general arguments lead us to a conjecture that a *universal estimate in the theorem of distortion is realized in the two-dimensional case.*

5. Isotopy

One of the key elements of the Teichmüller theory is the theorem of existence of the extreme (least nonconformal) quasiconformal mapping between two Riemann surfaces and the description of such mapping. This mapping has a constant coefficient of quasiconformality at any point. The starting element of the theorem is the Grötzsch lemma of linearity of the extreme quasiconformal mapping between two rectangles with a correspondence of vertices. (This lemma was known not only to Teichmüller.)

The Beltrami theorem with a description of the dependence of a normalized solution (mapping) on a parameter is useful not only in the complex dynamics. Such theorem enables one to deform (by an isotopy) a mapping into the identity one so that the coefficient of quasiconformality of a mapping at every point is continuous and tends monotonically to 1.

In the spatial case, such isotopy is possible not in all cases. But if, for example, the coefficient of quasiconformality of a mapping has an isolated local maximum, the mapping can be locally deformed by isotopy so that the common coefficient of quasiconformality decreases. The Teichmüller theorem indicates that, in the two-dimensional case, the elimination of "stresses" can be carried out to the complete constancy of the coefficient of quasiconformality of the extreme mapping.

6. Concluding comment

This short review is written on the basis of articles [14–16], where some details, explanations, and the reason for the very statement of the above-considered questions can be found, if necessary.

Here, we presented only those subjects that are close to the creative activity of Professor B. Bojarski.

One of the forthcoming issues of UMB will be dedicated to the memory of Georgii Dmitrievich Suvorov. I plan to present there several open questions that are related to the boundary behavior of mappings, i.e., the field of studies developed by G. D. Suvorov.

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